Research Article

$L^p$ Approximation by Multivariate Baskakov-Durrmeyer Operator

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The main aim of this paper is to introduce and study multivariate Baskakov-Durrmeyer operator, which is nontensor product generalization of the one variable. As a main result, the strong direct inequality of $L^p$ approximation by the operator is established by using a decomposition technique.

1. Introduction

Let $P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \; x \in [0, \infty), \; n \in \mathbb{N}$. The Baskakov operator defined by

$$ B_{n,1}(f, x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n}\right) $$

(1.1)

was introduced by Baskakov [1] and can be used to approximate a function $f$ defined on $[0, \infty)$. It is the prototype of the Baskakov-Kantorovich operator (see [2]) and the Baskakov-Durrmeyer operator defined by (see [3, 4])

$$ M_{n,1}(f, x) = \sum_{k=0}^{\infty} P_{n,k}(x)(n-1) \int_{0}^{\infty} P_{n,k}(t) f(t) dt, \; x \in [0, \infty), $$

(1.2)

where $f \in L^p[0, \infty)(1 \leq p < \infty)$.

By now, the approximation behavior of the Baskakov-Durrmeyer operator is well understood. It is characterized by the second-order Ditzian-Totik modulus (see [3])

$$ \omega^2_p(f, t)_p = \sup_{0<h\leq t} \left\| f(\cdot + 2h\varphi(\cdot)) - 2f(\cdot + h\varphi(\cdot)) + f(\cdot) \right\|_p, \; \varphi(x) = \sqrt{x(1+x)}. $$

(1.3)
More precisely, for any function defined on $L^p[0, \infty) (1 \leq p < \infty)$, there is a constant such that

$$
\|M_{n,1}(f) - f\|_p \leq \text{const.} \left( \omega^2_p \left( f, \frac{1}{\sqrt{n}} \right)_p + \frac{1}{n} \|f\|_p \right),
$$

(1.4)

$$
\omega^2_p(f,t)_p = O(t^{2\alpha}) \iff \|M_{n,1}(f) - f\|_p = O(n^{-\alpha}),
$$

(1.5)

where $0 < \alpha < 1$.

Let $T \subset \mathbb{R}^d (d \in \mathbb{N})$, which is defined by

$$
T := T_d := \{ x := (x_1, x_2, \ldots, x_d) : 0 \leq x_i < \infty, 1 \leq i \leq d \}. \quad (1.6)
$$

Here and in the following, we will use the standard notations

$$
x := (x_1, x_2, \ldots, x_d), \quad k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}_0^d,
$$

$$
x^k := x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \quad k! = k_1! k_2! \cdots k_d!, \quad |x| := \sum_{i=1}^d x_i, \quad |k| := \sum_{i=1}^d k_i,
$$

$$
\binom{n}{k} := \frac{n!}{k!(n-|k|)!}, \quad \sum_{k=0}^\infty := \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \cdots \sum_{k_d=0}^\infty.
$$

(1.7)

By means of the notations, for a function $f$ defined on $T$ the multivariate Baskakov operator is defined as (see [5])

$$
B_{n,d}(f, x) := \sum_{k=0}^\infty f \left( \frac{k}{n} \right) P_{n,k}(x),
$$

(1.8)

where

$$
P_{n,k}(x) = \binom{n + |k| - 1}{k} x^k (1 + |x|)^{-n-|k|}.
$$

(1.9)

Naturally, we can modify the multivariate Baskakov operator as multivariate Baskakov-Durrmeyer operator

$$
M_{n,d} f := M_{n,d}(f, x) := \sum_{k=0}^\infty P_{n,k}(x) \phi_{n,k,d}(f), \quad f \in L^p(T),
$$

(1.10)

where

$$
\phi_{n,k,d}(f) := \frac{\int_T P_{n,k}(u) f(u) du}{\int_T P_{n,k}(u) du} = (n-1)(n-2) \cdots (n-d) \int_T P_{n,k}(u) f(u) du.
$$

(1.11)
It is a multivariate generalization of the univariate Baskakov-Durrmeyer operators given in (1.2) and can be considered as a tool to approximate the function in \(L^p(T)\).

2. Main Result

We will show a direct inequality of \(L^p\) approximation by the Baskakov-Durrmeyer operator given in (1.10). By means of K-functional and modulus of smoothness defined in [5], we will extend (1.4) to the case of higher dimension by using a decomposition technique.

For \(x \in T\), we define the weight functions

\[
\varphi_i(x) = \sqrt{x_i(1 + |x|)}, \quad 1 \leq i \leq d.
\]  

(2.1)

Let

\[
D^r_i = \frac{\partial^r}{\partial x^r_i}, \quad r \in \mathbb{N}, \quad D^k = D^k_1 D^k_2 \cdots D^k_d, \quad k \in \mathbb{N}_0^d
\]  

(2.2)

denote the differential operators. For \(1 \leq p < \infty\), we define the weighted Sobolev space as follows:

\[
W^p_{q,r}(T) = \left\{ f \in L^p(T) : D^k f \in L_{loc}(T), \quad \varphi_r^i D^r_i f \in L^p(T) \right\},
\]  

(2.3)

where \(|k| \leq r\), \(k \in \mathbb{N}_0^d\), and \(T\) denotes the interior of \(T\). The Peetre K-functional on \(L^p(T) (1 \leq p < \infty)\), are defined by

\[
K^r_p(f, t') = \inf \left\{ \|f - g\|_p + t' \sum_{i=1}^d \|\varphi_r^i D^r_i g\|_p \right\}, \quad t > 0,
\]  

(2.4)

where the infimum is taken over all \(g \in W^p_{q,r}(T)\).

For any vector \(e\) in \(\mathbb{R}^d\), we write the \(r\)th forward difference of a function \(f\) in the direction of \(e\) as

\[
\Delta^r_{he} f(x) = \begin{cases} \sum_{i=0}^r \binom{r}{i} (-1)^i f(x + ihe), & x + r he \in T, \\ 0, & \text{otherwise.} \end{cases}
\]  

(2.5)

We then can define the modulus of smoothness of \(f \in L^p(T) (1 \leq p < \infty)\), as

\[
\omega_r^p(f, t) = \sup_{0 < \delta \leq t} \sum_{i=1}^d \|\Delta^r_{he} f\|_p,
\]  

(2.6)

where \(e_i\) denotes the unit vector in \(\mathbb{R}^d\), that is, its \(i\)th component is 1 and the others are 0.

In [5], the following result has been proved.
Lemma 2.1. There exists a positive constant, dependent only on $p$ and $r$, such that for any $f \in L^p(T)$, $1 \leq p < \infty$

$$\frac{1}{\text{const.}} \omega_p^r(f, t)_p \leq K_p^r(f, t^r)_p \leq \text{const.} \omega_p^r(f, t)_p. \quad (2.7)$$

Now we state the main result of this paper.

Theorem 2.2. If $f \in L^p(T)$, $1 \leq p < \infty$, then there is a positive constant independent of $n$ and $f$ such that

$$\|M_{n,d}f - f\|_p \leq \text{const.} \left( \omega_p^r(f, \frac{1}{\sqrt{n}})_p + \frac{1}{n} \|f\|_p \right). \quad (2.8)$$

Proof. Our proof is based on an induction argument for the dimension $d$. We will also use a decomposition method of the operator $M_{n,d}f$. We report the detailed proof only for two dimensions. The higher dimensional cases are similar.

Our proof depends on Lemma 2.1 and the following estimates:

$$\|M_{n,2}f - f\|_p \leq \text{const.} \begin{cases} \|f\|_{p'} & f \in L^p(T), \\ \frac{1}{n} \left( \sum_{i=1}^{2} \|q_i^D f\|_p + \|f\|_p \right) & f \in W_2^{p,2}(T). \end{cases} \quad (2.9)$$

The first estimate is evident as the $M_{n,d}f$ are positive and linear contractions on $L^p(T)(1 \leq p < \infty)$. We can demonstrate the second estimate by reducing it to the one dimensional inequality

$$\|M_{n,1}f - f\|_p \leq \frac{\text{const.}}{n} \left( \|q^2 f''\|_p + \|f\|_p \right), \quad (2.10)$$

which has been proved in [3].

Now we give the following decomposition formula:

$$M_{n,2}(f, x) = \sum_{k_i=0}^{\infty} \sum_{k_2=0}^{x_i} P_{n,k_i}(x_1) P_{n+k_i,k_2} \left( \frac{x_2}{1+x_1} \right) (n-1)(n-2)$$

$$\times \int_0^\infty P_{n,k_1}(u_1) P_{n+k_1,k_2} \left( \frac{u_2}{1+u_1} \right) f(u_1, u_2) du_1 du_2$$

$$= \sum_{k_i=0}^{\infty} P_{n,k_i}(x_1) (n-2) \int_0^\infty P_{n-1,k_i}(u_1) \sum_{k_2=0}^{\infty} P_{n+k_1,k_2} \left( \frac{x_2}{1+x_1} \right)$$

$$\times (n + k_1 - 1) \int_0^\infty P_{n+k_i,k_2}(t) f(u_1, (1+u_1)t) dt du_1$$

$$= \sum_{k_i=0}^{\infty} P_{n,k_i}(x_1) (n-2) \int_0^\infty P_{n-1,k_i}(u_1) M_{n+k_i,1}(\xi_{n_1}, z) du_1.$$
where
\[ g_{u_1}(t) = f(u_1, (1 + u_1)t), \quad 0 \leq t < \infty, \quad z = \frac{x_2}{1 + x_1}, \]
(2.12)

which can be checked directly and will play an important role in the following proof.

From the decomposition formula, it follows that
\[
M_{n,2}(f, x) - f(x) = \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2)
\times \left\{ \int_{0}^{\infty} P_{n-1,k_1}(u_1) \left( M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z) \right) du_1 \right\} + M_{n,1}^*(h(\cdot), x_1) - h(x_1)
:= J + L,
\]
(2.13)

where
\[
h(u_1) := h(u_1, x) := f(u_1, (1 + u_1)\frac{x_2}{1 + x_1}), \quad 0 \leq u_1 < \infty,
\]
(2.14)

Then by the Jensen’s inequality, we have
\[
\|J\|_p^p \leq \int_{\Gamma} \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) \left| (n-2) \int_{0}^{\infty} P_{n-1,k_1}(u_1) \left( M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z) \right) du_1 \right|^p dx
\leq \int_{\Gamma} \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) \left( n-2 \right) \int_{0}^{\infty} P_{n-1,k_1}(u_1) \left( M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z) \right)^p du_1 dx
\]
\[
= \int_{0}^{\infty} \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(1 + x_1)dx_1(n-2) \int_{0}^{\infty} P_{n-1,k_1}(u_1)
\times \left| (M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) \right|^p dz du_1
\leq \sum_{k_1=0}^{\infty} \frac{n + k_1 - 1}{n - 1} \int_{0}^{\infty} P_{n-1,k_1}(u_1) \left( M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z) \right)^p dz du_1
\leq \text{const} \sum_{k_1=0}^{\infty} \frac{n + k_1 - 1}{n - 1} \int_{0}^{\infty} P_{n-1,k_1}(u_1) \left( \frac{1}{n + k_1} \right)^p \left( \|\varphi^2 g_{u_1}''\|_p + \|g_{u_1}\|_p^p \right) du_1.
\]
(2.15)

However, by definition, one also has
\[
\varphi^2(t) g_{u_1}''(t) = t(1 + t)(1 + u_1)^2 D_2^2 f(u_1, (1 + u_1)t) = \left( \psi_2^2 D_2^2 f \right)(u_1, (1 + u_1)t).
\]
(2.16)
Therefore,

\[
\|J\|_p^p \leq \text{const.} \sum_{k_1=0}^\infty \frac{n+k_1-1}{(n-1)(n+k_1)^p} \int_0^\infty P_{n-1,k_1}(u_t) \times \left( \left| \left( \phi_2^2 D_2^2 f \right)(u_t, (1+u_t)t) \right|^p + \left| f(u_t, (1+u_t)t) \right|^p \right) dt \, du_t
\]

\[
= \text{const.} \sum_{k_1=0}^\infty \frac{n+k_1-1}{(n-1)(n+k_1)^p} \int_0^\infty \frac{1}{1+u_t} P_{n-1,k_1}(u_t) \times \int_0^\infty \left( \left| \left( \phi_2^2(u_t, u_{21})D_2^2 f(u_t, u_{22}) \right) \right|^p + \left| f(u_t, u_{22}) \right|^p \right) du_t \, du_{u_{22}}
\]

\[
\leq \frac{\text{const.}}{n^p} \sum_{k_1=0}^\infty \int_0^\infty P_{n,k_1}(u_1) \int_0^\infty \left( \left| \left( \phi_2^2(u_t, u_{21})D_2^2 f(u_t, u_{22}) \right) \right|^p + \left| f(u_t, u_{22}) \right|^p \right) du_t \, du_{u_{22}}
\]

\[
= \frac{\text{const.}}{n^p} \left( \| \phi_2^2 D_2^2 f \|_p^p + \| f \|_p^p \right).
\]

To estimate the second term \( L \), we use a similar method as to estimate (2.10) (see [3]) and can get

\[
\|L\|_p \leq \frac{\text{const.}}{n} \left( \| \phi^2 h'' \|_p + \| h \|_p \right).
\]

(2.18)

Denoting \( \phi_{12}(x) = \phi_{21}(x) := \sqrt{x_1 x_2}, D_{12}^2 := \phi^2/(\partial x_1 \partial x_2) \), and \( D_{21}^2 := \phi^2/(\partial x_2 \partial x_1) \), we have

\[
\left| \phi_2^2(s) h''(s) \right|
\]

\[
= \left| s(1+s) \left( D_{12}^2 f + \frac{x_2}{1+x_1} D_{12}^2 f + \frac{x_2}{1+x_1} D_{21}^2 f + \frac{x_2^2}{(1+x_1)^2} D_{22}^2 f \right) \times \left( s, (1+s) \frac{x_2}{1+x_1} \right) \right|
\]

\[
= \left| \left( \frac{1+x_1}{1+x_1+x_2} \phi_1^2 D_1^2 f + \phi_1^2 D_{12}^2 f + \phi_1^2 D_{21}^2 f + \frac{s x_2}{1+s} \frac{x_2}{1+x_1+x_2} \phi_2^2 D_2^2 f \right) \left( s, (1+s) \frac{x_2}{1+x_1} \right) \right|
\]

(2.19)

Recalling that \( \phi_{12}(x) \) is no bigger than \( \phi_1(x) \) or \( \phi_2(x) \), and the fact

\[
\left| D_{12}^2 f(x) \right| \leq \sup \left( \left| D_1^2 f(x) \right|, \left| D_2^2 f(x) \right| \right)
\]

(2.20)

proved in [6] (see [6, Lemma 2.1]), we obtain

\[
\left\| \phi^2 h'' \right\|_p \leq \text{const.} \sum_{i=1}^2 \left\| \phi_i^2 D_i^2 f \right\|_{p'}
\]

(2.21)
and hence
\[ \|L\|_p \leq \text{const.} \frac{1}{n} \left( \sum_{i=1}^{2} \|q_i D^2 f\|_p + \|f\|_p \right). \] (2.22)

The second inequality of (2.9) has thus been established, and the proof of Theorem 2.2 is finished.

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**References**