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Research Article

Size of Convergence Domains for Generalized Hausdorff Prime Matrices

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We show that there exit E-J generalized Hausdorff matrices and unbounded sequences x such that each matrix has convergence domain $c \oplus x$.

1. Introduction

The convergence domain of an infinite matrix $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$ will be denoted by (A) and is defined by $(A) := \{x = \{x_n\} \mid A_n(x) \in c\}$, where c denotes the space of convergence sequences, $A_n(x) := \sum_k a_{nk} x_k$. The necessary and sufficient conditions of Silverman and Toeplitz for a matrix to be conservative are $\lim_n a_{nk} = a_k$ exists for each k, $\lim_n \sum_{k=0}^{\infty} a_{nk} = t$ exists, and $\|A\| := \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$. A conservative matrix A is called multiplicative if each $a_k = 0$ and regular if, in addition, $t = 1$.

The E-J generalized Hausdorff matrices under consideration were defined independently by Endl ([1, 2]) and Jakimovski [3]. Each matrix $H_\mu^{(\alpha)}$ is a lower triangular matrix with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \tag{1.1}$$

where α is real number, $\{\mu_n\}$ is a real or complex sequence and Δ is forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$. We will consider here only nonnegative α. For $\alpha = 0$, one obtains an ordinary Hausdorff matrix.

From [1] or [3] a E-J generalized Hausdorff matrix (for $\alpha > 0$) is regular if and only if there exists a function $\chi \in BV[0,1]$ with $\chi(1) - \chi(0+) = 1$ such that

$$\mu_n^{(\alpha)} = \int_0^1 t^{n+\alpha} d\chi(t), \tag{1.2}$$

in which case χ is called the moment generating function, or mass function, for $H_\mu^{(\alpha)}$ and $\mu_n^{(\alpha)}$ is called moment sequence.

For ordinary Hausdorff summability [4], the necessary and sufficient conditions, for regularity are that function $\chi \in BV[0,1]$, $\chi(1) - \chi(0) = 1$, $\chi(0+) = \chi(0)$, and (1.2) is satisfied with $\alpha = 0$.

As noted in [5], the set of all multiplicative Hausdorff matrices forms a commutative Banach algebra that is also an integral domain, making it possible to define the concepts of unit, prime, divisibility, associate, multiple, and factor. Hille and Tamarkin ([6, 7]), using some techniques from [8], showed that every Hausdorff matrix with moment function

$$\mu(z) = \frac{z-a}{z+b}, \qquad R(a) > 0, \qquad R(b) > 0 \tag{1.3}$$

is prime. In 1967, Rhoades [9] showed that the convergence domain of every known prime Hausdorff matrix is of the form $c \oplus x$ for a particular unbounded sequence x.

Given any unbounded sequence x, Zeller [10] constructed a regular matrix A with convergence domain $(A) = c \oplus x$. It has been shown by Parameswaran [11] that if x is any unbounded sequence such that $\{x_n - x_{n-1}\}$ is bounded, divergent, and Borel summable, then no Hausdorff matrix H exists with $(H) = c \oplus x$.

The main result of this paper is to show that there exist E-J generalized Hausdorff matrices $H_\mu^{(\alpha)}$ whose moment sequences are

$$\mu_n^{(\alpha)} = \frac{n-a}{n+b+\alpha}, \qquad R(a) > 0, \qquad R(b) > 0, \tag{1.4}$$

and unbounded sequences $x^{(\alpha)}$ such that each matrix has convergent domain $c \oplus x^{(\alpha)}$.

Define the sequences $x^{(\alpha)}$ by

$$x_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} \quad \text{for } R(a) > 0, \tag{1.5}$$

where it is understood that if a is positive integer, then $x_n^{(\alpha)} = 0$ for $n = 0, 1, \ldots, a-1$.

If $\lambda_n^{(\alpha)}$ is the moment sequence defined by $(n-a)/(n+1+\alpha)$, $R(a) > 0$, then it is clear that $(H_\mu^{(\alpha)}) = (H_\lambda^{(\alpha)})$. Hence, it will be sufficient to prove the theorem by using $b = 1$, in (1.4). To have the convenience of regularity, we will use the sequence

$$\mu_n^{(\alpha)} = \frac{n-a}{-(a+\alpha)(n+1+\alpha)}, \tag{1.6}$$

since the constant $-1/(a+\alpha)$ does not affect the size of the convergence domain of $H_\mu^{(\alpha)}$.

2. Auxiliary Results

In order to prove the main theorem of this paper, we will need the following results.

Lemma 2.1. *Let $A, B \in \mathbb{C}$, $d, n \in \mathbb{N} \cup \{0\}$, $d \leq n$. Then, formally, for any n,*

$$\sum_{k=d}^{n} \frac{\Gamma(A+k)}{\Gamma(B+k)} = \frac{1}{A-B+1}\left[\frac{\Gamma(A+n+1)}{\Gamma(B+n)} - \frac{\Gamma(A+d)}{\Gamma(B-1+d)}\right]. \tag{2.1}$$

Proof. Lemma 2.1 appears as formula 12 on page 138 of [12]. □

Lemma 2.2. *For m, n integers $n > m+1 > a$, $x_n^{(\alpha)}$ as in* (1.5),

$$\sum_{k=m+1}^{n} \frac{1}{x_k^{(\alpha)}(k-a)} = \frac{1}{a+\alpha}\left(\frac{1}{x_m^{(\alpha)}} - \frac{1}{x_n^{(\alpha)}}\right). \tag{2.2}$$

Proof. Using Lemma 2.1,

$$\begin{aligned}
\sum_{k=m+1}^{n} \frac{1}{x_k^{(\alpha)}(k-a)} &= \sum_{k=m+1}^{n} \frac{\Gamma(k-a)}{\Gamma(k+\alpha+1)} \\
&= \frac{1}{-a-(\alpha+1)+1}\left[\frac{\Gamma(n+1-a)}{\Gamma(n+1+\alpha)} - \frac{\Gamma(m+1-a)}{\Gamma(m+1+\alpha)}\right] \\
&= \frac{1}{a+\alpha}\left(\frac{1}{x_m^{(\alpha)}} - \frac{1}{x_n^{(\alpha)}}\right).
\end{aligned} \tag{2.3}$$

□

Lemma 2.3. *For $0 \leq r \leq a$*

$$-\sum_{j=a+1}^{n-1} \left(h_{nj}^{(\alpha)}\right)^{-1} h_{jr}^{(\alpha)} = \frac{x_n^{(\alpha)}}{\Gamma(a+\alpha+1)} - \frac{(a+\alpha+1)}{n-a}. \tag{2.4}$$

Proof. $\mu_n^{(\alpha)}$ can be written as

$$\mu_n^{(\alpha)} = \frac{-1}{a+\alpha} + \frac{a+\alpha+1}{(a+\alpha)(n+\alpha+1)}, \tag{2.5}$$

so that, for $0 \le k < n$, $h_{nk}^{(\alpha)} = (a+\alpha+1)/(a+\alpha)(n+1+\alpha)$. From Lemma 2.1 and (3.11),

$$
\begin{aligned}
-\sum_{j=a+1}^{n-1}\left(h_{nj}^{(\alpha)}\right)^{-1}h_{jr}^{(\alpha)} &= -\sum_{j=a+1}^{n-1}\frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)^2}{x_j^{(\alpha)}(j-a)(a+\alpha)(j+\alpha+1)} \\
&= (a+\alpha+1)^2 x_n^{(\alpha)}\sum_{j=a+1}^{n-1}\frac{\Gamma(j-a)}{\Gamma(j+2+\alpha)} \\
&= \frac{(a+\alpha+1)^2 x_n^{(\alpha)}}{(a+\alpha+1)}\left(\frac{1}{\Gamma(a+\alpha+2)}-\frac{\Gamma(n-a)}{\Gamma(n+1+\alpha)}\right) \\
&= \frac{x_n^{(\alpha)}}{\Gamma(a+\alpha+1)}-\frac{(a+\alpha+1)}{n-a}.
\end{aligned}
\tag{2.6}
$$

□

3. Main Result

Theorem 3.1. *If for fixed a and b the matrix $H_\mu^{(\alpha)}$ is defined by* (1.4) *and a sequence $x^{(\alpha)}$ by* (1.5), *then $(H_\mu^{(\alpha)}) = c \oplus x^{(\alpha)}$.*

Proof. We will first show that $c \oplus x^{(\alpha)} \subseteq (H_\mu^{(\alpha)})$.

We can write the matrix $H_\mu^{(\alpha)} = (-1/(a+\alpha))(I - H_\lambda^{(\alpha)})$, where the diagonal entries of $H_\lambda^{(\alpha)}$ are

$$
\lambda_n^{(\alpha)} = \frac{a+\alpha+1}{n+\alpha+1}. \tag{3.1}
$$

For each n and k,

$$
\begin{aligned}
\Delta^{n-k}\lambda_k^{(\alpha)} &= (a+\alpha+1)\int_0^1 t^{k+\alpha}(1-t)^{n-k}dt \\
&= \frac{(a+\alpha+1)\Gamma(k+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha+2)}.
\end{aligned}
\tag{3.2}
$$

Therefore,

$$
\begin{aligned}
\left(H_\lambda^{(\alpha)}\right)_{n,k} &= \binom{n+\alpha}{n-k}\Delta^{n-k}\lambda_k^{(\alpha)} \\
&= \binom{n+\alpha}{n-k}\frac{(a+\alpha+1)\Gamma(k+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha+2)} \\
&= \frac{a+\alpha+1}{n+\alpha+1}.
\end{aligned}
\tag{3.3}
$$

Define $y_n = -u_n/(a+\alpha)$, where

$$u_n = x_n^{(\alpha)} - \sum_{k=0}^{n} h_{n,k}^{(\alpha)} x_k^{(\alpha)}. \tag{3.4}$$

From Lemma 2.1,

$$\begin{aligned} u_n &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{a+\alpha+1}{(n+\alpha+1)(\alpha+a+1)}\left[\frac{\Gamma(n+\alpha+2)}{\Gamma(n-a+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(-a)}\right] \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)}[1-1] + \frac{\Gamma(\alpha+1)}{\Gamma(-a)(n+\alpha+1)} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(-a)(n+\alpha+1)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.5}$$

This argument is valid provided a is not a positive integer. If a is a positive integer, then $x_k^{(\alpha)} = 0$ for $0 \le k \le a-1$.

Then, $u_n = 0$ for $0 \le n \le a-1$, and for $n \ge a$, from Lemma 2.1, we get

$$\begin{aligned} u_n &= x_n^{(\alpha)} - \frac{(a+\alpha+1)}{(n+\alpha+1)} \sum_{k=a}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-a+1)} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{(a+\alpha+1)}{(n+\alpha+1)}\left[\frac{\Gamma(a+\alpha+1)}{\Gamma(1)}\right] - \frac{1}{n+\alpha+1}\left[\frac{\Gamma(n+\alpha+2)}{\Gamma(n-a+1)} - \frac{\Gamma(a+\alpha+2)}{\Gamma(1)}\right] \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{\Gamma(a+\alpha+2)}{(n+1+\alpha)} - \frac{\Gamma(n+\alpha+2)}{(n+1+\alpha)\Gamma(n-a+1)} + \frac{\Gamma(a+\alpha+2)}{(n+1+\alpha)} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)}\left[1 - \frac{(n+1+\alpha)}{(n+1+\alpha)}\right] \\ &= 0 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.6}$$

Since $H_\mu^{(\alpha)}$ is regular, $c \subseteq (H_\mu^{(\alpha)})$. Thus, $c \oplus x^{(\alpha)} \subseteq (H_\mu^{(\alpha)})$.

To prove the converse, we will use Zeller's technique to construct a regular matrix A with $(A) = c \oplus x^{(\alpha)}$ and then show that $(H_\mu^{(\alpha)}) \subseteq (A)$.

Set $P_0 = 0$ and define a sequence $\{P_n\}$ inductively by selecting P_{n+1} to be smallest integer $P > P_n$ such that $|x_P^{(\alpha)}| \ge 2\,|x_{P_n}^{(\alpha)}|$. (Such a construction is clearly possible, since $x^{(\alpha)}$ is not bounded.) Let $q_n^{(\alpha)} = 1 - x_{P_{n-1}}^{(\alpha)}/x_{P_n}^{(\alpha)}$, $n = 1, 2, \ldots$. Define a matrix B by

$$\begin{aligned} b_{00} &= 1, \\ b_{n,n-1} &= \frac{1}{q_n^{(\alpha)}}, \quad n \ge 1, \\ b_{n,n} &= -\frac{x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} x_{P_n}^{(\alpha)}}, \quad n \ge 1, \\ b_{n,k} &= 0 \quad \text{otherwise}. \end{aligned} \tag{3.7}$$

Now, define the matrix A as follows:

$$a_{P_n,P_k} = b_{nk},$$
$$a_{P_n,k} = 0, \qquad k \neq P_i \text{ for any integer } i, \tag{3.8}$$
$$a_{nn} = 1, \qquad n \neq P_i \text{ for any integer } i.$$

If $n \neq P_i$ for any integer i, then there exists an integer r such that $P_r < n < P_{r+1}$. For this r, define

$$a_{n,P_{r-1}} = \frac{x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}},$$
$$a_{n,P_r} = \frac{-x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}. \tag{3.9}$$

Set $a_{nk} = 0$ otherwise. From [10], A is regular and $(A) = c \oplus x^{(\alpha)}$. There are three cases to consider, based on whether a is real number and not a positive integer, a is positive integer, or a is complex.

Proof of Case I. If a is real and not a positive integer, the E-J generalized Hausdorff matrix $H_\mu^{(\alpha)}$ generated by (1.6) has a unique two sided inverse $(H_\mu^{(\alpha)})^{-1} = ((h_{nk}^{(\alpha)})^{-1})$ with generating sequence

$$\frac{1}{\mu_n^{(\alpha)}} = \frac{-(a+\alpha)(n+1+\alpha)}{(n-a)} = -(a+\alpha) - \frac{(a+\alpha)(a+\alpha+1)}{n-a}. \tag{3.10}$$

For $k < n$,

$$\begin{aligned}\left(h_{nk}^{(\alpha)}\right)^{-1} &= \binom{n+\alpha}{n-k}\Delta^{n-k}\frac{1}{\mu_k^{(\alpha)}}\\ &= \frac{-(a+\alpha)(a+\alpha+1)\Gamma(n+\alpha+1)\Gamma(k-a)}{\Gamma(k+\alpha+1)\Gamma(n-a+1)}\\ &= \frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)}{x_k^{(\alpha)}(k-a)},\end{aligned} \tag{3.11}$$

$$\left(h_{nn}^{(\alpha)}\right)^{-1} = \frac{-(a+\alpha)(n+1+\alpha)}{n-a}. \tag{3.12}$$

To show that $(H_\mu^{(\alpha)}) \subseteq (A)$, it will be sufficent to show that $D = A(H_\mu^{(\alpha)})^{-1}$ is a regular matrix. Each column of $(H_\mu^{(\alpha)})^{-1}$ is essentially a scalar multiple of (1.5), so it is obvious that each column of $(H_\mu^{(\alpha)})^{-1}$ belongs to the convergence domain of A. However, it will be necessary to calculate the terms of D explicitly, since we must show that $t = 1$ and that D has finite norm.

If $k \neq P_i$ for any integer i, and r denotes the integer such that $P_{r-1} < k < P_r$, then from the definition of A,

$$
\begin{aligned}
d_{P_n,k} &= \sum_{j=r}^{n} a_{P_n,P_j}\left(h_{P_j,k}^{(\alpha)}\right)^{-1} \\
&= b_{n,n-1}\left(h_{P_{n-1},k}^{(\alpha)}\right)^{-1} + b_{nn}\left(h_{P_n,k}^{(\alpha)}\right)^{-1} \\
&= 0.
\end{aligned}
\tag{3.13}
$$

If $k = P_r$ for $r < n - 1$, then

$$
d_{P_n,P_r} = \sum_{j=r}^{n} a_{P_n,P_j}\left(h_{P_j,P_r}^{(\alpha)}\right)^{-1} = 0. \tag{3.14}
$$

For $k = P_{n-1}$,

$$
\begin{aligned}
d_{P_n,P_{n-1}} &= a_{P_n,P_{n-1}}\left(h_{P_{n-1},P_{n-1}}^{(\alpha)}\right)^{-1} + a_{P_n,P_n}\left(h_{P_n,P_{n-1}}^{(\alpha)}\right)^{-1} \\
&= \frac{1}{q_n^{(\alpha)}}\left(\frac{-(a+\alpha)(P_{n-1}+\alpha+1)}{P_{n-1}-a}\right) + \left(\frac{-x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} x_{P_n}^{(\alpha)}}\right)\left(\frac{-(a+\alpha)(a+\alpha+1)x_{P_n}^{(\alpha)}}{x_{P_{n-1}}^{(\alpha)}(P_{n-1}-a)}\right) \\
&= \frac{-(a+\alpha)}{q_n^{(\alpha)}}.
\end{aligned}
\tag{3.15}
$$

For $P_{n-1} < k < P_n$,

$$
d_{P_n,k} = a_{P_n,P_n}\left(h_{P_n,k}^{(\alpha)}\right)^{-1} = \frac{(a+\alpha)(1+a+\alpha)x_{P_{n-1}}^{(\alpha)}}{(k-a)q_n^{(\alpha)} x_k^{(\alpha)}}, \tag{3.16}
$$

$$
d_{P_n,P_n} = a_{P_n,P_n}\left(h_{P_n,P_n}^{(\alpha)}\right)^{-1} = \frac{(a+\alpha)(1+\alpha+P_n)x_{P_{n-1}}^{(\alpha)}}{(P_n-a)q_n^{(\alpha)} x_{P_n}^{(\alpha)}}. \tag{3.17}
$$

For $n \neq P_i$ for any i, if we now let r denote the integer such that $P_r < n < P_{r+1}$, then for $0 < k < P_{r-1}$,

$$
\begin{aligned}
d_{nk} &= \sum_{j=k}^{n} a_{n,j}\left(h_{j,k}^{(\alpha)}\right)^{-1} \\
&= a_{n,P_{r-1}}\left(h_{P_{r-1},k}^{(\alpha)}\right)^{-1} + a_{n,P_r}\left(h_{P_r,k}^{(\alpha)}\right)^{-1} + a_{n,n}\left(h_{n,k}^{(\alpha)}\right)^{-1} = 0.
\end{aligned}
\tag{3.18}
$$

For $k = P_{r-1}$,

$$\begin{aligned}
d_{n,P_{r-1}} &= a_{n,P_{r-1}}\left(h^{(\alpha)}_{P_{r-1},P_{r-1}}\right)^{-1} + a_{n,P_r}\left(h^{(\alpha)}_{P_r,P_{r-1}}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,P_{r-1}}\right)^{-1} \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{P_{r-1}-a}\left(\frac{P_{r-1}+\alpha+1}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} - \frac{(a+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}\frac{x^{(\alpha)}_{P_r}}{x^{(\alpha)}_{P_{r-1}}} + \frac{a+\alpha+1}{x^{(\alpha)}_{P_{r-1}}}\right) \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{P_{r-1}-a}\left(\frac{P_{r-1}+\alpha+1}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} - \frac{(a+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}\right) \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}.
\end{aligned} \tag{3.19}$$

For $P_{r-1} < k < P_r$,

$$\begin{aligned}
d_{nk} &= a_{n,P_r}\left(h^{(\alpha)}_{P_r,k}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,k}\right)^{-1} \\
&= \frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}x^{(\alpha)}_{P_{r-1}}}{x_k^{(\alpha)}\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)(k-a)}.
\end{aligned} \tag{3.20}$$

For $k = P_r$,

$$\begin{aligned}
d_{n,P_r} &= a_{n,P_r}\left(h^{(\alpha)}_{P_r,P_r}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,P_r}\right)^{-1} \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{(P_r-a)}\left[\frac{-(P_r+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} + \frac{a+\alpha+1}{x^{(\alpha)}_{P_r}}\right] \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{(P_r-a)x^{(\alpha)}_{P_r}\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)}\left[-(P_r+\alpha+1)x^{(\alpha)}_{P_r} + \left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)(a+\alpha+1)\right].
\end{aligned} \tag{3.21}$$

The quantity in brackets is equal to $-(P_r-a)x^{(\alpha)}_{P_r} - x^{(\alpha)}_{P_{r-1}}(a+\alpha+1)$, giving

$$d_{n,P_r} = \frac{(a+\alpha)x_n^{(\alpha)}}{x^{(\alpha)}_{P_r-}x^{(\alpha)}_{P_{r-1}}} + \frac{x_n^{(\alpha)}}{x^{(\alpha)}_{P_r}}\frac{(a+\alpha)(a+\alpha+1)x^{(\alpha)}_{P_{r-1}}}{(P_r-a)\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)}. \tag{3.22}$$

For $P_r < k < n$,

$$d_{n,k} = a_{n,n}\left(h^{(\alpha)}_{n,k}\right)^{-1} = \frac{-x_n^{(\alpha)}}{x_k^{(\alpha)}}\frac{(a+\alpha)(a+\alpha+1)}{(k-a)}, \tag{3.23}$$

and finally,

$$d_{n,n} = \frac{-(a+\alpha)(n+1+\alpha)}{n-a}. \tag{3.24}$$

By using (3.13)–(3.17),

$$\begin{aligned}\sum_{k=0}^{P_n} d_{P_n,k} &= d_{P_n,P_{n-1}} + \sum_{k=P_{n-1}+1}^{P_n-1} d_{P_n,k} + d_{P_n,P_n}\\ &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + x^{(\alpha)}_{P_{n-1}}(a+\alpha+1)\sum_{k=P_{n-1}+1}^{P_n-1}\frac{1}{x_k^{(\alpha)}(k-a)} + \frac{(P_n+\alpha+1)x^{(\alpha)}_{P_{n-1}}}{(P_n-a)x^{(\alpha)}_{P_n}}\right].\end{aligned} \tag{3.25}$$

By using Lemma 2.2, and noting that

$$\begin{aligned}&\frac{P_n+\alpha+1}{P_n-a} = 1 + \frac{1+a+\alpha}{P_n-a},\\ &\sum_{k=0}^{P_n} d_{P_n,k} = \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + x^{(\alpha)}_{P_{n-1}}\frac{(a+\alpha+1)}{a+\alpha}\left(\frac{1}{x^{(\alpha)}_{P_{n-1}}} - \frac{1}{x^{(\alpha)}_{P_n-1}}\right) + \frac{1+a+\alpha}{P_n-a}\frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n}} + \frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n}}\right]\\ &\qquad = \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + \frac{a+\alpha+1}{a+\alpha} - \frac{a+\alpha+1}{a+\alpha}\frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n-1}} + \frac{1+a+\alpha}{P_n-a}\frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n}} + \frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n}}\right].\end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned}-\frac{a+\alpha+1}{a+\alpha}\frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n-1}} + \frac{1+a+\alpha}{P_n-a}\frac{x^{(\alpha)}_{P_{n-1}}}{x^{(\alpha)}_{P_n}} &= (a+\alpha+1)\left[-\frac{x^{(\alpha)}_{P_{n-1}}}{(a+\alpha)x^{(\alpha)}_{P_n-1}} + \frac{x^{(\alpha)}_{P_{n-1}}}{(P_n-a)x^{(\alpha)}_{P_n}}\right]\\ &= (a+\alpha+1)\left[-\frac{x^{(\alpha)}_{P_{n-1}}\Gamma(P_n-a)}{(a+\alpha)\Gamma(P_n+\alpha)} + \frac{x^{(\alpha)}_{P_{n-1}}\Gamma(P_n-a+1)}{(P_n-a)\Gamma(P_n+\alpha+1)}\right]\\ &= (a+\alpha+1)\frac{\Gamma(P_n-a)}{\Gamma(P_n+\alpha)}\left[\frac{-x^{(\alpha)}_{P_{n-1}}}{(a+\alpha)} + \frac{x^{(\alpha)}_{P_{n-1}}}{(P_n+\alpha)}\right]\\ &= -\frac{(a+\alpha+1)\Gamma(P_n-a+1)\ x^{(\alpha)}_{P_{n-1}}}{\Gamma(P_n+\alpha+1)(a+\alpha)}.\end{aligned} \tag{3.27}$$

Finally,

$$\begin{aligned}
\sum_{k=0}^{P_n} d_{P_n,k} &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+\frac{(a+\alpha+1)}{a+\alpha}+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}-\frac{(a+\alpha+1)\Gamma(P_n-a+1)x_{P_{n-1}}^{(\alpha)}}{(a+\alpha)\Gamma(P_n+\alpha+1)}\right] \\
&= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+\frac{(a+\alpha+1)}{a+\alpha}\left(1-\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right)+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right] \\
&= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+(1+a+\alpha)\left(\frac{q_n^{(\alpha)}}{a+\alpha}\right)+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right] \\
&= -\frac{(a+\alpha)}{q_n^{(\alpha)}}\left(1-\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right)+(1+a+\alpha)=1.
\end{aligned} \tag{3.28}$$

For $n \neq P_i$ for any i, r the integer such that $P_r < n < P_{r+1}$, and using (3.18)–(3.24), we have

$$\begin{aligned}
\sum_{k=0}^{n} d_{nk} = \frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}&\left[-1+x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\sum_{k=P_{r-1}+1}^{P_r-1}\frac{1}{x_k^{(\alpha)}(k-a)}+1+\frac{x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}(P_r-a)}\right] \\
&+\sum_{k=P_r+1}^{n-1}\frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)}{x_k^{(\alpha)}(k-a)}-\frac{(a+\alpha)(n+1+\alpha)}{(n-a)}.
\end{aligned} \tag{3.29}$$

Writing $(n+1+\alpha)/(n-a) = 1+(x_n^{(\alpha)}(1+a+\alpha)/x_n^{(\alpha)}(n-a))$ and using Lemma 2.2, the quantity in brackets, which we call I_1, takes the form

$$\begin{aligned}
I_1 &= x_{P_{r-1}}^{(\alpha)}\frac{(a+\alpha+1)}{(a+\alpha)}\left(\frac{1}{x_{P_{r-1}}^{(\alpha)}}-\frac{1}{x_{P_r-1}^{(\alpha)}}\right)+\frac{x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}(P_r-a)} \\
&= x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\left(\frac{1}{(a+\alpha)x_{P_{r-1}}^{(\alpha)}}-\frac{1}{(a+\alpha)x_{P_r-1}^{(\alpha)}}+\frac{1}{x_{P_r}^{(\alpha)}(P_r-a)}\right).
\end{aligned} \tag{3.30}$$

The sum

$$\begin{aligned}
-\frac{1}{(a+\alpha)x_{P_r-1}^{(\alpha)}}+\frac{1}{x_{P_r}^{(\alpha)}(P_r-a)} &= -\frac{\Gamma(P_r-a)}{(a+\alpha)\Gamma(P_r+\alpha)}+\frac{\Gamma(P_r-a+1)}{(P_r-a)\Gamma(P_r+\alpha+1)} \\
&= -\frac{1}{(a+\alpha)x_{P_r}^{(\alpha)}}.
\end{aligned} \tag{3.31}$$

Thus,

$$\begin{aligned}\frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}I_1 &= \frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\left(\frac{1}{(a+\alpha)x_{P_{r-1}}^{(\alpha)}}-\frac{1}{(a+\alpha)x_{P_r}^{(\alpha)}}\right)\\ &= \frac{x_n^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}}.\end{aligned} \tag{3.32}$$

Finally,

$$\begin{aligned}\sum_{k=0}^{n} d_{nk} &= \frac{x_n^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}}-(a+\alpha)(1+a+\alpha)x_n^{(\alpha)}\left[\sum_{k=P_r+1}^{n-1}\frac{1}{x_k^{(\alpha)}(k-a)}+\frac{1}{x_n^{(\alpha)}(n-a)}\right]-(a+\alpha)\\ &= \frac{(a+\alpha+1)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}}-(a+\alpha)(1+a+\alpha)x_n^{(\alpha)}\left[\frac{1}{a+\alpha}\left(\frac{1}{x_{P_r}^{(\alpha)}}-\frac{1}{x_n^{(\alpha)}}\right)\right]-(a+\alpha)\\ &= \frac{(a+\alpha+1)}{x_{P_r}^{(\alpha)}}\left[x_n^{(\alpha)}-\left(x_n^{(\alpha)}-x_{P_r}^{(\alpha)}\right)\right]-(a+\alpha)\\ &= 1.\end{aligned} \tag{3.33}$$

Clearly, D has null columns. It remains to show that D has finite norm.

For all integers, $n \geq [a]+1$, $x_n^{(\alpha)}$ is positive and $(1/2) \leq q_n^{(\alpha)} \leq 1$. From (3.25),

$$\begin{aligned}\sum_{k=0}^{P_n}\left|d_{P_n,k}\right| &= \left|d_{P_n,P_{n-1}}\right|+\sum_{k=P_{n-1}+1}^{P_n-1}\left|d_{P_n,k}\right|+\left|d_{P_n,P_n}\right|\\ &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[1+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right]+1+a+\alpha.\end{aligned} \tag{3.34}$$

Since $|x_{P_n}^{(\alpha)}| \geq 2|x_{P_{n-1}}^{(\alpha)}|$, then, $x_{P_{n-1}}^{(\alpha)}/x_{P_n}^{(\alpha)} \leq 1/2$, and the above sum is bounded by $4\alpha+4a+1$. From (3.29),

$$\sum_{k=0}^{n}|d_{nk}| = \frac{2(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}+(a+\alpha)+\frac{2(a+\alpha+1)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}}-(a+\alpha+1). \tag{3.35}$$

From choice of n, $|x_n^{(\alpha)}| < 2|x_{P_r}^{(\alpha)}|$. Again, using the fact that $|x_{P_r}^{(\alpha)}| \geq 2|x_{P_{r-1}}^{(\alpha)}|$, we have

$$\sum_{k=0}^{n}|d_{nk}| < \frac{2(a+\alpha)\left(2\left|x_{P_r}^{(\alpha)}\right|\right)}{\left|x_{P_r}^{(\alpha)}\right|-\left|x_{P_r}^{(\alpha)}\right|/2}+4(a+\alpha+1)+2(a+\alpha)+1 = 14(a+\alpha)+5. \tag{3.36}$$

Since there are only a finite number of rows of D with $n < [a] + 1$, D has finite norm and is regular. □

Proof of Case II. If a is a positive integer, $\mu_a^{(\alpha)} = 0$, and $H_\mu^{(\alpha)}$ fails to have a two-sided inverse. However, if we define a new matrix $F = (f_{nk})$ with $f_{a,a} = 1$ and which agrees with $H_\mu^{(\alpha)}$ elsewhere, then F does possess a unique two-sided inverse. Morever, $(F) = (H_\mu^{(\alpha)})$ and, for $k > a$, $f_{nk}^{-1} = (h_{nk}^{(\alpha)})^{-1}$, where the $(h_{nk}^{(\alpha)})^{-1}$ are computed using (3.11) and (3.12).

From (1.5), $x_n^{(\alpha)} = 0$ for $0 \le n < a$. Consequently, $P_0 = 0$, $P_1 = a$ and $P_2 = a+1$. Now, let $E := AF^{-1} = (e_{nk})$. To prove that E is regular, we are concerned with the behavior of the e_{nk} for all n sufficiently large. We will restrict our attention to $n > a+1$. Since $f_{nk}^{-1} = (h_{nk}^{(\alpha)})^{-1}$ for all $k > a$, it is clear that $e_{nk} = d_{nk}$ for $k > a$. If we can show that $e_{nk} = 0$ for all $0 \le k \le a$ and $n > a+1$, then it will follow that E is regular, since D is.

For $n > a+1$,

$$\begin{gathered}\sum_{j=a}^{n} f_{nj}^{-1} f_{ja} = 0, \\ f_{na}^{-1} f_{aa} = -\sum_{j=a+1}^{n} f_{nj}^{-1} f_{ja} = -\sum_{j=a+1}^{n-1} \left(h_{nj}^{(\alpha)}\right)^{-1} h_{ja}^{(\alpha)} - \left(h_{nn}^{(\alpha)}\right)^{-1} h_{na}^{(\alpha)}.\end{gathered} \tag{3.37}$$

Since $f_{a,a} = 1$ and $(h_{nn}^{(\alpha)})^{-1} h_{na}^{(\alpha)} = -(a+\alpha+1)/(n-a)$, $f_{na}^{-1} = x_n^{(\alpha)}/\Gamma(a+\alpha+1)$. By induction it is showed that $f_{n,a-r}^{-1} = k_r^{(\alpha)}(a) x_n^{(\alpha)}$, where $k_r^{(\alpha)}(a)$ is a function of a.

For $n > a+1$, $P_{n-1} > P_a \ge P_1 = a \ge r$

$$\begin{aligned} e_{P_n,r} &= \sum_{j=r}^{P_n} a_{P_n,j} f_{jr}^{-1} = b_{n,n-1} f_{P_{n-1},r}^{-1} + b_{n,n} f_{P_n,r}^{-1} \\ &= \frac{1}{q_n^{(\alpha)}} \left(k_{a-r}^{(\alpha)}(a) x_{P_{n-1}}^{(\alpha)} - \frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}} \left(x_{P_n}^{(\alpha)} k_{a-r}^{(\alpha)}(a) \right) \right) = 0. \end{aligned} \tag{3.38}$$

For $n > a+1$, $n \ne P_i$ for any integer i, $0 \le r \le a$, and s the integer such that $P_s < n < P_{s+1}$,

$$\begin{aligned} e_{n,r} &= \sum_{j=r}^{n} a_{n,j} f_{jr}^{-1} = a_{n,P_{s-1}} f_{P_{s-1},r}^{-1} + a_{n,P_s} f_{P_s,r}^{-1} + a_{n,n} f_{n,r}^{-1} \\ &= \frac{x_n^{(\alpha)}}{x_{P_s}^{(\alpha)} - x_{P_{s-1}}^{(\alpha)}} \left(k_{a-r}^{(\alpha)}(a) x_{P_{s-1}}^{(\alpha)} - k_{a-r}^{(\alpha)}(a) x_{P_s}^{(\alpha)} \right) + k_{a-r}^{(\alpha)}(a) x_n^{(\alpha)} = 0. \end{aligned} \tag{3.39}$$

□

Proof of Case III. If a is complex, then none of the $\mu_n^{(\alpha)}$ vanish, and we may use the matrix D of Case I. It will be sufficient to show that D has finite norm. From (3.25),

$$\sum_{k=0}^{P_n} |d_{P_n,k}| = \left| \frac{-(a+\alpha)}{q_n^{(\alpha)}} \right| + \left| \frac{(a+\alpha)(P_n+\alpha+1) x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} (P_n - a) x_{P_n}^{(\alpha)}} \right| + \sum_{k=P_{n-1}+2}^{P_n - 1} |d_{P_n,k}| + |d_{P_n,P_{n-1}+1}|. \tag{3.40}$$

Again, $|x_{P_n}^{(\alpha)}| \geq 2|x_{P_{n-1}}^{(\alpha)}|$. It can be shown that $1/2 \leq |q_n^{(\alpha)}| \leq 3/2$. Since

$$|d_{P_n,P_{n-1}+1}| = \left|\frac{(a+\alpha)(a+\alpha+1)}{q_n^{(\alpha)}(P_{n-1}+1+\alpha)}\right|, \tag{3.41}$$

the first two and last terms of (3.40), are clearly bounded in n.

For $P_{n-1}+1 < k < P_n$, using (3.16),

$$\begin{aligned}
|d_{P_n,k}| &= \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)\Gamma(k-a)}{\Gamma(P_{n-1}+1-a)\Gamma(k+1+\alpha)q_n^{(\alpha)}}\right| \\
&= \left|\frac{(a+\alpha)(a+\alpha+1)(k-a-1)\cdots(P_{n-1}-a+1)}{\Gamma(k+1+\alpha)q_n^{(\alpha)}}\right|\Gamma(P_{n-1}+1+\alpha) \\
&\leq \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}}\right| \\
&\quad \cdot \left|\frac{|P_{n-1}+1-a|(|P_{n-1}+1-a|+1)\cdots(|P_{n-1}+1-a|+k-P_{n-1}-2)}{\Gamma(k+1+\alpha)}\right| \\
&= \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\frac{\Gamma(k+w)}{\Gamma(k+1+\alpha)},
\end{aligned} \tag{3.42}$$

where $w = |P_{n-1}+1-a| - P_{n-1} - 1$. $w < 0$ for all n sufficiently large. From Lemma 2.1, we can write

$$\begin{aligned}
\sum_{k=P_{n-1}+2}^{P_n-1} |d_{P_n,k}| &\leq \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\left[\frac{\Gamma(x+w)}{(w-\alpha)\Gamma(x+\alpha)}\right]_{P_{n-1}+2}^{P_n} \\
&< \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\frac{\Gamma(P_{n-1}+2+w)}{(\alpha-w)\Gamma(P_{n-1}+2+\alpha)},
\end{aligned} \tag{3.43}$$

and the sum is uniformly bounded in n, since $-w$ is bounded away from zero.

If $n \neq P_i$, for any i, then from (3.29),

$$\begin{aligned}
\sum_{k=0}^{n} |d_{nk}| &= \left|\frac{-(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}\right| + \sum_{k=P_{r-1}+1}^{P_r-1}\left|\frac{(a+\alpha)(a+\alpha+1)x_{P_{r-1}}^{(\alpha)}x_n^{(\alpha)}}{\left(x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}\right)x_k^{(\alpha)}(k-a)}\right| \\
&\quad + \left|\frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}\right| + \left|\frac{(a+\alpha)(a+\alpha+1)x_{P_{r-1}}^{(\alpha)}x_n^{(\alpha)}}{\left(x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}\right)x_{P_r}^{(\alpha)}(P_r-a)}\right| \\
&\quad + \sum_{k=P_r+1}^{n-1}\left|\frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}}{x_k^{(\alpha)}(k-a)}\right| + \left|\frac{-(a+\alpha)(n+1+\alpha)}{n-a}\right|.
\end{aligned} \tag{3.44}$$

Terms 1, 3, 4, and 6 of (3.44) are clearly bounded in n. Recalling that $q_r^{(\alpha)} = 1 - x_{P_{r-1}}^{(\alpha)}/x_{P_r}^{(\alpha)}$, the first summation may be written in the form

$$\left|\frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}}{q_r^{(\alpha)}x_{P_r}^{(\alpha)}}\right|\sum_{k=P_{r-1}+1}^{P_r-1}\left|\frac{x_{P_{r-1}}^{(\alpha)}}{x_k^{(\alpha)}(k-a)}\right|. \tag{3.45}$$

The summation is identical with the one in (3.40), and the above expression is uniformly bounded, since $|x_n^{(\alpha)}| < 2|x_{P_r}^{(\alpha)}|$. Using an argument similar to the one used in establishing (3.40), the second summation of (3.44) can be shown to be uniformly bounded. □

□

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