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*Research Article*

# An Upper Bound on the Critical Value $\beta^*$ Involved in the Blasius Problem

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Utilizing the Schauder fixed point theorem to study existence on positive solutions of an integral equation, we obtain an upper bound of the critical value $\beta^*$ involved in the Blasius problem, in particular, $\beta^* < -18733/10^5 = -0.18733$. Previous results only presented a lower bound $\beta^* \geq -1/2$ and numerical investigations $\beta^* \doteq -0.3541$.

## 1. Introduction

The following third-order nonlinear differential equation arising in the boundary-layer problems

$$f'''(\eta) + f(\eta) f''(\eta) = 0 \quad \text{on } [0, \infty) \tag{1.1}$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = \beta, \qquad f'(\infty) = 1, \tag{1.2}$$

called the Blasius problem [1], has been used to describe the steady two-dimensional flow of a slightly viscous incompressible fluid past a flat plate, where $\eta$ is the similarity boundary-layer ordinate, $f(\eta)$ is the similarity stream function, and $f'(\eta)$ and $f''(\eta)$ are the velocity and the shear stress functions, respectively.

Problem (1.1)-(1.2) also arises in the study of the mixed convection in porous media [2]. The mixed convection parameter is given by $\beta = 1 + \varepsilon$, with $\varepsilon = R_a/P_e$ where $R_a$ is the

*Rayleigh* number and $P_e$ the *Péclet* number. The case of $\beta < 0$ corresponds to a flat plate moving at steady speed opposite to that of a uniform mainstream [3].

The boundary value problem (1.1)-(1.2) has been widely studied analytically. Weyl [4] proved that (1.1)-(1.2) has one and only one solution for $\beta = 0$; Coppel [5] studied the case of $\beta > 0$; the cases of $0 < \beta < 1$ [6] and $\beta > 1$ [7] were also investigated, respectively. Also, see [8]. Blasius problem is a special case of the Falkner-Skan equation, for $\beta = 0$; we may refer to [9–13] for some recent results on the Falkner-Skan equation.

Very recently, Brighi et al. [14] summarized historical study on the Blasius problem and analyzed the case $\beta < 0$ in details, in which the shape and the number of solutions were determined. We may refer to [14] and the references therein for more recent results.

However, up to today, we know only that there exists a critical value $\beta^* \in [-1/2, 0)$ such that (1.1)-(1.2) has at least a solution for $\beta \geq \beta^*$, no solution for $\beta < \beta^*$ [15]. Numerical results showed that $\beta^* \doteq -0.3541$ [15].

An open question is what is exactly $\beta^*$? To our knowledge, there is little study on it.

In this paper, we will study the open question mentioned above by studying the existence on positive solutions of an integral equation and present an upper bound of $\beta^*$, in particular, $\beta^* < -18733/10^5 = -0.18733$.

## 2. An Upper Bound of $\beta^*$

By the basic fact in [14], we know easily that if $f$ is a solution of (1.1)-(1.2), then $f'' > 0$ for $\eta \in [0, \infty)$. In this case, the most powerful method is the so-called Crocco transformation (see [14, 15]), which consists of choosing $t = f'$ as independent variable and expressing $z = f''$ as a function of $t$. Differentiating $z(f') = f''$ (the variable $t$ is omitted for simplicity), we obtain $z'(f')f'' = f''' = -ff''$; hence $z'(f') = -f$. Differentiating once again, we obtain $z''(f')f'' = -f'$. Then (1.1)-(1.2) becomes the Crocco equation [14]

$$\frac{d^2z}{dt^2} = -\frac{t}{z}, \quad \beta \leq t < 1 \tag{2.1}$$

with the boundary conditions

$$z'(\beta) = 0, \qquad z(1) = 0. \tag{2.2}$$

Integrating (2.1) from $\beta$ to $t$, we have

$$z'(t) = -\int_\beta^t \frac{s}{z(s)} ds \quad \text{on } [\beta, 1). \tag{2.3}$$

Integrating this equality from $t$ to 1, we obtain the following integral equation that is equivalent to (2.1)-(2.2):

$$z(t) = \int_t^1 \frac{s(1-s)}{z(s)} ds + (1-t) \int_\beta^t \frac{s}{z(s)} ds \quad \text{for } t \in [\beta, 1). \tag{2.4}$$

Let $g(\beta) = 1/3 - 8(1-\beta)\beta^2$ for $\beta \in [-1/2, 0]$, then $g'(\beta) = -8(2\beta - 3\beta^2) > 0$ for $\beta \in [-1/2, 0]$. By direct computation

$$g\left(-\frac{1}{5}\right) < 0, \qquad g\left(-\frac{18733}{10^5}\right) > 0. \tag{2.5}$$

Hence there exists $\widetilde{\beta} \in (-1/5, -18733/10^5)$ such that $g(\widetilde{\beta}) = 0$ and $g(\beta) > 0$ for $\beta \in (\widetilde{\beta}, 0)$.

We shall prove that (1.1)-(1.2) has at least a solution for $\beta \in [\widetilde{\beta}, 0)$.

Let $\beta \in [\widetilde{\beta}, 0)$ and $C[\beta, 1]$ be the Banach space of continuous functions on $[\beta, 1]$ with the norm $\|z\| = \max\{|z(t)| : t \in [\beta, 1]\}$ and $S : C[\beta, 1] \to C[\beta, 1]$ with $Sz(t) = \max\{z(t), c(t)\}$, where $c(t) = c_\beta(1-t)$ for $t \in [\beta, 1]$ and

$$c_\beta = \frac{\sqrt{3}/3 - \sqrt{g(\beta)}}{4(1-\beta)}. \tag{2.6}$$

Clearly, $Sz(t) \geq c(t)$ for $z \in C[\beta, 1]$ and $0 < c_\beta \leq \sqrt{3}/12$.

*Notation.* One has

$$Az(t) = \int_t^1 \frac{s(1-s)}{Sz(s)} ds, \quad Bz(t) = \int_\beta^t \frac{s}{Sz(s)} ds \quad \text{for } \beta \leq t < 1. \tag{2.7}$$

We consider the following integral equation of the form

$$z(t) = Az(t) + (1-t)Bz(t) \quad \text{for } \beta \leq t < 1. \tag{2.8}$$

**Lemma 2.1.** *The integral equation* (2.8) *has a solution* $z \in C[\beta, 1]$.

*Proof.* Let $C = \{z \in C[\beta, 1] : \|z\| \leq 2M\}$ with $M = \int_\beta^1 ((1-s)|s|/c(s) ds)$. We define an operator $T$ on $C$ by setting

$$Tz(t) = \begin{cases} Az(t) + (1-t)Bz(t) & \text{if } t \in [\beta, 1), \\ 0 & \text{if } t = 1. \end{cases} \tag{2.9}$$

Since

$$\begin{gathered} Az(t) = \int_t^1 \frac{s(1-s)}{Sz(s)}\, ds \leq \int_t^1 \frac{s}{c_\beta}\, ds = \frac{1-t^2}{2c_\beta} \quad \text{for } t \in (0,1), \\ \int_0^t \frac{s}{Sz(s)}\, ds \leq \int_0^t \frac{1}{c_\beta(1-s)}\, ds = -\frac{\ln(1-t)}{c_\beta} \quad \text{for } t \in (0,1), \\ \lim_{t \to 1^-} (1-t)\ln(1-t) = 0, \end{gathered} \tag{2.10}$$

we know that $\lim_{t\to 1^-} Tz(t) = 0$ and then $T$ maps $C$ into $C[\beta, 1]$. We show that $T$ is continuous and compact from $C$ into $C$.

Let $z_n \in C$, $z \in C$, and $\lim_{n\to+\infty}\|z_n - z\| = 0$. Since $1-t \le 1-s$ for $\beta \le s \le t \le 1$, we have

$$\begin{aligned}|Tz_n(t) - Tz(t)| &\le |Az_n(t) - Az(t)| + (1-t)|Bz_n(t) - Bz(t)| \\ &\le \int_\beta^1 \left| \frac{s(1-s)}{Sz_n(s)} - \frac{s(1-s)}{Sz(s)} \right| ds \\ &\quad + \int_\beta^1 \left| \left( \frac{s(1-s)}{Sz_n(s)} - \frac{s(1-s)}{Sz(s)} \right) \frac{1-t}{1-s} \right| ds \\ &\le 2\int_\beta^1 \left| \frac{s(1-s)}{Sz_n(s)} - \frac{s(1-s)}{Sz(s)} \right| ds.\end{aligned} \tag{2.11}$$

Since

$$\lim_{n\to+\infty} \frac{s(1-s)}{Sz_n(s)} = \frac{(1-s)s}{Sz(s)} \quad \text{for } s \in [\beta, 1) \tag{2.12}$$

and $Sz(t) \ge c(t)$, the Lebesgue dominated convergence theorem, the dominated function $F(s) = 1/c_\beta$ for $s \in [\beta, 1]$ implies that $\|Tz_n - Tz\| \to 0$, that is, $T$ is continuous.

By $d(Tz(t))/dt = -\int_\beta^t (s/Sz(s))ds$, we have

$$\left| \frac{d(Tz(t))}{dt} \right| \le \int_\beta^t \frac{|s|}{Sz(s)} ds \le \int_\beta^t \frac{|s|}{c(s)} ds \quad \text{for } \beta \le t < 1. \tag{2.13}$$

Noticing that

$$\int_\beta^1 \int_\beta^t \frac{|s|}{c(s)} ds\, dt = \int_\beta^1 \int_s^1 \frac{|s|}{c(s)} dt\, ds = \int_\beta^1 \frac{(1-s)|s|}{c(s)} ds = M < \infty, \tag{2.14}$$

we have $\int_\beta^1 |d(Tz(s))/ds| ds \le M$. This, together with the absolute continuity of the Lebesgue integral, implies that $T(C) = \{Tz(t) : z \in C\}$ is equicontinuous.

On the other hand,

$$\begin{aligned}|Tz(t)| &\le \int_t^1 \frac{|s|(1-s)}{Sz(s)} ds + \int_\beta^t \frac{|s|(1-t)}{Sz(s)} ds \\ &\le \int_\beta^1 \frac{|s|(1-s)}{c(s)} ds + \int_\beta^1 \frac{|s|(1-s)}{c(s)} ds = 2M.\end{aligned} \tag{2.15}$$

It follows from the Schauder fixed point theorem that there exists $z \in C$ such that (2.8) holds. □

**Theorem 2.2.** *The problem* (1.1)-(1.2) *has at least a solution for* $\beta \in [\tilde{\beta}, 0)$ *and then* $\beta^* < -18733/10^5 = -0.18733$.

*Proof.* We first prove that the function $z$ obtained in Lemma 2.1 is a solution of (2.4) for $\beta \in [\tilde{\beta}, 0)$. Clearly, we have only to prove $Sz(t) = z(t)$ for $t \in [\beta, 1]$, that is, $z(t) \geq c(t)$ for $t \in [\beta, 1]$.

First of all, we prove that there exists $t \in (\beta, 1)$ such that $z(t) > c(t)$. In fact, if $z(t) \leq c(t)$ for $t \in (\beta, 1)$, then by $Sz(t) = c_\beta(1-t)$

$$c_\beta(1-\beta) \geq z(\beta) = \int_\beta^1 \frac{s(1-s)}{Sz(s)} ds = \int_\beta^1 \frac{s(1-s)}{c(s)} ds = \frac{1}{2c_\beta}\left(1-\beta^2\right). \tag{2.16}$$

This implies that $c_\beta^2 \geq (1+\beta)/2 \geq (1-1/5)/2 = 2/5$, which contradicts $c_\beta \leq \sqrt{3}/12$.

From the relations

$$z'(t) = -\int_\beta^t \frac{s}{Sz(s)} ds, \qquad z''(t) = -\frac{t}{Sz(t)}, \tag{2.17}$$

we know that $z$ is convex and increasing on $[\beta, 0]$ and concave on $[0, 1]$. Moreover, since $z(1) = 0$, there exists $\tilde{t} \in (0, 1)$ such that $z(\tilde{t}) = \max\{z(t) : t \in [\beta, 1]\}$.

For $t \in [\tilde{t}, 1)$, we have $Bz(t) \geq Bz(\tilde{t}) = -z'(\tilde{t}) = 0$. Then, from (2.8) we deduce that $Az(t) \leq z(t) \leq Sz(t)$ for $t \in [\tilde{t}, 1)$ and hence

$$Az(t)(-Az(t))' \leq t(1-t) \quad \text{for } t \in \left[\tilde{t}, 1\right). \tag{2.18}$$

Integrating the last inequality for $\tilde{t}$ to 1 and using $Az(1) = 0$, we know that

$$\frac{\left[Az\left(\tilde{t}\right)\right]^2}{2} \leq \int_{\tilde{t}}^1 s(1-s) ds \leq \int_0^1 s(1-s) ds = \frac{1}{6}. \tag{2.19}$$

And then $z(\tilde{t}) = Az(\tilde{t}) \leq \sqrt{3}/3$. This, together with $c(t) \leq c_\beta \leq \sqrt{3}/12$ for $t \in [0, 1]$, implies that $Sz(t) \leq \sqrt{3}/3$ for $t \in [0, 1]$. Hence

$$\int_0^1 \frac{s(1-s)}{Sz(s)} ds \geq \int_0^1 \frac{s(1-s)}{\sqrt{3}/3} ds = \frac{\sqrt{3}}{6}. \tag{2.20}$$

Noticing that $Sz(t) \geq c(t)$ and $t(1-t) < 0$ for $t \in (\beta, 0)$, we obtain

$$\int_\beta^0 \frac{s(1-s)}{Sz(s)} ds \geq \int_\beta^0 \frac{s(1-s)}{c(s)} ds = -\frac{\beta^2}{2c_\beta}. \tag{2.21}$$

Then

$$z(\beta) = \int_\beta^1 \frac{s(1-s)}{Sz(s)} ds = \int_\beta^0 \frac{s(1-s)}{Sz(s)} ds + \int_0^1 \frac{s(1-s)}{Sz(s)} ds \geq \frac{\sqrt{3}}{6} - \frac{\beta^2}{2c_\beta}. \tag{2.22}$$

By direct computation, we have $\sqrt{3}/6 - \beta^2/2c_\beta = c_\beta(1-\beta)$ and then $z(\beta) \geq c(\beta)$. Since $z$ is convex and increasing on $[\beta, 0]$ and concave on $[0, 1]$ with $z(1) = 0$, we immediately get $z(t) \geq c(t)$ for $t \in [\beta, 1]$. Hence $Sz = z$ and $z$ is a positive solution of (2.4).

Since any positive solution of (2.1)-(2.2) is a solution of (1.1)-(1.2) [14] and (2.1)-(2.2) is equivalent to (2.4), hence (1.1)-(1.2) has at least a solution for $\beta \in [\widetilde{\beta}, 0)$ and we obtain the desired result $\beta^* \leq \widetilde{\beta} < -18733/10^5 = -0.18733$. □

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