

Research Article

Some Sublinear Dynamic Integral Inequalities on Time Scales

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We study some nonlinear dynamic integral inequalities on time scales by introducing two adjusting parameters, which provide improved bounds on unknown functions. Our results include many existing ones in the literature as special cases and can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales.

1. Introduction

Following Hilger's landmark paper [1], there have been plenty of references focused on the theory of time scales in order to unify continuous and discrete analysis, where a time scale is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist; for example, $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{H}_n$, the space of the harmonic numbers.

Recently, many authors have extended some continuous and discrete integral inequalities to arbitrary time scales. For example, see [2–14] and the references cited therein. The purpose of this paper is to further investigate some sublinear integral inequalities on time scales that have been studied in a recent paper [6]. By introducing two adjusting parameters α and β , we first generalize a basic inequality that plays a fundamental role in the proofs of the main results in [6]. Then, we provide improved bounds on unknown functions, which include many existing results in [6, 14] as special cases and can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales.

2. Time Scale Essentials

The definitions below merely serve as a preliminary introduction to the time scale calculus; they can be found in the context of a much more robust treatment than is allowed here in the text [15, 16] and the references therein.

Definition 2.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup\mathbb{T}$ (resp. $\rho(t)$ at t for $t > \inf\mathbb{T}$) by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}, \quad (\rho(t) = \sup\{s < t : t \in \mathbb{T}\}), \quad t \in \mathbb{T}. \quad (2.1)$$

Also define $\sigma(\sup\mathbb{T}) = \sup\mathbb{T}$, if $\sup\mathbb{T} < \infty$, and $\rho(\inf\mathbb{T}) = \inf\mathbb{T}$, if $\inf\mathbb{T} > -\infty$. The graininess functions are given by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$. The set \mathbb{T}^κ is derived from \mathbb{T} as follows: if \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Throughout this paper, the assumption is made that \mathbb{T} inherits from the standard topology on the real numbers \mathbb{R} . The jump operators σ and ρ allow the classification of points in a time scale in the following way. If $\sigma(t) > t$, the point t is right-scattered, while if $\rho(t) < t$, then t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, the point t is right-dense; if $t > \inf\mathbb{T}$ and $\rho(t) = t$ then t , is left-dense. Points that are right-dense and left-dense at the same time are called dense. The composition $f \circ \sigma$ is often denoted f^σ .

Definition 2.2. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous (denoted $f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$) if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points.

Every right-dense continuous function has a delta antiderivative [15, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists. Likewise every left-dense continuous function f on the time scale, denoted $f \in C_{\text{ld}}(\mathbb{T}, \mathbb{R})$, has a nabla antiderivative [15, Theorem 8.45]

Definition 2.3. Fix $t \in \mathbb{T}$, and let $y : \mathbb{T}^\kappa \rightarrow \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t such that, for all $s \in U$,

$$\left| [y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s] \right| \leq \epsilon|\sigma(t) - s|. \quad (2.2)$$

Call $y^\Delta(t)$ the (delta) derivative of y at t . It is easy to see that f^Δ is the usual derivative f' for $\mathbb{T} = \mathbb{R}$ and the usual forward difference Δf for $\mathbb{T} = \mathbb{Z}$.

Definition 2.4. If $F^\Delta(t) = f(t)$, then define the (Cauchy) delta integral by

$$\int_a^b f(s) \Delta s = F(b) - F(a). \quad (2.3)$$

Definition 2.5. Say $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive provided that $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. Denote by $\mathcal{R}_+(\mathbb{T}, \mathbb{R})$ the set of all regressive and rd-continuous functions p satisfying $1 + \mu(t)p(t) > 0$

on \mathbb{T} . For $h > 0$, define the cylinder transformation $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ by $\xi_h(z) = (1/h) \operatorname{Log}(1+zh)$, where Log is the principal logarithm function, $\mathbb{C}_h = \{z \in \mathbb{C} : z \neq -1/h\}$, and $\mathbb{Z}_h = \{z \in \mathbb{C} : -\pi/h < \operatorname{Im}(z) \leq \pi/h\}$. For $h = 0$, define $\xi_0(z) = z$. Define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right), \quad t, s \in \mathbb{T}. \tag{2.4}$$

3. Main Results

In the sequel, we always assume that $0 < \lambda < 1$ is a constant, \mathbb{T} is a time scale with $t_0 \in \mathbb{T}$. The following sublinear integral inequalities on time scales will be considered:

$$x(t) \leq a(t) + b(t) \int_{t_0}^t [g(s)x(s) + h(s)x^\lambda(s)] \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{I}$$

$$x(t) \leq a(t) + b(t) \int_{t_0}^t w(t, s) [g(s)x(s) + h(s)x^\lambda(s)] \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{II}$$

$$x(t) \leq a(t) + b(t) \int_{t_0}^t f(s, x^\lambda(s)) \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{III}$$

where $a, b, g, h, x : \mathbb{T}^\kappa \rightarrow \mathbb{R}_+ = [0, \infty)$ are rd-continuous functions, $w : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ is continuous, and $f : \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ is continuous.

If we let $x(t) = u^p(t)$ and $\lambda p = q$, then inequalities (I)–(III) reduce to those inequalities studied in [6]. We say inequalities (I)–(III) are sublinear since $0 < \lambda < 1$. In the sequel, some generalized and improved bounds on unknown functions $x(t)$ will be provided by introducing two adjusting parameters α and β .

Before establishing our main results, we need the following lemmas.

Lemma 3.1 ([15, Theorem 6.1, page 255]). *Let $y, q \in C_{\text{rd}}$ and $p \in \mathcal{R}_+(\mathbb{T}, \mathbb{R})$. Then*

$$y^\Delta(t) \leq p(t)y(t) + q(t), \quad t \in \mathbb{T}, \tag{3.1}$$

Implies that

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(s))q(s) \Delta s, \quad t \in \mathbb{T}. \tag{3.2}$$

Lemma 3.2. *Let c and x are nonnegative functions, $0 < \lambda < 1$ is a constant. Then, for any positive function k ,*

$$cx^\lambda \leq \lambda k^{\lambda-1} c^\alpha x + (1 - \lambda)k^\lambda c^\beta \tag{3.3}$$

holds, where α and β are nonnegative constants satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$.

Proof. For nonnegative constants a and b , positive constants p and q with $1/p + 1/q = 1$, the basic inequality in [17]

$$\frac{a}{p} + \frac{b}{q} \geq a^{1/p} b^{1/q} \quad (3.4)$$

holds. Let $1/p = \lambda$, $1/q = 1 - \lambda$, $a = k^{\lambda-1} c^\alpha$, and $b = k^\lambda c^\beta$. Then, inequality (3.3) is valid. \square

Remark 3.3. When $c = 1$, Lemma 3.2 reduces to Lemma 3.1 with $\lambda = q/p$ in [6].

Lemma 3.4 ([15, Theorem 1.117, page 46]). *Suppose that for each $\epsilon > 0$ there exists a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left| w(\sigma(t), \tau) - w(s, \tau) - w_1^\Delta(t, \tau)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \quad s \in U, \quad (3.5)$$

where $w : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ is continuous at (t, t) , $t \in \mathbb{T}^\kappa$ with $t > t_0$ and $w_1^\Delta(t, \cdot)$ (the derivative of w with respect to the first variable) is rd-continuous on $[t_0, \sigma(t)]$. Then

$$v(t) := \int_{t_0}^t w(t, \tau) \Delta \tau \quad (3.6)$$

implies that

$$v^\Delta(t) = \int_{t_0}^t w_1^\Delta(t, \tau) \Delta \tau + w(\sigma(t), t). \quad (3.7)$$

Now, let us give the main results of this paper.

Theorem 3.5. *Assume that $a, b, g, h, x : \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ are rd-continuous functions. Then, for any rd-continuous function $k(t) > 0$ on \mathbb{T}^κ , any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (I) implies that*

$$x(t) \leq a(t) + b(t) \int_{t_0}^t e_P(t, \sigma(s)) Q(s) \Delta s, \quad t \in \mathbb{T}^\kappa, \quad (3.8)$$

where

$$\begin{aligned} P(t) &= b(t) \left[g(t) + \lambda k^{\lambda-1}(t) h^\alpha(t) \right], \\ Q(t) &= a(t) \left[g(t) + \lambda k^{\lambda-1}(t) h^\alpha(t) \right] + (1 - \lambda) k^\lambda(t) h^\beta(t). \end{aligned} \quad (3.9)$$

Proof. Set

$$y(t) = \int_{t_0}^t \left[g(s)x(s) + h(s)x^\lambda(s) \right] \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (3.10)$$

Then, $y(t_0) = 0$ and (I) can be restated as

$$x(t) \leq a(t) + b(t)y(t), \quad t \in \mathbb{T}^\kappa. \quad (3.11)$$

Based on a straightforward computation and Lemma 3.2, we have

$$\begin{aligned} y^\Delta(t) &= g(t)x(t) + h(t)x^\lambda(t) \\ &\leq g(t)x(t) + \lambda k^{\lambda-1}(t)h^\alpha(t)x(t) + (1-\lambda)k^\lambda(t)h^\beta(t), \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (3.12)$$

Combining (3.11) and (3.12) yields

$$\begin{aligned} y^\Delta(t) &\leq \left[g(t) + \lambda k^{\lambda-1}(t)h^\alpha(t) \right] \left[a(t) + b(t)y(t) \right] + (1-\lambda)k^\lambda(t)h^\beta(t) \\ &= P(t)y(t) + Q(t), \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (3.13)$$

Note that $y, Q \in \mathcal{C}_{rd}$ and $P \in \mathcal{R}_+$. By Lemma 3.1, (3.11), and (3.13), we get (3.8). \square

Remark 3.6. For given $k(t) > 0$, by choosing different constants α and β , some improved bounds on $x(t)$ can be obtained. For example, when $h(t)$ is sufficiently large, we may set $\alpha = 0$ since the value of $e_p(t, s)$ changes drastically. Similarly, we may set $\beta = 0$ for sufficiently small $h(t)$.

Remark 3.7. When $k(t) = k > 0$, $\alpha = \beta = 1$, Theorem 3.5 reduces to Theorem 3.2 in [6]. For some particular cases of \mathbb{T} , $k(t)$, α , and β , Theorem 3.5 reduces to Corollary 3.3, Corollary 3.4 in [6], Theorem 1(a_1), and Theorem 3(c_1) in [14].

Theorem 3.8. *Assume that $a, b, g, h, x : \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ are rd-continuous functions. Let $w(t, s)$ be defined as in Lemma 3.4 such that $w_1^\Delta(t, s) \geq 0$ for $t \geq s$ and (3.5) holds. Then, for any rd-continuous function $k(t) > 0$, any nonnegative constants α and β satisfying $\lambda\alpha + (1-\lambda)\beta = 1$, inequality (II) implies that*

$$x(t) \leq a(t) + b(t) \int_{t_0}^t e_A(t, \sigma(s))B(s) \Delta s, \quad t \in \mathbb{T}^\kappa, \quad (3.14)$$

where

$$\begin{aligned} A(t) &= w(\sigma(t), t)P(t) + \int_{t_0}^t w_1^\Delta(t, s)P(s) \Delta s, \\ B(t) &= w(\sigma(t), t)Q(t) + \int_{t_0}^t w_1^\Delta(t, s)Q(s) \Delta s, \end{aligned} \quad (3.15)$$

$P(t)$ and $Q(t)$ are the same as in Theorem 3.5.

Proof. Define a function

$$z(t) = \int_{t_0}^t k(t, s) \Delta s, \quad t \in \mathbb{T}^\kappa, \quad (3.16)$$

where

$$k(t, s) = w(t, s) [g(s)x(s) + h(s)x^\lambda(s)]. \quad (3.17)$$

Then, $z(t_0) = 0$, $z(t)$ is nondecreasing, and

$$x(t) \leq a(t) + b(t)z(t), \quad t \in \mathbb{T}^\kappa. \quad (3.18)$$

Similar to the arguments in Theorem 3.5, by Lemmas 3.2 and 3.4 we have

$$\begin{aligned} z^\Delta(t) &= k(\sigma(t), t) + \int_{t_0}^t k_1^\Delta(t, s) \Delta s \\ &= w(\sigma(t), t) [g(t)x(t) + h(t)x^\lambda(t)] + \int_{t_0}^t w_1^\Delta(t, s) [g(s)x(s) + h(s)x^\lambda(s)] \Delta s \\ &\leq w(\sigma(t), t) [P(t)z(t) + Q(t)] + \int_{t_0}^t w_1^\Delta(t, s) [P(s)z(s) + Q(s)] \Delta s \\ &\leq \left[w(\sigma(t), t)P(t) + \int_{t_0}^t w_1^\Delta(t, s)P(s) \Delta s \right] z(t) + \left[w(\sigma(t), t)Q(t) + \int_{t_0}^t w_1^\Delta(t, s)Q(s) \Delta s \right] \\ &= A(t)z(t) + B(t), \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (3.19)$$

Note that $z, B \in \mathcal{C}_{rd}$ and $A \in \mathcal{R}_+$. By Lemma 3.1, we get (3.14). \square

Theorem 3.9. Assume that a, b, x are nonnegative rd-continuous functions defined on \mathbb{T}^κ . Let $f : \mathbb{T}^\kappa \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function satisfying

$$0 \leq f(t, x) - f(t, y) \leq \phi(t, y)(x - y) \quad (3.20)$$

for $t \in \mathbb{T}^\kappa$ and $x \geq y \geq 0$, where $\phi : \mathbb{T}^\kappa \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function. Then, for any rd-continuous function $k(t) > 0$, any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (III) implies that

$$x(t) \leq a(t) + b(t) \int_{t_0}^t e_M(t, \sigma(s)) N(s) \Delta s, \quad t \in \mathbb{T}^\kappa, \quad (3.21)$$

where

$$\begin{aligned} M(t) &= \lambda k^{\lambda-1}(t)h^\alpha(t)b(t)\phi\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right], \\ N(t) &= \lambda k^{\lambda-1}(t)h^\alpha(t)a(t)\phi\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right] + f\left[t, (1-\lambda)k^\lambda(t)\right]. \end{aligned} \quad (3.22)$$

Proof. Define a function $u(t)$ by

$$u(t) = \int_{t_0}^t f\left[s, x^\lambda(s)\right] \Delta s. \quad (3.23)$$

Then, $u(t_0) = 0$, and $x(t) \leq a(t) + b(t)u(t)$. According to the straightforward computation, from (3.20) we get

$$\begin{aligned} u^\Delta(t) &= f\left[t, x^\lambda(t)\right] \\ &\leq f\left[t, \lambda k^{\lambda-1}(t)h^\alpha(t)x(t) + (1-\lambda)k^\lambda(t)h^\beta(t)\right] \\ &\leq \lambda k^{\lambda-1}(t)h^\alpha(t)\phi\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right]x(t) + f\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right] \\ &\leq \lambda k^{\lambda-1}(t)h^\alpha(t)\phi\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right][a(t) + b(t)u(t)] + f\left[t, (1-\lambda)k^\lambda(t)h^\beta(t)\right] \\ &= M(t)u(t) + N(t), \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (3.24)$$

Note that $u, N \in \mathcal{C}_{\text{rd}}$ and $M \in \mathcal{R}_+$. By Lemma 3.1, we get (3.21). \square

Remark 3.10. For some particular cases of \mathbb{T} , $k(t)$, α and β , Theorems 3.8 and 3.9 include Theorem 3.8, Theorem 3.14, Corollary 3.9, Corollary 3.10 in [6], Theorem 1(a_3), Theorem 3(c_3) and Theorem 4(d_1) in [14] as special cases.

Remark 3.11. Some other integral inequalities on time scales were studied in [8, 9] by using Lemma 3.1 in [6]. Since Lemma 3.1 generalizes and improves Lemma 3.1, similar to the arguments in this paper, the results in [8, 9] can also be generalized and improved based on Lemma 3.1.

4. Applications

To illustrate the usefulness of the results, we state the corresponding theorems in the previous section for the special cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Corollary 4.1. *Let $\mathbb{T} = \mathbb{R}$, and let $a, b, g, h, x : [t_0, \infty) \rightarrow \mathbb{R}_+$ be continuous. Then, for any continuous function $k(t) > 0$ on $[t_0, \infty)$, any nonnegative constants α and β satisfying $\lambda\alpha + (1-\lambda)\beta = 1$, inequality (I) implies that*

$$x(t) \leq a(t) + b(t) \int_{t_0}^t \exp\left(\int_s^t P(\tau) d\tau\right) Q(s) ds, \quad t \geq t_0, \quad (4.1)$$

where $P(t)$ and $Q(t)$ are defined as in Theorem 3.5.

Corollary 4.2. Let $\mathbb{T} = \mathbb{Z}$ and $a, b, g, h, x : \mathbb{N}_0 = \{t_0, t_0 + 1, \dots\} \rightarrow \mathbb{R}_+$. Then, for any function $k(t) > 0$ on \mathbb{N}_0 , any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (I) implies that

$$x(t) \leq a(t) + b(t) \sum_{s=t_0}^{t-1} \left(\prod_{\tau=s+1}^{t-1} (1 + P(\tau)) \right) Q(s), \quad t \in \mathbb{N}_0, \quad (4.2)$$

where $P(t)$ and $Q(t)$ are defined as in Theorem 3.5.

Corollary 4.3. Assume that $\mathbb{T} = \mathbb{R}$ and $a, b, g, h, x : [t_0, \infty) \rightarrow \mathbb{R}_+$ are continuous. Let $\omega(t, s)$ be defined as in Lemma 3.4 such that $\omega_1^\Delta(t, s) \geq 0$ for $t \geq s$ and (3.5) holds. Then, for any continuous function $k(t) > 0$ on $[t_0, \infty)$, any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (II) implies that

$$x(t) \leq a(t) + b(t) \int_{t_0}^t \exp \left(\int_s^t A(\tau) d\tau \right) B(s) ds, \quad t \geq t_0, \quad (4.3)$$

where $A(t)$ and $B(t)$ are the same as in Theorem 3.8.

Corollary 4.4. Assume that $\mathbb{T} = \mathbb{Z}$ and $a, b, g, h, x : \mathbb{N}_0 \rightarrow \mathbb{R}_+$. Let $\omega(t, s)$ be defined as in Lemma 3.4 such that $\omega_1^\Delta(t, s) \geq 0$ for $t \geq s$ and (3.5) holds. Then, for any function $k(t) > 0$ on \mathbb{N}_0 , any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (II) implies that

$$x(t) \leq a(t) + b(t) \sum_{s=t_0}^{t-1} \left(\prod_{\tau=s+1}^{t-1} (1 + A(\tau)) \right) B(s), \quad t \in \mathbb{N}_0, \quad (4.4)$$

where $A(t)$ and $B(t)$ are the same as in Theorem 3.8.

Corollary 4.5. Assume that $\mathbb{T} = \mathbb{R}$ and a, b, x are nonnegative continuous functions. Let $f : [t_0, \infty) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function satisfying (3.20). Then, for any continuous function $k(t) > 0$ on $[t_0, \infty)$, any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (III) implies that

$$x(t) \leq a(t) + b(t) \int_{t_0}^t \exp \left(\int_s^t M(\tau) d\tau \right) N(s) ds, \quad t \geq t_0, \quad (4.5)$$

where $M(t)$ and $N(t)$ are defined as in Theorem 3.9.

Corollary 4.6. Assume that $\mathbb{T} = \mathbb{Z}$ and a, b, x are nonnegative functions on \mathbb{N}_0 . Let $f : \mathbb{N}_0 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function satisfying (3.20). Then, for any function $k(t) > 0$ on \mathbb{N}_0 , any nonnegative constants α and β satisfying $\lambda\alpha + (1 - \lambda)\beta = 1$, inequality (III) implies that

$$x(t) \leq a(t) + b(t) \sum_{s=t_0}^{t-1} \left(\prod_{\tau=s+1}^{t-1} (1 + M(\tau)) \right) N(s), \quad t \in \mathbb{N}_0, \quad (4.6)$$

where $M(t)$ and $N(t)$ are defined as in Theorem 3.9.

Remark 4.7. It is not difficult to provide similar results for other specific time scales of interest. For example, consider the time scale $\mathbb{T} = \{0, 1, q, q^2, \dots\}$ with $q > 1$. Note that $\sigma(t) = qt$ and $\mu(t) = (q - 1)t$ for any $t \in \mathbb{T}$; we have

$$e_p(t, \sigma(s)) = \prod_{\tau=qs}^{t-1} [1 + (q - 1)\tau p(\tau)]^{1/(q-1)\tau} \tag{4.7}$$

for $t > s \geq t_0$ and $t, s, \tau \in \mathbb{T}$. Thus, Theorems 3.5–3.9 can be easily applied.

Finally, we apply Theorem 3.5 to a numerical example. Consider the following initial value problem on time scales:

$$x^\Delta(t) = H(t, x(t), x^\lambda(t)), \quad x(t_0) = x_0, \quad t \in \mathbb{T}^\kappa, \tag{4.8}$$

where $H : \mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying

$$\left| H(t, x(t), x^\lambda(t)) \right| \leq g(t)|x(t)| + h(t)|x^\lambda(t)|, \quad t \in \mathbb{T}, \tag{4.9}$$

where $g(t)$ and $h(t)$ are nonnegative rd-continuous functions on \mathbb{T}^κ . Then, by Theorem 3.5, we see that the solution of (4.8) satisfies

$$|x(t)| \leq |x_0| + \int_{t_0}^t e_{\tilde{p}}(t, \sigma(s)) \tilde{Q}(s) \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{4.10}$$

where

$$\tilde{P}(t) = g(t) + \lambda h^\alpha(t), \quad \tilde{Q}(t) = |x_0| [g(t) + \lambda h^\alpha(t)] + (1 - \lambda) h^\beta(t), \tag{4.11}$$

α, β are nonnegative constants, and $\lambda\alpha + (1 - \lambda)\beta = 1$.

In fact, the solution of (4.8) satisfies the following integral inequality:

$$x(t) = x_0 + \int_{t_0}^t H(s, x(s), x^\lambda(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{4.12}$$

It yields

$$|x(t)| \leq |x_0| + \int_{t_0}^t \left[g(s)|x(s)| + h(s)|x^\lambda(s)| \right] \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{4.13}$$

Using Theorem 3.5 with $k(t) = 1$, $a(t) = |x_0|$, and $b(t) = 1$, we see that (4.13) implies (4.10).

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