Research Article

An Application of Hybrid Steepest Descent Methods for Equilibrium Problems and Strict Pseudocontractions in Hilbert Spaces

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Received 9 December 2010; Accepted 13 February 2011

Academic Editor: Shusen Ding

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We use the hybrid steepest descent methods for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudocontraction mapping in the setting of real Hilbert spaces. We proved strong convergence theorems of the sequence generated by our proposed schemes.

1. Introduction

Let H be a real Hilbert space and C a closed convex subset of H , and let ϕ be a bifunction of $C \times C$ into R , where R is the set of real numbers. The equilibrium problem for $\phi : C \times C \rightarrow R$ is to find $x \in C$ such that

$$\text{EP} : \phi(x, y) \geq 0 \quad \forall y \in C \quad (1.1)$$

denoted the set of solution by $\text{EP}(\phi)$. Given a mapping $T : C \rightarrow H$, let $\phi(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$, then $z \in \text{EP}(\phi)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimizations, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem, see, for instance, [1, 2].

A mapping T of C into itself is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. In 2007, Plubtieng and Punpaeng [3], S. Takahashi and W. Takahashi [4], and Tada and W. Takahashi [5] considered iterative methods for finding an element of $\text{EP}(\phi) \cap F(T)$.

Recall that an operator A is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \quad (1.2)$$

In 2006, Marino and Xu [6] introduced the general iterative method and proved that for a given $x_0 \in H$, the sequence $\{x_n\}$ is generated by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \quad (1.3)$$

where T is a self-nonexpansive mapping on H , f is a contraction of H into itself with $\beta \in (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfies certain conditions, and A is a strongly positive bounded linear operator on H and converges strongly to a fixed-point x^* of T which is the unique solution to the following variational inequality:

$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0$, for $x \in F(T)$, and is also the optimality condition for some minimization problem. A mapping $S : C \rightarrow H$ is said to be k -strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \quad (1.4)$$

Note that the class of k -strict pseudo-contraction strictly includes the class of nonexpansive mapping, that is, S is nonexpansive if and only if S is 0-strictly pseudocontractive; it is also said to be pseudocontractive if $k = 1$. Clearly, the class of k -strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

The set of fixed points of S is denoted by $F(S)$. Very recently, by using the general approximation method, Qin et al. [7] obtained a strong convergence theorem for finding an element of $F(S)$. On the other hand, Ceng et al. [8] proposed an iterative scheme for finding an element of $EP(\phi) \cap F(S)$ and then obtained some weak and strong convergence theorems. Based on the above work, Y. Liu [9] introduced two iteration schemes by the general iterative method for finding an element of $EP(\phi) \cap F(S)$.

In 2001, Yamada [10] introduced the following hybrid iterative method for solving the variational inequality:

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n), \quad n \geq 0, \quad (1.5)$$

where F is k -Lipschitzian and η -strongly monotone operator with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$, then he proved that if $\{\lambda_n\}$ satisfies appropriate conditions, the $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of variational inequality

$$\langle F\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F_{ix}(T), \quad \tilde{x} \in F_{ix}(T). \quad (1.6)$$

Motivated and inspired by these facts, in this paper, we introduced two iteration methods by the hybrid iterative method for finding an element of $EP(\phi) \cap F(S)$, where $S : C \rightarrow H$ is a k -strictly pseudocontractive non-self mapping, and then obtained two strong convergence theorems.

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H . We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x . $x_n \rightarrow x$ implies that $\{x_n\}$ converges strongly to x . For any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (2.1)$$

Such a $P_C x$ is called the metric projection of H onto C . It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$, $u = p_C x \Leftrightarrow \langle x - u, u - y \rangle \geq 0$, for all $y \in C$.

It is widely known that H satisfies Opial's condition [11], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad (2.2)$$

holds for every $y \in H$ with $y \neq x$. In order to solve the equilibrium problem for a bifunction $\phi : C \times C \rightarrow R$, let us assume that ϕ satisfies the following conditions:

- (A1) $\phi(x, x) = 0$, for all $x \in C$,
- (A2) ϕ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$, for all $x, y \in C$,
- (A3) For all $x, y, z \in C$.

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y); \quad (2.3)$$

- (A4) For each fixed $x \in C$, the function $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 2.1 (see [12]). *Let ϕ be a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3) and (A4) then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (2.4)$$

Further, if $T_r x = \{z \in C : \phi(z, y) + 1/r \langle y - z, z - x \rangle \geq 0, \forall y \in C\}$, then the following hold:

- (1) T_r is single-valued,
- (2) T_r is firmly nonexpansive, that is,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H; \quad (2.5)$$

- (3) $F(T_r) = EP(\phi)$,
- (4) $EP(\phi)$ is nonempty, closed and convex.

Lemma 2.2 (see [13]). *If $S : C \rightarrow H$ is a k -strict pseudo-contraction, then the fixed-point set $F(S)$ is closed convex, so that the projection $P_{F(S)}$ is well defined.*

Lemma 2.3 (see [14]). *Let $S : C \rightarrow H$ be a k -strict pseudo-contraction. Define $T : C \rightarrow H$ by $Tx = \lambda x + (1 - \lambda)Sx$ for each $x \in C$, then, as $\lambda \in [k, 1)$, T is nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.4 (see [15]). *In a Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle, \quad \forall x, y \in H. \quad (2.6)$$

Lemma 2.5 (see [16]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0, \quad (2.7)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$, such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.
Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

Throughout the rest of this paper, we always assume that F is a L -lipschitzian continuous and η -strongly monotone operator with $L, \eta > 0$ and assume that $0 < \mu < 2\eta/L^2$. $\tau = \mu(\eta - \mu L^2/2)$. Let $\{T_{\lambda_n}\}$ be mappings defined as Lemma 2.1. Define a mapping $S_n : C \rightarrow H$ by $S_n x = \beta_n x + (1 - \beta_n)Sx$, for all $x \in C$, where $\beta_n \in [k, 1)$, then, by Lemma 2.3, S_n is nonexpansive. We consider the mapping G_n on H defined by

$$G_n x = (I - \alpha_n \mu F) S_n T_{\lambda_n} x, \quad x \in H, \quad n \in N, \quad (3.1)$$

where $\alpha_n \in (0, 1)$. By Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \|G_n x - G_n y\| &\leq (1 - \alpha_n \tau) \|T_{\lambda_n} x - T_{\lambda_n} y\| \\ &\leq (1 - \alpha_n \tau) \|x - y\|. \end{aligned} \quad (3.2)$$

It is easy to see that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed-point $x_n^F \in H$ such that

$$x_n^F = (I - \alpha_n \mu F) S_n T_{\lambda_n} x_n^F. \quad (3.3)$$

For simplicity, we will write x_n for x_n^F provided no confusion occurs. Next, we prove that the sequence $\{x_n\}$ converges strongly to a $q \in F(S) \cap \text{EP}(\phi)$ which solves the variational inequality

$$\langle Fq, p - q \rangle \geq 0, \quad \forall p \in F(S) \cap \text{EP}(\phi). \quad (3.4)$$

Equivalently, $q = P_{F(S) \cap \text{EP}(\phi)}(I - \mu F)q$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Let $S : C \rightarrow H$ be a k -strictly pseudocontractive nonself mapping such that $F(S) \cap \text{EP}(\phi) \neq \emptyset$. Let $F : H \rightarrow H$ be an L -Lipschitzian continuous and η -strongly monotone operator on H with $L, \eta > 0$ and $0 < \mu < 2\eta/L^2$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} \phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \beta_n u_n + (1 - \beta_n) S u_n, \\ x_n &= (I - \alpha_n \mu F) y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \quad (3.5)$$

where $u_n = T_{\lambda_n} x_n$, $y_n = S_n u_n$, and $\{\lambda_n\} \subset (0, +\infty)$ satisfy $\liminf_{n \rightarrow \infty} \lambda_n > 0$ if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ and $\lim_{n \rightarrow \infty} \beta_n = \lambda$,

then $\{x_n\}$ converges strongly to a point $q \in F(S) \cap \text{EP}(\phi)$ which solves the variational inequality (3.4).

Proof. First, take $p \in F(S) \cap \text{EP}(\phi)$. Since $u_n = T_{\lambda_n} x_n$ and $p = T_{\lambda_n} p$, from Lemma 2.1, for any $n \in \mathbb{N}$, we have

$$\|u_n - p\| = \|T_{\lambda_n} x_n - T_{\lambda_n} p\| \leq \|x_n - p\|. \quad (3.6)$$

Then, since $S_n p = p$, we obtain that

$$\|y_n - p\| = \|S_n u_n - S_n p\| \leq \|u_n - p\| \leq \|x_n - p\|. \quad (3.7)$$

Further, we have

$$\begin{aligned} \|x_n - p\| &= \|-\alpha_n \mu F p + (I - \mu \alpha_n F) y_n - (I - \mu \alpha_n F) p\| \\ &\leq \alpha_n \|-\mu F(p)\| + (1 - \alpha_n \tau) \|y_n - p\|. \end{aligned} \quad (3.8)$$

It follows that $\|x_n - p\| \leq \|\mu F(p)\|/\tau$.

Hence, $\{x_n\}$ is bounded, and we also obtain that $\{u_n\}$ and $\{y_n\}$ are bounded. Notice that

$$\begin{aligned}\|u_n - y_n\| &\leq \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \alpha_n \|\mu F y_n\|.\end{aligned}\quad (3.9)$$

By Lemma 2.1, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{\lambda_n} x_n - T_{\lambda_n} p\|^2 \leq \langle x_n - p, u_n - p \rangle \\ &= \frac{1}{2} (\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2).\end{aligned}\quad (3.10)$$

It follows that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \quad (3.11)$$

Thus, from Lemma 2.4, (3.7), and (3.11), we obtain that

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n(-\mu F p) + (I - \mu \alpha_n F) y_n - (I - \mu \alpha_n F) p\|^2 \\ &\leq (1 - \alpha_n \tau)^2 \|y_n - p\|^2 + 2\alpha_n \langle -\mu F p, x_n - p \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|u_n - p\|^2 + 2\alpha_n \langle -\mu F p, x_n - p \rangle \\ &\leq (1 - \alpha_n \tau)^2 (\|x_n - p\|^2 - \|x_n - u_n\|^2) + 2\alpha_n \|\mu F p\| \|x_n - p\| \\ &= (1 - 2\alpha_n \tau + (\alpha_n \tau)^2) \|x_n - p\|^2 \\ &\quad - (1 - \alpha_n \tau)^2 \|x_n - u_n\|^2 + \|2\alpha_n \mu F p\| \|x_n - p\| \\ &\leq \|x_n - p\|^2 + (\alpha_n \tau)^2 \|x_n - p\|^2 - (1 - \alpha_n \tau)^2 \|x_n - u_n\|^2 + 2\alpha_n \|\mu F p\| \|x_n - p\|.\end{aligned}\quad (3.12)$$

It follows that

$$(1 - \alpha_n \tau)^2 \|x_n - u_n\|^2 \leq (\alpha_n \tau)^2 \|x_n - p\|^2 + 2\alpha_n \|\mu F p\| \|x_n - p\|. \quad (3.13)$$

Since $\alpha_n \rightarrow 0$, therefore

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.14)$$

From (3.9), we derive that

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (3.15)$$

Define $T : C \rightarrow H$ by $Tx = \lambda x + (1 - \lambda)Sx$, then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.3. We note that

$$\|Tu_n - u_n\| \leq \|Tu_n - y_n\| + \|y_n - u_n\| \leq |\lambda - \beta_n| \|u_n - Su_n\| + \|y_n - u_n\|. \quad (3.16)$$

So by (3.15) and $\beta_n \rightarrow \lambda$, we obtain that

$$\lim_{n \rightarrow \infty} \|Tu_n - u_n\| = 0. \quad (3.17)$$

Since $\{u_n\}$ is bounded, so there exists a subsequence $\{u_{n_i}\}$ which converges weakly to q . Next, we show that $q \in F(S) \cap EP(\phi)$. Since C is closed and convex, C is weakly closed. So we have $q \in C$. Let us show that $q \in F(S)$. Assume that $q \notin F(T)$, Since $u_{n_i} \rightharpoonup q$ and $q \neq Tq$, it follows from the Opial's condition that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|u_{n_i} - q\| &< \liminf_{n \rightarrow \infty} \|u_{n_i} - Tq\| \\ &\leq \liminf_{n \rightarrow \infty} (\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tq\|) \\ &\leq \liminf_{n \rightarrow \infty} \|u_{n_i} - q\|. \end{aligned} \quad (3.18)$$

This is a contradiction. So, we get $q \in F(T)$ and $q \in F(S)$.

Next, we show that $q \in EP(\phi)$. Since $u_n = T_{\lambda_n} x_n$, for any $y \in C$, we obtain

$$\phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0. \quad (3.19)$$

From (A2), we have

$$\frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq \phi(y, u_n). \quad (3.20)$$

Replacing n by n_i , we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \geq \phi(y, u_{n_i}). \quad (3.21)$$

Since $(u_{n_i} - x_{n_i})/\lambda_{n_i} \rightarrow 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A4) that $0 \geq \phi(y, q)$, for all $y \in C$. Let $z_t = ty + (1 - t)q$ for all $t \in (0, 1]$ and $y \in C$, then we have $z_t \in C$ and hence $\phi(z_t, q) \leq 0$. Thus, from (A1) and (A4), we have

$$0 = \phi(z_t, z_t) \leq t\phi(z_t, y) + (1 - t)\phi(z_t, q) \leq t\phi(z_t, y), \quad (3.22)$$

and hence $0 \leq \phi(z_t, y)$. From (A3), we have $0 \leq \phi(q, y)$ for all $y \in C$ and hence $q \in \text{EP}(\phi)$. Therefore, $q \in F(S) \cap \text{EP}(\phi)$. On the other hand, we note that

$$x_n - q = -\alpha_n \mu Fq + (I - \mu \alpha_n F)y_n - (I - \mu \alpha_n F)q. \quad (3.23)$$

Hence, we obtain

$$\begin{aligned} \|x_n - q\|^2 &= \langle -\alpha_n \mu Fq, x_n - q \rangle + \langle (I - \mu \alpha_n F)y_n - (I - \mu \alpha_n F)q, x_n - q \rangle \\ &\leq \alpha_n \langle -\mu Fq, x_n - q \rangle + (1 - \alpha_n \tau) \|x_n - q\|^2. \end{aligned} \quad (3.24)$$

It follows that

$$\|x_n - q\|^2 \leq \frac{1}{\tau} \langle -\mu Fq, x_n - q \rangle. \quad (3.25)$$

This implies that

$$\|x_n - q\|^2 \leq \frac{\langle -\mu Fq, x_n - q \rangle}{\tau}. \quad (3.26)$$

In particular,

$$\|x_{n_i} - q\|^2 \leq \frac{\langle -\mu Fq, x_{n_i} - q \rangle}{\tau}. \quad (3.27)$$

Since $x_{n_i} \rightharpoonup q$, it follows from (3.27) that $x_{n_i} \rightarrow q$ as $i \rightarrow \infty$. Next, we show that q solves the variational inequality (3.4).

As a matter of fact, we have

$$\begin{aligned} x_n &= (I - \alpha_n \mu F)y_n \\ &= (I - \alpha_n \mu F)S_n T_{\lambda_n} x_n, \end{aligned} \quad (3.28)$$

and we have

$$\mu Fx_n = -\frac{1}{\alpha_n} \{ (I - S_n T_{\lambda_n})x_n - \mu \alpha_n (Fx_n - FS_n T_{\lambda_n} x_n) \}. \quad (3.29)$$

Hence, for $p \in F(S) \cap \text{EP}(\phi)$,

$$\begin{aligned} \langle (\mu F)x_n, x_n - p \rangle &= -\frac{1}{\alpha_n} \langle \{ (I - S_n T_{\lambda_n})x_n - \mu \alpha_n (Fx_n - FS_n T_{\lambda_n} x_n) \}, x_n - p \rangle \\ &= -\frac{1}{\alpha_n} \langle (I - S_n T_{\lambda_n})x_n - (I - S_n T_{\lambda_n})p, x_n - p \rangle + \mu \langle (Fx_n - FS_n T_{\lambda_n} x_n), x_n - p \rangle. \end{aligned} \quad (3.30)$$

Since $I - S_n T_{\lambda_n}$ is monotone (i.e., $\langle x - y, (I - S_n T_{\lambda_n})x - (I - S_n T_{\lambda_n})y \rangle \geq 0$, for all $x, y \in H$). This is due to the nonexpansivity of $S_n T_{\lambda_n}$.

Now replacing n in (3.30) with n_i and letting $i \rightarrow \infty$, we obtain

$$\begin{aligned} \langle (\mu F)q, q - p \rangle &= \lim_{i \rightarrow \infty} \langle \mu F x_{n_i}, x_{n_i} - p \rangle \\ &\leq \lim_{i \rightarrow \infty} \mu \langle F x_{n_i} - F S_n T_{\lambda_n} x_{n_i}, x_{n_i} - p \rangle = 0. \end{aligned} \quad (3.31)$$

That is, $q \in F(S) \cap \text{EP}(\phi)$ is a solution of (3.4). To show that the sequence $\{x_n\}$ converges strongly to q , we assume that $x_{n_k} \rightarrow \hat{x}$. Similiary to the proof above, we derive $\hat{x} \in F(S) \cap \text{EP}(\phi)$. Moreover, it follows from the inequality (3.31) that

$$\langle (\mu F)q, q - \hat{x} \rangle \leq 0. \quad (3.32)$$

Interchange q and $\hat{x}$ to obtain

$$\langle (\mu F)\hat{x}, \hat{x} - q \rangle \leq 0. \quad (3.33)$$

Adding up (3.32) and (3.33) yields

$$(\mu\eta) \|q - \hat{x}\|^2 \leq \langle q - \hat{x}, (\mu F)q - (\mu F)\hat{x} \rangle \leq 0. \quad (3.34)$$

Hence, $q = \hat{x}$, and therefore $x_n \rightarrow q$ as $n \rightarrow \infty$,

$$\langle (I - \mu F)q - q, q - p \rangle \geq 0, \forall p \in F(S) \cap \text{EP}(\phi). \quad (3.35)$$

This is equivalent to the fixed-point equation

$$P_{F(S) \cap \text{EP}(\phi)}(I - \mu F)q = q. \quad (3.36)$$

□

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Let $S : C \rightarrow H$ be a k -strictly pseudocontractive nonself mapping such that $F(S) \cap \text{EP}(\phi) \neq \emptyset$. Let $F : H \rightarrow H$ be an L -Lipschitzian continuous and η -strongly monotone operator on H with $L, \eta > 0$. Suppose that $0 < \mu < 2\eta/L^2$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{aligned} \phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \beta_n u_n + (1 - \beta_n) S u_n, \\ x_{n+1} &= (I - \alpha_n \mu F) y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \quad (3.37)$$

where $u_n = T_{\lambda_n}x_n$, $y_n = S_n u_n$ if $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
- (ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ and $\lim_{n \rightarrow \infty} \beta_n = \lambda$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
- (iii) $\{\lambda_n\} \in (0, +\infty)$, $\lim_{n \rightarrow \infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $q \in F(S) \cap \text{EP}(\phi)$ which solves the variational inequality (3.4).

Proof. We first show that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(S) \cap \text{EP}(\phi)$ to derive that

$$\begin{aligned} \|x_{n+1} - p\| &= \|- \alpha_n \mu F p + (I - \mu \alpha_n F) y_n - (I - \mu \alpha_n F) p\| \\ &\leq \alpha_n \|\mu F(p)\| + (1 - \alpha_n \tau) \|x_n - p\| \\ &\leq (1 - \alpha_n \tau) \|x_n - p\| + \alpha_n \|\mu F(p)\|. \end{aligned} \quad (3.38)$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{\tau} \|\mu F(p)\| \right\}, \quad \forall n \in N, \quad (3.39)$$

and hence $\{x_n\}$ is bounded. From (3.6) and (3.7), we also derive that $\{u_n\}$ and $\{y_n\}$ are bounded. Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$. We have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(I - \alpha_n \mu F) y_n - (I - \alpha_{n-1} \mu F) y_{n-1}\| \\ &= \|(I - \alpha_n \mu F) y_n - (I - \alpha_n \mu F) y_{n-1} + (I - \alpha_n \mu F) y_{n-1} - (I - \alpha_{n-1} \mu F) y_{n-1}\| \\ &\leq (1 - \alpha_n \tau) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu F y_{n-1}\| \\ &\leq (1 - \alpha_n \tau) \|y_n - y_{n-1}\| + K |\alpha_n - \alpha_{n-1}|, \end{aligned} \quad (3.40)$$

where

$$K = \sup \{ \|\mu F y_n\| : n \in N \} < \infty. \quad (3.41)$$

On the other hand, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|S_n u_n - S_{n-1} u_{n-1}\| \\ &\leq \|S_n u_n - S_n u_{n-1}\| + \|S_n u_{n-1} - S_{n-1} u_{n-1}\| \\ &\leq \|u_n - u_{n-1}\| + \|S_n u_{n-1} - S_{n-1} u_{n-1}\|. \end{aligned} \quad (3.42)$$

From $u_{n+1} = T_{\lambda_{n+1}}x_{n+1}$ and $u_n = T_{\lambda_n}x_n$, we note that

$$\phi(u_{n+1}, y) + \frac{1}{\lambda_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C, \quad (3.43)$$

$$\phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.44)$$

Putting $y = u_n$ in (3.43) and $y = u_{n+1}$ in (3.44), we have

$$\begin{aligned} \phi(u_{n+1}, u_n) + \frac{1}{\lambda_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0, \\ \phi(u_n, u_{n+1}) + \frac{1}{\lambda_n} \langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0. \end{aligned} \quad (3.45)$$

So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}} \right\rangle \geq 0, \quad (3.46)$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{\lambda_n}{\lambda_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.47)$$

Since $\lim_{n \rightarrow \infty} \lambda_n > 0$, without loss of generality, let us assume that there exists a real number a such that $\lambda_n > a > 0$ for all $n \in N$. Thus, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\} \\ \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{a} |\lambda_{n+1} - \lambda_n| M_0, \end{aligned} \quad (3.48)$$

where $M_0 = \sup\{\|u_n - x_n\| : n \in N\}$. Next, we estimate $\|S_n u_{n-1} - S_{n-1} u_{n-1}\|$. Notice that

$$\begin{aligned} \|S_n u_{n-1} - S_{n-1} u_{n-1}\| &= \|(\beta_n u_{n-1} + (1 - \beta_n) S u_{n-1}) - (\beta_{n-1} u_{n-1} + (1 - \beta_{n-1}) S u_{n-1})\| \\ &\leq |\beta_n - \beta_{n-1}| \|u_{n-1} - S u_{n-1}\|. \end{aligned} \quad (3.49)$$

From (3.48), (3.49), and (3.42), we obtain that

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{M_0}{a} |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| \|u_{n-1} - S u_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| M_1 + |\beta_n - \beta_{n-1}| M_1, \end{aligned} \quad (3.50)$$

where M_1 is an appropriate constant such that

$$M_1 \geq \frac{M_0}{a} + \|u_{n-1} - Su_{n-1}\|, \quad \forall n \in N. \quad (3.51)$$

From (3.41) and (3.50), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq K|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n\tau)(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + |\beta_n - \beta_{n-1}|M_1) \\ &\leq (1 - \alpha_n\tau)\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|), \end{aligned} \quad (3.52)$$

where $M = \max[K, M_1]$. Hence, few by Lemma 2.5, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.53)$$

From (3.48) and (3.50), $|\lambda_n - \lambda_{n-1}| \rightarrow 0$ and $|\beta_n - \beta_{n-1}| \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.54)$$

Since

$$x_{n+1} = (I - \alpha_n \mu F)y_n, \quad (3.55)$$

it follows that

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n \|\mu F y_n\|. \end{aligned} \quad (3.56)$$

From $\alpha_n \rightarrow 0$ and (3.53), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.57)$$

For $p \in F(S) \cap \text{EP}(\phi)$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{\lambda_n} x_n - T_{\lambda_n} p\|^2 \leq \langle x_n - p, u_n - p \rangle \\ &= \frac{1}{2} (\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2). \end{aligned} \quad (3.58)$$

This implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \quad (3.59)$$

Then, from (3.7) and (3.59), we derive that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|\mu\alpha_n Fp + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)p\|^2 \\
 &\leq (1 - \alpha_n \tau)^2 \|y_n - p\|^2 + \alpha_n^2 \|\mu Fp\|^2 + 2\alpha_n \|\mu Fp\| \|y_n - p\| \\
 &\leq \|u_n - p\|^2 + \alpha_n^2 \|\mu Fp\|^2 + 2\alpha_n \|\mu Fp\| \|y_n - p\| \\
 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + \alpha_n^2 \|\mu Fp\|^2 + 2\alpha_n \|\mu Fp\| \|y_n - p\|.
 \end{aligned} \tag{3.60}$$

Since $\alpha_n \rightarrow 0$, $\|x_n - x_{n+1}\| \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{3.61}$$

From (3.57) and (3.61), we obtain that

$$\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.62}$$

Define $T : C \rightarrow H$ by $Tx = \lambda x + (1 - \lambda)Sx$, then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.3. Notice that

$$\begin{aligned}
 \|Tu_n - u_n\| &\leq \|Tu_n - y_n\| + \|y_n - u_n\| \\
 &\leq |\lambda - \beta_n| \|u_n - Su_n\| + \|y_n - u_n\|.
 \end{aligned} \tag{3.63}$$

By (3.62) and $\beta_n \rightarrow \lambda$, we obtain that

$$\lim_{n \rightarrow \infty} \|Tu_n - u_n\| = 0. \tag{3.64}$$

Next, we show that $\limsup_{n \rightarrow \infty} \langle \mu Fq, q - x_n \rangle \leq 0$, where $q = P_{F(S) \cap \text{EP}(\phi)}(I - \mu F)q$ is a unique solution of the variational inequality (3.4). Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i \rightarrow \infty} \langle \mu Fq, q - x_{n_i} \rangle = \limsup_{n \rightarrow \infty} \langle \mu Fq, q - x_n \rangle. \tag{3.65}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w .

Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From (3.61) and (3.64), we obtain $x_{n_i} \rightharpoonup w$ and $Tu_{n_i} \rightharpoonup w$. By the same argument as in the proof of Theorem 3.1, we have $w \in F(S) \cap \text{EP}(\phi)$. Since $q = P_{F(S) \cap \text{EP}(\phi)}(I - \mu F)q$, it follows that

$$\limsup_{n \rightarrow \infty} \langle \mu Fq, q - x_n \rangle = \langle \mu Fq, q - w \rangle \leq 0. \tag{3.66}$$

From $x_{n+1} - q = -\alpha_n \mu Fq + (I - \mu \alpha_n F)y_n - (I - \mu \alpha_n F)q$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|(I - \mu \alpha_n F)y_n - (I - \mu \alpha_n F)q\|^2 + 2\alpha_n \langle -\mu Fq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \langle -\mu Fq, x_{n+1} - q \rangle. \end{aligned} \quad (3.67)$$

This implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \{1 - 2\alpha_n \tau + (\alpha_n \tau)^2\} \|x_n - q\|^2 + 2\alpha_n \langle -\mu Fq, x_{n+1} - q \rangle \\ &= (1 - 2\alpha_n \tau) \|x_n - q\|^2 + (\alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \langle -\mu Fq, x_{n+1} - q \rangle \\ &= (1 - 2\alpha_n \tau) \|x_n - q\|^2 + 2\alpha_n \tau \left\{ \frac{\alpha_n \tau^2}{2\tau} M^* + \frac{1}{\tau} \langle -\mu Fq, x_{n+1} - q \rangle \right\} \\ &= (1 - \gamma_n) \|x_n - q\|^2 + \gamma_n \delta_n, \end{aligned} \quad (3.68)$$

where $M^* = \sup\{\|x_n - q\|^2 : n \in N\}$, $\gamma_n = 2\alpha_n \tau$, and $\delta_n = (\alpha_n \tau^2 / 2\tau) M^* + (1/\tau) \langle -\mu Fq, x_{n+1} - q \rangle$.

It is easy to see that $\gamma_n \rightarrow 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ by (3.66). Hence by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to q . $\square$

Acknowledgments

M. Tian was supported in part by the Science Research Foundation of Civil Aviation University of China (no. 2010kys02). He was also Supported in part by The Fundamental Research Funds for the Central Universities (Grant no. ZXH2009D021).

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