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Research Article

Ostrowski Type Inequalities in the Grushin Plane

Heng-Xing Liu and Jing-Wen Luan

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

Correspondence should be addressed to Jing-Wen Luan, jwluan@whu.edu.cn

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Motivated by the work of B.-S. Lian and Q.-H. Yang (2010) we proved an Ostrowski inequality associated with Carnot-Carathéodory distance in the Grushin plane. The procedure is based on a representation formula. Using the same representation formula, we prove some Hardy type inequalities associated with Carnot-Carathéodory distance in the Grushin plane.

1. Introduction

The classical Ostrowski inequality [1] is as follows:

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \left(\frac{1}{4}+\frac{(x-((a+b)/2))^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty \tag{1.1}$$

for $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality. Inequality (1.1) was extended from intervals to rectangles and general domains in $\mathbb{R}^n$ (see [2–5]). Recently, it has been proved by the same authors [6] that there exists an Ostrowski inequality on the 3-dimension Heisenberg group associated with horizontal gradient and Carnot-Carathéodory distance, and it is also a sharp inequality.

The aim of this note is to establish some Ostrowski type inequality in the Grushin plane, known as the simplest example of sub-Riemannian metric associated with Grushin operator (cf. [7–10]). Recall that in the Grushin plane, the sub-Riemannian metric is given by the vectors

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial y} \tag{1.2}$$

and satisfies $[X_1, X_2] = \partial/\partial y$. By Chow's conditions, the Carnot-Carathéodory distance $d_{cc}(u, v)$ between any two points $u, v \in \mathbb{R}^2$ is finite (cf. [11]). We denote $d_{cc}(u) = d_{cc}(o, u)$, where $o = (0, 0)$ is the origin. Define on $\mathbb{R}^2$ the dilation δ_λ as

$$\delta_\lambda u = \delta_\lambda(x, y) := \left(\lambda x, \lambda^2 y\right), \quad u = (x, y) \in \mathbb{R}^2. \tag{1.3}$$

For simplicity, we will write it as $\lambda u = (\lambda x, \lambda^2 y)$. It is not difficult to check that X_1 and X_2 are homogeneous of degree one with respect to the dilation. The Jacobian determinant of δ_λ is λ^Q, where $Q = 1 + 2 = 3$ is the homogeneous dimension. The Carnot-Carathéodory distance d_{cc} satisfies

$$d_{cc}(\lambda(x, y)) = \lambda d_{cc}(x, y), \quad \lambda > 0. \tag{1.4}$$

Let B_R be the Carnot-Carathéodory ball centered at the origin o and of radius $R > 0$. Let $\Sigma = \partial B_1$ be the corresponding unit sphere. Let $d\sigma$ be the surface measure on Σ. Given any $o \neq u = (x, y) \in \mathbb{R}^2$, set $x^* = x/d_{cc}(u)$, $y^* = y/d_{cc}^2(u)$, and $u^* = (x^*, y^*)$. For $f \in C(\overline{B_R(o)})$, let

$$\tilde{f}(r) = \frac{1}{|\Sigma|}\int_{\Sigma} f(ru^*)d\sigma, \quad 0 < r \leq R, \tag{1.5}$$

be the averages of f over the unit sphere, where $|\Sigma|$ denote the volume of the Σ. Then, we can state our result as follows.

Theorem 1.1. *Let $f \in C^1(\overline{B_R})$. Then for $u = (x, y) = ru^*$, there holds*

$$\left| f(u) - \frac{1}{|B_R|}\int_{B_R} f(v)dv \right| \leq \mathcal{N}(f) + \left(\frac{3}{4}R - r + \frac{r^4}{2R^Q}\right)\|\nabla_L f\|_\infty, \tag{1.6}$$

where

$$\mathcal{N}(f) := \sup_{u \in B_R}\left| f(u) - \tilde{f}(r)\right| = \left\| f - \tilde{f}\right\|_\infty, \tag{1.7}$$

$|B_R|$ denote the volume of the B_R, and ∇_L is the gradient operator defined by $\nabla_L = (X_1, X_2)$. The constants in (1.6) *are the best possible, equality that can be attained for nontrivial radial functions at any $r \in [0, R]$.*

We also obtain the following Hardy type inequalities in the Grushin plane. We refer to [12] the Hardy inequalities associated with nonisotropic gauge induced by the fundamental solution.

Theorem 1.2. *Let $f \in C_0^\infty(\mathbb{R}^2)$. There holds, for $1 < p < 3$,*

$$\int_{\mathbb{R}^2} \left|\nabla_L f(u)\right|^p du \geq \left(\frac{3-p}{p}\right)^p \int_{\mathbb{R}^2} \frac{|f|^p}{d_{cc}^p(u)} du. \tag{1.8}$$

2. Geodesics in the Grushin Plane

In this section, we will follow [13] to give a parametrization of Grushin plane using the geodesics. Recall that the Grushin operator is given by

$$\Delta_L = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}. \tag{2.1}$$

The associated Hamiltonian function $H(x, y, \xi, \theta)$ is of the form

$$H(x, y, \xi, \eta) = \frac{1}{2}\left(\xi^2 + x^2\eta^2\right). \tag{2.2}$$

It is known that geodesics in the Grushin plane are solutions of the Hamiltonian system (cf. [8])

$$\begin{aligned}
\dot{x}(s) &= \frac{\partial H}{\partial \xi} = \xi(s), \\
\dot{\xi}(s) &= -\frac{\partial H}{\partial x} = -x\eta^2(s), \\
\dot{y}(s) &= \frac{\partial H}{\partial \eta} = x^2\eta, \\
\dot{\eta}(s) &= -\frac{\partial H}{\partial y} = 0, \quad \text{that is}, \eta(s) = \eta(0).
\end{aligned} \tag{2.3}$$

Taking the initial date $(x(0), y(0)) = (0, 0)$ and $(\xi(0), \eta(0)) = (A, \phi)$, one can find the solutions (cf. [8])

$$\begin{aligned}
x(s) &= A\frac{\sin \phi s}{\phi}, \\
y(s) &= A^2\frac{2\phi s - \sin 2\phi s}{4\phi},
\end{aligned} \tag{2.4}$$

where the time s is exactly the Carnot-Carathéodory distance. Letting $\phi \to 0$, we get the Euclidean geodesics

$$\left(x(s), y(s)\right) = (As, 0) \tag{2.5}$$

and hence the correct normalization is $A^2 = 1$.

Set

$$\Omega = \left\{(\phi, \rho) \in \mathbb{R}^2 : -\pi \leq \phi\rho \leq \pi, \rho \geq 0\right\} \tag{2.6}$$

and define $\Phi : \Omega \to \mathbb{R}^2$ by $\Phi(\phi,\rho) = (x(\phi,\rho), y(\phi,\rho))$, where

$$x(\phi,\rho) = A\frac{\sin\phi\rho}{\phi},$$
$$y(\phi,\rho) = \frac{2\phi\rho - \sin 2\phi\rho}{4\phi^2} \tag{2.7}$$

with $A^2 = 1$. If $A = 1$, the range of Φ is $[0,+\infty) \times \mathbb{R}$; if $A = -1$, the range of Φ is $(-\infty, 0] \times \mathbb{R}$. Furthermore, if one fixes $\rho > 0$, (2.7) with $A = \pm 1$ and $-\pi/\rho \le \phi \le \pi/\rho$ parameterize ∂B_ρ.

On the other hand, the Carnot-Carathéodory distance d_{cc} satisfies (cf. [10, Theorem 2.6]), for $x \ne 0$,

$$d_{cc}(x,y) = \frac{\theta}{\sin\theta}|x|, \tag{2.8}$$

where $\theta = \mu^{-1}(2y/x^2)$

$$\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta = \frac{2\theta - \sin 2\theta}{2\sin^2\theta} : (-\pi,\pi) \longrightarrow \mathbb{R} \tag{2.9}$$

is a diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$ (cf. [14]), and μ^{-1} is the inverse function of μ. From (2.7), we have

$$\mu(\theta) = \frac{2y}{x^2} = \frac{2\phi\rho - \sin 2\phi\rho}{2\sin^2\phi\rho} = \mu(\phi\rho). \tag{2.10}$$

Therefore,

$$\theta = \phi\rho \tag{2.11}$$

since μ is a diffeomorphism.

We finally recall the polar coordinates in the Grushin plane associated with d_{cc}. The following coarea formula has been proved in [15]:

$$\int_{\mathbb{R}^2} f(u)|\nabla_L d_{cc}(u)|du = \int_{-\infty}^{+\infty}\int_{\{d_{cc}(u)=\lambda\}} f(u)dP(E_\lambda)d\lambda, \tag{2.12}$$

where $E_\lambda = \{u \in \mathbb{R}^2 : d_{cc}(u) > \lambda\}$ and $P(E_\lambda)$ is the perimeter-measure. Notice that $|\nabla_L d_{cc}(u)| = 1$ a.e. (cf. [15]); and $P(E_\lambda) = \lambda^2 P(E_1)$ (cf. [9, Proposition 2.2]); we have the following polar coordinates in the Grushin plane, for all $f \in L^1(\mathbb{R}^2)$:

$$\int_{\mathbb{R}^2} f(u)du = \int_0^{+\infty}\int_\Sigma f(\lambda u^*)\lambda^2 d\sigma d\lambda. \tag{2.13}$$

3. The Proofs

To prove the main result, we first need the following representation formula.

Lemma 3.1. *Let $R_2 > R_1 > 0$ and $f \in C^1(B_{R_2} \setminus B_{R_1})$. There holds*

$$\int_\Sigma f(R_2 u^*) d\sigma - \int_\Sigma f(R_1 u^*) d\sigma = \int_{B_{R_2} \setminus B_{R_1}} \langle \nabla_L f, \nabla_L d_{cc} \rangle \cdot \frac{1}{d_{cc}^2} du. \tag{3.1}$$

Proof. Let u^* be a point on the sphere, that is, $u^* = (x^*, y^*)$, where $d_{cc}(x^*, y^*) = 1$. We consider for $0 < R_1 < R_2$ the following difference using the fundamental theorem of calculus:

$$\begin{aligned}\int_\Sigma f(R_2 \xi^*) d\sigma - \int_\Sigma f(R_1 \xi^*) d\sigma &= \int_\Sigma \int_{R_1}^{R_2} \frac{d}{d\rho} f(\rho \xi^*) d\rho d\sigma \\ &= \int_\Sigma \int_{R_1}^{R_2} \left(\frac{\partial f(u)}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial f(u)}{\partial y} \cdot \frac{\partial y}{\partial \rho} \right) d\rho d\sigma,\end{aligned} \tag{3.2}$$

where $u = (x, y) = \rho u^*$. Using (2.7), we have

$$\begin{aligned}\frac{\partial f(u)}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial f(u)}{\partial y} \cdot \frac{\partial y}{\partial \rho} &= A \cos \phi \rho \frac{\partial f(u)}{\partial x} + \frac{\sin^2 \phi \rho}{\phi} \frac{\partial f(u)}{\partial y} \\ &= A \cos \phi \rho \frac{\partial f(u)}{\partial x} + A \sin \phi \rho \cdot x \frac{\partial f(u)}{\partial y} \\ &= A \cos \phi \rho X_1 f(u) + A \sin \phi \rho X_2 f(u).\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and rewriting the expression into a solid integral using the polar coordinates, we get

$$\int_\Sigma f(R_2 \xi^*) d\sigma - \int_\Sigma f(R_1 \xi^*) d\sigma = \int_{B_{R_2} \setminus B_{R_1}} \frac{A \cos \phi \rho X_1 f(u) + A \sin \phi \rho X_2 f(u)}{d_{\mathrm{cc}}^2} du. \tag{3.4}$$

To finish our proof, it is enough to show that

$$X_1 d_{\mathrm{cc}}(u) = A \cos \phi \rho, \qquad X_2 d_{\mathrm{cc}}(u) = A \sin \phi \rho \tag{3.5}$$

in $\mathbb{R} \setminus \{0\} \times \mathbb{R}$. This is just the following Lemma 3.2. The proof of Lemma 3.1 is now complete. □

Lemma 3.2. *There hold, for $x \neq 0$,*

$$X_1 d_{cc}(u) = A \cos \phi \rho, \qquad X_2 d_{cc}(u) = A \sin \phi \rho. \tag{3.6}$$

Proof. Recall that if $x \neq 0$, then

$$d_{cc}(u) = d_{cc}(x, y) = \frac{\theta}{\sin\theta}|x|, \tag{3.7}$$

where $\theta = \mu^{-1}(2y/x^2)$. The simple calculation shows

$$\begin{gathered}
\mu'(\theta) = \frac{2\sin\theta - 2\theta\cos\theta}{\sin^3\theta}, \\
\frac{\partial\theta}{\partial x} = \frac{1}{\mu'(\theta)} \cdot \frac{-4y}{x^3}, \qquad \frac{\partial\theta}{\partial y} = \frac{1}{\mu'(\theta)} \cdot \frac{-2}{x^2}.
\end{gathered} \tag{3.8}$$

Therefore, if $x \neq 0$, then

$$\begin{aligned}
X_1 d_{cc}(u) &= \frac{\partial d_{cc}(u)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\theta}{\sin\theta}|x|\right) = \frac{x}{|x|} \cdot \frac{\theta}{\sin\theta} + |x| \cdot \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \cdot \frac{\partial\theta}{\partial x} \\
&= \frac{x}{|x|} \cdot \frac{\theta}{\sin\theta} - |x|\sin\theta \cdot \frac{-2y}{x^3} = \frac{x}{|x|} \cdot \frac{\theta}{\sin\theta} - \frac{|x|}{x} \cdot \sin\theta \cdot \mu(\theta) \\
&= A\frac{\theta}{\sin\theta} - A\sin\theta \cdot \left(\frac{\theta}{\sin^2\theta} - \cot\theta\right), \\
&= A\cos\theta.
\end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned}
X_2 d_{cc}(u) &= x\frac{\partial d_{cc}(u)}{\partial y} = x\frac{\partial}{\partial y}\left(\frac{\theta}{\sin\theta}|x|\right) = x|x| \cdot \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \cdot \frac{\partial\theta}{\partial y} \\
&= \frac{|x|}{x} \cdot \sin\theta = A\sin\theta.
\end{aligned} \tag{3.10}$$

Therefore, we obtain, by (2.11),

$$X_1 d_{cc}(u) = A\cos\phi\rho, \quad X_2 d_{cc}(u) = A\sin\phi\rho. \tag{3.11}$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 1.1. We have, by Lemma 3.1, since $|B_R| = \int_0^R \int_\Sigma s^2 d\sigma ds = |\Sigma|R^3/3$,

$$\begin{aligned}
&\left| f(u) - \frac{1}{|B_R|}\int_{B_R} f(v)dv \right| \\
&\quad \le \left| f(u) - \tilde{f}(r) \right| + \left| \frac{1}{|\Sigma|}\int_\Sigma f(ru^*)d\sigma - \frac{3}{|\Sigma|R^3}\int_0^R \int_\Sigma f(su^*)s^2 d\sigma ds \right| \\
&\quad = \mathcal{N}_c(f) + (*),
\end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
(*) &= \left| \frac{1}{|\Sigma|}\int_{\Sigma} f(ru^*)d\sigma - \frac{3}{|\Sigma|R^3}\int_0^R\int_{\Sigma} f(su^*)s^2 d\sigma ds \right| \\
&= \frac{3}{|\Sigma|R^3}\left| \int_0^R \left(\int_{\Sigma} f(ru^*)d\sigma - \int_{\Sigma} f(su^*)d\sigma \right) s^2 ds \right| \\
&\le \frac{3}{|\Sigma|R^3}\int_0^R \left| \int_{\Sigma'} f(ru^*)d\sigma - \int_{\Sigma'} f(s\xi^*)d\sigma \right| s^2 ds \\
&\le \frac{3}{R^3}\cdot \int_0^R |r-s|s^2 ds \cdot \|\nabla_L f\|_\infty \\
&= \left(\frac{3}{4}R - r + \frac{r^4}{2R^3} \right)\|\nabla_L f\|_\infty .
\end{aligned} \tag{3.13}
$$

To see that the estimate in (1.8) is sharp, we consider the function $f(u) = f(d_{\mathrm{cc}}(u)) = |r - d_{\mathrm{cc}}(u)|$ and that r is fixed in $[0, R]$. Notice that $|\nabla_L f(u)| = 1$ a.e.; we have $|\nabla_L f(u)|_\infty = 1$. We look at inequality (1.6) evaluating the function at r. Since $f(r) = 0$, the left-hand side of (1.6) is

$$
\text{L.H.S. } (1.6) = \frac{3}{R^3}\cdot \int_0^R |r-s|s^2 ds = \frac{3}{4}R - r + \frac{r^4}{2R^3} \tag{3.14}
$$

and the right-hand side of (1.8) is

$$
\text{R.H.S. } (1.8) = \frac{3}{4}R - r + \frac{r^4}{2R^3}. \tag{3.15}
$$

Thus the equality holds in (1.6). This completes the proof of the sharpness of inequality (1.6). The proof of the theorem is now complete. □

Proof of Theorem 1.2. Let $\epsilon > 0$. Then $0 \le f_\epsilon := (|f|^2 + \epsilon^2)^{p/2} - \epsilon^p \in C_0^\infty(\mathbb{R}_2)$. In fact, f_ϵ has the same support as f. Putting $f_\epsilon d_{\mathrm{cc}}^{3-p}(u)$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0+$, we get, since $d_{\mathrm{cc}}(o) = 0$,

$$
\int_{\mathbb{R}^2} \langle \nabla_L f_\epsilon, \nabla_L d_{\mathrm{cc}} \rangle \cdot \frac{1}{d_{\mathrm{cc}}^{p-1}} + (3-p)\int_{\mathbb{R}^2} \frac{f_\epsilon}{d_{\mathrm{cc}}^{p}} = 0. \tag{3.16}
$$

Here we use the fact $|\nabla_L d_{cc}(u)| = 1$ a.e. Therefore,

$$(3-p)\int_{\mathbb{R}^2} \frac{f_\epsilon}{d_{cc}^p} = -p\int_{\mathbb{R}^2} \left(|f|^2+\epsilon^2\right)^{(p-2)/2} f\langle \nabla_L f, \nabla_L d_{cc}\rangle \cdot \frac{1}{d_{cc}^{p-1}}$$
$$\le p\int_{\mathbb{R}^2} \frac{\left(|f|^2+\epsilon^2\right)^{(p-2)/2} |f|\cdot|\nabla_L f|}{d_{cc}^{p-1}} \tag{3.17}$$
$$\le p\int_{\mathbb{R}^2} \frac{\left(|f|^2+\epsilon^2\right)^{(p-1)/2} \cdot |\nabla_L f|}{d_{cc}^{p-1}}.$$

By dominated convergence, letting $\epsilon \to 0+$, we have

$$(3-p)\int_{\mathbb{R}^2} \frac{|f|^p}{d_{cc}^p} \le p\int_{\mathbb{R}^2} \frac{|f|^{p-1}\cdot|\nabla_L f|}{d_{cc}^{p-1}}. \tag{3.18}$$

By Hölder's inequality,

$$(3-p)\int_{\mathbb{R}^2} \frac{|f|^p}{d_{cc}^p} \le p\left(\int_{\mathbb{R}^2} \frac{|f|^p}{d_{cc}^p}\right)^{(p-1)/p} \left(\int_{\mathbb{R}^2} |\nabla_L f|^p\right)^{1/p}. \tag{3.19}$$

Canceling and raising both sides to the power p, we get (1.8). □

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