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Research Article

Generalization of an Inequality for Integral Transforms with Kernel and Related Results

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We establish a generalization of the inequality introduced by Mitrinović and Pečarić in 1988. We prove mean value theorems of Cauchy type for that new inequality by taking its difference. Furthermore, we prove the positive semidefiniteness of the matrices generated by the difference of the inequality which implies the exponential convexity and logarithmic convexity. Finally, we define new means of Cauchy type and prove the monotonicity of these means.

1. Introduction

Let $K(x,t)$ be a nonnegative kernel. Consider a function $u : [a,b] \to \mathbb{R}$, where $u \in U(v,K)$, and the representation of u is

$$u(x) = \int_a^b K(x,t)v(t)dt \tag{1.1}$$

for any continuous function v on $[a,b]$. Throughout the paper, it is assumed that all integrals under consideration exist and that they are finite.

The following theorem is given in [1] (see also [2, page 235]).

Theorem 1.1. *Let $u_i \in U(v,K)$ $(i = 1,2)$ and $r(t) \geq 0$ for all $t \in [a,b]$. Also let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a function such that $\phi(x)$ is convex and increasing for $x > 0$. Then*

$$\int_a^b r(x)\phi\left(\left|\frac{u_1(x)}{u_2(x)}\right|\right)dx \leq \int_a^b s(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)dx, \tag{1.2}$$

where

$$s(x) = v_2(x) \int_a^b \frac{r(t)K(t,x)}{u_2(t)} dt, \quad u_2(t) \neq 0. \tag{1.3}$$

The following definition is equivalent to the definition of convex functions.

Definition 1.2 (see [2]). Let $I \subseteq \mathbb{R}$ be an interval, and let $\phi : I \to \mathbb{R}$ be convex on I. Then, for $s_1, s_2, s_3 \in I$ such that $s_1 < s_2 < s_3$, the following inequality holds:

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0. \tag{1.4}$$

Let us recall the following definition.

Definition 1.3 (see [3, page 373]). A function $h : (a,b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geq 0 \tag{1.5}$$

for all $n \in \mathbb{N}$ and all choices of $\xi_i \in \mathbb{R}, x_i + x_j \in (a,b),\ i,j = 1,\ldots,n$.

The following proposition is useful to prove the exponential convexity.

Proposition 1.4 (see [4]). *Let $h : (a,b) \to \mathbb{R}$. The following statements are equivalent.*

(i) *h is exponentially convex.*

(ii) *h is continuous, and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0 \tag{1.6}$$

for every $n \in \mathbb{N}, \xi_i \in (a,b)$, and $x_i \in (a,b)$, $1 \leq i \leq n$.

Corollary 1.5. *If $h : (a,b) \to \mathbb{R}^+$ is exponentially convex, then h is* log-*convex; that is,*

$$h(\lambda x + (1-\lambda)y) \leq h(x)^{\lambda} h(y)^{1-\lambda} \quad \forall x,y \in (a,b),\ \lambda \in [0,1]. \tag{1.7}$$

This paper is organized in this manner. In Section 2, we give the generalization of Mitrinović-Pečarić inequality and prove the mean value theorems of Cauchy type. We also introduce the new type of Cauchy means. In Section 3, we give the proof of positive semidefiniteness of matrices generated by the difference of that inequality obtained from the generalization of Mitrinović-Pečarić inequality and also discuss the exponential convexity. At the end, we prove the monotonicity of the means.

2. Main Results

Theorem 2.1. *Let $u_i \in U(v,K)$ $(i=1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $I \subseteq \mathbb{R}$ be an interval, let $\phi : I \to \mathbb{R}$ be convex, and let $u_1(x)/u_2(x)$, $v_1(x)/v_2(x) \in I$. Then*

$$\int_a^b r(x)\phi\left(\frac{u_1(x)}{u_2(x)}\right)dx \leq \int_a^b q(x)\phi\left(\frac{v_1(x)}{v_2(x)}\right)dx, \tag{2.1}$$

where

$$q(x) = v_2(x)\int_a^b \frac{r(t)K(t,x)}{u_2(t)}dt, \quad u_2(t) \neq 0. \tag{2.2}$$

Proof. Since $u_1 = \int_a^b K(x,t)v_1(t)dt$ and $v_2(t) > 0$, we have

$$\begin{aligned}
\int_a^b r(x)\phi\left(\frac{u_1(x)}{u_2(x)}\right)dx &= \int_a^b r(x)\phi\left(\frac{1}{u_2(x)}\int_a^b K(x,t)v_1(t)dt\right)dx \\
&= \int_a^b r(x)\phi\left(\frac{1}{u_2(x)}\int_a^b K(x,t)v_2(t)\frac{v_1(t)}{v_2(t)}dt\right)dx \\
&= \int_a^b r(x)\phi\left(\int_a^b \frac{K(x,t)v_2(t)}{u_2(x)}\frac{v_1(t)}{v_2(t)}dt\right)dx.
\end{aligned} \tag{2.3}$$

By Jensen's inequality, we get

$$\begin{aligned}
\int_a^b r(x)\phi\left(\frac{u_1(x)}{u_2(x)}\right)dx &\leq \int_a^b r(x)\left(\int_a^b \frac{K(x,t)v_2(t)}{u_2(x)}\phi\left(\frac{v_1(t)}{v_2(t)}\right)dt\right)dx \\
&= \int_a^b\left(\int_a^b \frac{r(x)K(x,t)v_2(t)}{u_2(x)}\phi\left(\frac{v_1(t)}{v_2(t)}\right)dt\right)dx \\
&= \int_a^b \phi\left(\frac{v_1(t)}{v_2(t)}\right)v_2(t)\left(\int_a^b \frac{r(x)K(x,t)}{u_2(x)}dx\right)dt \\
&= \int_a^b q(t)\phi\left(\frac{v_1(t)}{v_2(t)}\right)dt.
\end{aligned} \tag{2.4}$$

□

Remark 2.2. If ϕ is strictly convex on I and $v_1(x)/v_2(x)$ is nonconstant, then the inequality in (2.1) is strict.

Remark 2.3. Let us note that Theorem 1.1 follows from Theorem 2.1. Indeed, let the condition of Theorem 1.1 be satisfied, and let $\widetilde{u}_i \in U(|v|, K)$; that is,

$$\widetilde{u}_1(x) = \int_a^b K(x,t)|v_1(t)|dt. \tag{2.5}$$

So, by Theorem 2.1, we have

$$\int_a^b q(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)dx = \int_a^b q(x)\phi\left(\frac{|v_1(x)|}{v_2(x)}\right)dx \geq \int_a^b r(x)\phi\left(\frac{\widetilde{u}_1(x)}{u_2(x)}\right)dx. \tag{2.6}$$

On the other hand, ϕ is increasing function, we have

$$\begin{aligned}\phi\left(\frac{\widetilde{u}_1(x)}{u_2(x)}\right) &= \phi\left(\frac{1}{u_2(x)}\int_a^b K(x,t)|v_1(t)|dt\right) \\ &\geq \phi\left(\frac{1}{u_2(x)}\left|\int_a^b K(x,t)v_1(t)dt\right|\right) \\ &= \phi\left(\frac{|u_1(x)|}{u_2(x)}\right) = \phi\left(\left|\frac{u_1(x)}{u_2(x)}\right|\right).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we get (1.2).

If $f \in C([a,b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}dt. \tag{2.8}$$

We will use the following kernel in the upcoming corollary:

$$K_I(x,t) = \begin{cases} \dfrac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}, & a \leq t \leq x, \\ 0, & x < t \leq b. \end{cases} \tag{2.9}$$

Corollary 2.4. *Let $u_i \in C([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $I \subseteq \mathbb{R}$ be an interval, let $\phi : I \to \mathbb{R}$ be convex, $u_1(x)/u_2(x)$, $I_a^\alpha u_1(x)/I_a^\alpha u_2(x) \in I$, and $u_1(x), u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$. Then*

$$\int_a^b r(x)\phi\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right)dx \leq \int_a^b \phi\left(\frac{u_1(t)}{u_2(t)}\right)Q_I(t)dt, \tag{2.10}$$

where

$$Q_I(t) = \frac{u_2(t)}{\Gamma(\alpha)}\int_t^b \frac{r(x)(x-t)^{\alpha-1}}{I_a^\alpha u_2(x)}dx, \quad I_a^\alpha u_2(x) \neq 0. \tag{2.11}$$

Let $AC([a,b])$ be space of all absolutely continuous functions on $[a,b]$. By $AC^n([a,b])$, we denote the space of all functions $g \in C^n([a,b])$ with $g^{(n-1)} \in AC([a,b])$.

Let $\alpha \in \mathbb{R}^+$ and $g \in AC^n([a,b])$. Then the Caputo fractional derivative (see [5, p. 270]) of order α for a function g is defined by

$$D^{\alpha}_{*a}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \tag{2.12}$$

where $n = [\alpha]+1$; the notation of $[\alpha]$ stands for the largest integer not greater than α.

Here we use the following kernel in the upcoming corollary:

$$K_D(x,t) = \begin{cases} \dfrac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a \le t \le x, \\ 0, & x < t \le b. \end{cases} \tag{2.13}$$

Corollary 2.5. *Let $u_i \in AC^n([a,b])$ $(i=1,2)$, and $r(x) \ge 0$ for all $x \in [a,b]$. Also let $I \subseteq \mathbb{R}$ be an interval, let $\phi : I \to \mathbb{R}$ be convex, $u_1^{(n)}(t)/u_2^{(n)}(t)$, $D^{\alpha}_{*a}u_1(x)/D^{\alpha}_{*a}u_2(x) \in I$, and $u_1(x), u_2(x)$ have Caputo fractional derivative of order $\alpha > 0$. Then*

$$\int_a^b r(x)\phi\left(\frac{D^{\alpha}_{*a}u_1(x)}{D^{\alpha}_{*a}u_2(x)}\right)dx \le \int_a^b \phi\left(\frac{u_1^{(n)}(t)}{u_2^{(n)}(t)}\right)Q_D(t)dt, \tag{2.14}$$

where

$$Q_D(t) = \frac{u_2^{(n)}(t)}{\Gamma(n-\alpha)}\int_t^b \frac{r(x)(x-t)^{n-\alpha-1}}{D^{\alpha}_{*a}u_2(x)}dx, \quad D^{\alpha}_{*a}u_2(x) \ne 0. \tag{2.15}$$

Let $L_1(a,b)$ be the space of all functions integrable on (a,b). For $\beta \in \mathbb{R}^+$, we say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D^{\beta}_a f$ in $[a,b]$ if and only if $D^{\beta-k}_a f \in C([a,b])$ for $k = 1,\dots,[\beta]+1$, $D^{\beta-1}_a f \in AC([a,b])$, and $D^{\beta}_a \in L_\infty(a,b)$.

The next lemma is very useful to give the upcoming corollary [6] (see also [5, p. 449]).

Lemma 2.6. *Let $\beta > \alpha \ge 0$, $f \in L_1(a,b)$ has an L_∞ fractional derivative $D^{\beta}_a f$ in $[a,b]$, and*

$$D^{\beta-k}_a f(a) = 0, \quad k = 1,\dots,[\beta]+1. \tag{2.16}$$

Then

$$D^{\alpha}_a f(s) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^s (s-t)^{\beta-\alpha-1}D^{\beta}_a f(t)dt \tag{2.17}$$

for all $a \le s \le b$.

Clearly

$$\begin{aligned} &D_a^\alpha f \text{ is in } AC([a,b]) \quad \text{for } \beta-\alpha \geq 1, \\ &D_a^\alpha f \text{ is in } C([a,b]) \quad \text{for } \beta-\alpha \in (0,1), \end{aligned} \tag{2.18}$$

hence

$$\begin{aligned} &D_a^\alpha f \in L_\infty(a,b), \\ &D_a^\alpha f \in L_1(a,b). \end{aligned} \tag{2.19}$$

Now we use the following kernel in the upcoming corollary:

$$K_L(s,t) = \begin{cases} \dfrac{(s-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq t \leq s, \\ 0, & s < t \leq b. \end{cases} \tag{2.20}$$

Corollary 2.7. *Let $\beta > \alpha \geq 0$, $u_i \in L_1(a,b)$ $(i=1,2)$ has an L_∞ fractional derivative $D_a^\beta u_i$ in $[a,b]$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $D_a^{\beta-k} u_i(a) = 0$ for $k = 1,\dots,[\beta]+1$ $(i=1,2)$, let $\phi : I \to \mathbb{R}$ be convex, and $D_a^\alpha u_1(x)/D_a^\alpha u_2(x)$, $D_a^\beta u_1(x)/D_a^\beta u_2(x) \in I$. Then*

$$\int_a^b r(x)\phi\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right)dx \leq \int_a^b \phi\left(\frac{D_a^\beta u_1(t)}{D_a^\beta u_2(t)}\right)Q_L(t)dt, \tag{2.21}$$

where

$$Q_L(t) = \frac{D_a^\beta u_2(t)}{\Gamma(\beta-\alpha)} \int_t^b \frac{r(x)(x-t)^{\beta-\alpha-1}}{D_a^\alpha u_2(x)}dx, \quad D_a^\alpha u_2(x) \neq 0. \tag{2.22}$$

Lemma 2.8. *Let $f \in C^2(I)$, and let I be a compact interval, such that*

$$m \leq f''(x) \leq M, \quad \forall x \in I. \tag{2.23}$$

Consider two functions ϕ_1, ϕ_2 defined as

$$\begin{aligned} \phi_1(x) &= \frac{Mx^2}{2} - f(x), \\ \phi_2(x) &= f(x) - \frac{mx^2}{2}. \end{aligned} \tag{2.24}$$

Then ϕ_1 and ϕ_2 are convex on I.

Proof. We have

$$\begin{aligned}\phi_1''(x) &= M - f''(x) \geq 0,\\ \phi_2''(x) &= f''(x) - m \geq 0,\end{aligned} \tag{2.25}$$

that is ϕ_1, ϕ_2 are convex on I. □

Theorem 2.9. *Let $f \in C^2(I)$, let I be a compact interval, $u_i \in U(v, K)$ $(i = 1, 2)$, and $r(x) \geq 0$ for all $x \in [a, b]$. Also let $u_1(x)/u_2(x)$, $v_1(x)/v_2(x) \in I$, $v_1(x)/v_2(x)$ be nonconstant, and let $q(x)$ be given in* (2.2). *Then there exists $\xi \in I$ such that*

$$\begin{aligned}&\int_a^b \left(q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x) f\left(\frac{u_1(x)}{u_2(x)}\right)\right) dx\\ &\quad= \frac{f''(\xi)}{2} \int_a^b \left(q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right) dx.\end{aligned} \tag{2.26}$$

Proof. Since $f \in C^2(I)$ and I is a compact interval, therefore, suppose that $m = \min f''$, $M = \max f''$. Using Theorem 2.1 for the function ϕ_1 defined in Lemma 2.8, we have

$$\int_a^b r(x)\left(\frac{M}{2}\left(\frac{u_1(x)}{u_2(x)}\right)^2 - f\left(\frac{u_1(x)}{u_2(x)}\right)\right) dx \leq \int_a^b q(x)\left(\frac{M}{2}\left(\frac{v_1(x)}{v_2(x)}\right)^2 - f\left(\frac{v_1(x)}{v_2(x)}\right)\right) dx. \tag{2.27}$$

From Remark 2.2, we have

$$\int_a^b \left(q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right) dx > 0. \tag{2.28}$$

Therefore, (2.27) can be written as

$$\frac{2\int_a^b (q(x) f(v_1(x)/v_2(x)) - r(x) f(u_1(x)/u_2(x)))dx}{\int_a^b \left(q(x)(v_1(x)/v_2(x))^2 - r(x)(u_1(x)/u_2(x))^2\right)dx} \leq M. \tag{2.29}$$

We have a similar result for the function ϕ_2 defined in Lemma 2.8 as follows:

$$\frac{2\int_a^b (q(x) f(v_1(x)/v_2(x)) - r(x) f(u_1(x)/u_2(x)))dx}{\int_a^b \left(q(x)(v_1(x)/v_2(x))^2 - r(x)(u_1(x)/u_2(x))^2\right)dx} \geq m. \tag{2.30}$$

Using (2.29) and (2.30), we have

$$m \leq \frac{2\int_a^b (q(x) f(v_1(x)/v_2(x)) - r(x) f(u_1(x)/u_2(x)))dx}{\int_a^b \left(q(x)(v_1(x)/v_2(x))^2 - r(x)(u_1(x)/u_2(x))^2\right)dx} \leq M. \tag{2.31}$$

By Lemma 2.8, there exists $\xi \in I$ such that

$$\frac{\int_a^b (q(x)f(v_1(x)/v_2(x)) - r(x)f(u_1(x)/u_2(x)))dx}{\int_a^b \left(q(x)(v_1(x)/v_2(x))^2 - r(x)(u_1(x)/u_2(x))^2\right)dx} = \frac{f''(\xi)}{2}. \tag{2.32}$$

This is the claim of the theorem. □

Let us note that a generalized mean value Theorem 2.9 for fractional derivative was given in [7]. Here we will give some related results as consequences of Theorem 2.9.

Corollary 2.10. *Let $f \in C^2(I)$, let I be a compact interval, $u_i \in C([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x)$, $I_a^\alpha u_1(x)/I_a^\alpha u_2(x) \in I$, let $u_1(x)/u_2(x)$ be nonconstant, let $Q_I(t)$ be given in (2.11), and $u_1(x)$, $u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\begin{aligned}\int_a^b &\left(Q_I(x)f\left(\frac{u_1(x)}{u_2(x)}\right) - r(x)f\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right)\right)dx \\ &= \frac{f''(\xi)}{2}\int_a^b \left(Q_I(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2 - r(x)\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right)^2\right)dx.\end{aligned} \tag{2.33}$$

Corollary 2.11. *Let $f \in C^2(I)$, let I be compact interval, $u_i \in AC^n([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t)$, $D_{*a}^\alpha u_1(x)/D_{*a}^\alpha u_2(x) \in I$, let $u_1^{(n)}(x)/u_2^{(n)}(x)$ be nonconstant, let $Q_D(t)$ be given in (2.15), and $u_1(x), u_2(x)$ have Caputo derivative of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\begin{aligned}\int_a^b &\left(Q_D(x)f\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right) - r(x)f\left(\frac{D_{*a}^\alpha u_1(x)}{D_{*a}^\alpha u_2(x)}\right)\right)dx \\ &= \frac{f''(\xi)}{2}\int_a^b \left(Q_D(x)\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right)^2 - r(x)\left(\frac{D_{*a}^\alpha u_1(x)}{D_{*a}^\alpha u_2(x)}\right)^2\right)dx.\end{aligned} \tag{2.34}$$

Corollary 2.12. *Let $\beta > \alpha \geq 0$, $f \in C^2(I)$, let I be a compact interval, $u_i \in L_1(a,b)$ $(i = 1,2)$ has an L_∞ fractional derivative, and $r(x) \geq 0$ for all $x \in [a,b]$. Let $D_a^{\beta-k}u_i(a) = 0$ for $k = 1,\ldots,[\beta]+1$ $(i = 1,2)$, $D_a^\alpha u_1(x)/D_a^\alpha u_2(x)$, $D_a^\beta u_1(x)/D_a^\beta u_2(x) \in I$, let $D_a^\beta u_1(x)/D_a^\beta u_2(x)$ be nonconstant, and let $Q_L(t)$ be given in (2.22). Then there exists $\xi \in I$ such that*

$$\begin{aligned}\int_a^b &\left(Q_L(x)f\left(\frac{D_a^\beta u_1(x)}{D_a^\beta u_2(x)}\right) - r(x)f\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right)\right)dx \\ &= \frac{f''(\xi)}{2}\int_a^b \left(Q_L(x)\left(\frac{D_a^\beta u_1(x)}{D_a^\beta u_2(x)}\right)^2 - r(x)\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right)^2\right)dx.\end{aligned} \tag{2.35}$$

Theorem 2.13. *Let $f, g \in C^2(I)$, let I be a compact interval, $u_i \in U(v, K)$ $(i = 1, 2)$, and $r(x) \geq 0$ for all $x \in [a, b]$. Also let $u_1(x)/u_2(x)$, $v_1(x)/v_2(x) \in I$, $v_1(x)/v_2(x)$ be nonconstant, and let $q(x)$ be given in* (2.2). *Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b q(x) f(v_1(x)/v_2(x))dx - \int_a^b r(x) f(u_1(x)/u_2(x))dx}{\int_a^b q(x) g(v_1(x)/v_2(x))dx - \int_a^b r(x) g(u_1(x)/u_2(x))dx} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.36}$$

It is provided that denominators are not equal to zero.

Proof. Let us take a function $h \in C^2(I)$ defined as

$$h(x) = c_1 f(x) - c_2 g(x), \tag{2.37}$$

where

$$\begin{aligned} c_1 &= \int_a^b q(x) g\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) g\left(\frac{u_1(x)}{u_2(x)}\right) dx, \\ c_2 &= \int_a^b q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx. \end{aligned} \tag{2.38}$$

By Theorem 2.9 with $f = h$, we have

$$0 = \left(\frac{c_1}{2} f''(\xi) - \frac{c_2}{2} g''(\xi)\right)\left(\int_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 dx - \int_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2 dx\right). \tag{2.39}$$

Since

$$\int_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 dx - \int_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2 dx \neq 0, \tag{2.40}$$

so we have

$$c_1 f''(\xi) - c_2 g''(\xi) = 0. \tag{2.41}$$

This implies that

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.42}$$

This is the claim of the theorem. □

Let us note that a generalized Cauchy mean-valued theorem for fractional derivative was given in [8]. Here we will give some related results as consequences of Theorem 2.13.

Corollary 2.14. *Let $f, g \in C^2(I)$, let I be a compact interval, $u_i \in C([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x)$, $I_a^\alpha u_1(x)/I_a^\alpha u_2(x) \in I$, let $u_1(x)/u_2(x)$ be nonconstant, let $Q_I(t)$ be given in (2.11), and $u_1(x)$, $u_2(x)$ have Riemann-Liouville fractional derivative of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b Q_I(x) f(u_1(x)/u_2(x))dx - \int_a^b r(x) f(I_a^\alpha u_1(x)/I_a^\alpha u_2(x))dx}{\int_a^b Q_I(x) g(u_1(x)/u_2(x))dx - \int_a^b r(x) g(I_a^\alpha u_1(x)/I_a^\alpha u_2(x))dx} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.43}$$

It is provided that denominators are not equal to zero.

Corollary 2.15. *Let $f, g \in C^2(I)$, let I be a compact interval, $u_i \in AC^n([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t), D_{*a}^\alpha u_1(x)/D_{*a}^\alpha u_2(x) \in I$, let $u_1^{(n)}(x)/u_2^{(n)}(x)$ be nonconstant, let $Q_D(t)$ be given in (2.15), and $u_1(x)$, $u_2(x)$ have Caputo fractional derivative of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b Q_D(x) f\left(u_1^{(n)}(x)/u_2^{(n)}(x)\right)dx - \int_a^b r(x) f(D_{*a}^\alpha u_1(x)/D_{*a}^\alpha u_2(x))dx}{\int_a^b Q_D(x) g\left(u_1^{(n)}(x)/u_2^{(n)}(x)\right)dx - \int_a^b r(x) g(D_{*a}^\alpha u_1(x)/D_{*a}^\alpha u_2(x))dx} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.44}$$

It is provided that denominators are not equal to zero.

Corollary 2.16. *Let $\beta > \alpha \geq 0$, $f, g \in C^2(I)$, let I be a compact interval, $u_i \in L_1(a,b)$ $(i = 1,2)$ has an L_∞ fractional derivative $D_a^\beta u_i$ in $[a,b]$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $D_a^{\beta-k} u_i(a) = 0$ for $k = 1,\ldots,[\beta]+1$ $(i = 1,2)$, $D_a^\alpha u_1(x)/D_a^\alpha u_2(x), D_a^\beta u_1(x)/D_a^\beta u_2(x) \in I$, let $D_a^\beta u_1(x)/D_a^\beta u_2(x)$ be nonconstant, and let $Q_L(t)$ be given in (2.22). Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b Q_L(x) f\left(D_a^\beta u_1(x)/D_a^\beta u_2(x)\right)dx - \int_a^b r(x) f(D_a^\alpha u_1(x)/D_a^\alpha u_2(x))dx}{\int_a^b Q_L(x) g\left(D_a^\beta u_1(x)/D_a^\beta u_2(x)\right)dx - \int_a^b r(x) g(D_a^\alpha u_1(x)/D_a^\alpha u_2(x))dx} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.45}$$

It is provided that denominators are not equal to zero.

Corollary 2.17. *Let $I \subseteq \mathbb{R}^+$, let I be a compact interval, $u_i \in U(v,K)$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Let $u_1(x)/u_2(x), v_1(x)/v_2(x) \in I$, let $v_1(x)/v_2(x)$ be nonconstant, and let $q(x)$ be given in (2.2). Then, for $s, t \in \mathbb{R} \setminus \{0,1\}$ and $s \neq t$, there exists $\xi \in I$ such that*

$$\xi = \left(\frac{s(s-1)}{t(t-1)} \frac{\int_a^b q(x)(v_1(x)/v_2(x))^t dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x))^s dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx} \right)^{1/(t-s)}. \tag{2.46}$$

Proof. We set $f(x) = x^t$ and $g(x) = x^s$, $t \neq s$, $s, t \neq 0, 1$. By Theorem 2.13, we have

$$\frac{\int_a^b q(x)(v_1(x)/v_2(x))^t dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x))^s dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx} = \frac{t(t-1)\xi^{t-2}}{s(s-1)\xi^{s-2}}. \tag{2.47}$$

This implies that

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\int_a^b q(x)(v_1(x)/v_2(x))^t dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x))^s dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx}. \tag{2.48}$$

This implies that

$$\xi = \left(\frac{s(s-1)}{t(t-1)} \frac{\int_a^b q(x)(v_1(x)/v_2(x))^t dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x))^s dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx} \right)^{1/(t-s)}. \tag{2.49}$$

□

Remark 2.18. Since the function $\xi \to \xi^{t-s}$ is invertible and from (2.46), we have

$$m \leq \left(\frac{s(s-1)}{t(t-1)} \frac{\int_a^b q(x)(v_1(x)/v_2(x))^t dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x))^s dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx} \right)^{1/(t-s)} \leq M. \tag{2.50}$$

Now we can suppose that f''/g'' is an invertible function, then from (2.36) we have

$$\xi = \left(\frac{f''}{g''} \right)^{-1} \left(\frac{\int_a^b q(x)(v_1(x)/v_2(x)) dx - \int_a^b r(x)(u_1(x)/u_2(x))^t dx}{\int_a^b q(x)(v_1(x)/v_2(x)) dx - \int_a^b r(x)(u_1(x)/u_2(x))^s dx} \right). \tag{2.51}$$

We see that the right-hand side of (2.49) is mean, then for distinct $s, t \in \mathbb{R}$ it can be written as

$$M_{s,t} = \left(\frac{\Lambda_t}{\Lambda_s} \right)^{1/(t-s)} \tag{2.52}$$

as mean in broader sense. Moreover, we can extend these means, so in limiting cases for $s,t \neq 0,1$,

$$
\begin{aligned}
&\lim_{t\to s} M_{s,t}\\
&= M_{s,s}\\
&= \exp\left(\frac{\int_a^b q(x)\mathcal{A}(x)^s\log\mathcal{A}(x)dx - \int_a^b r(x)\mathcal{B}(x)^s\log\mathcal{B}(x)dx}{\int_a^b q(x)\mathcal{A}(x)^s dx - \int_a^b r(x)\mathcal{B}(x)^s dx} - \frac{2s-1}{s(s-1)}\right),\\
&\lim_{s\to 0} M_{s,s}\\
&= M_{0,0} = \exp\left(\frac{\int_a^b q(x)\log^2\mathcal{A}(x)dx - \int_a^b r(x)\log^2\mathcal{B}(x)dx}{2\left[\int_a^b q(x)\log\mathcal{A}(x)dx - \int_a^b r(x)\log\mathcal{B}(x)dx\right]} + 1\right),\\
&\lim_{s\to 1} M_{s,s}\\
&= M_{1,1}\\
&= \exp\left(\frac{\int_a^b q(x)\mathcal{A}(x)\log^2\mathcal{A}(x)dx - \int_a^b r(x)\mathcal{B}(x)\log^2\mathcal{B}(x)dx}{2\left[\int_a^b q(x)\mathcal{A}(x)\log\mathcal{A}(x)dx - \int_a^b r(x)\mathcal{B}(x)\log\mathcal{B}(x)dx\right]} - 1\right),\\
&\lim_{t\to 0} M_{s,t}\\
&= M_{s,0} = \left(\frac{\int_a^b q(x)\mathcal{A}(x)^s dx - \int_a^b r(x)\mathcal{B}(x)^s dx}{\left[\int_a^b q(x)\log\mathcal{A}(x)dx - \int_a^b r(x)\log\mathcal{B}(x)dx\right]s(s-1)}\right)^{(1/s)},\\
&\lim_{t\to 1} M_{s,t}\\
&= M_{s,1}\\
&= \left(\frac{\left[\int_a^b q(x)\mathcal{A}(x)\log\mathcal{A}(x)dx - \int_a^b r(x)\mathcal{B}(x)\log\mathcal{B}(x)dx\right]s(s-1)}{\int_a^b q(x)\mathcal{A}(x)^s dx - \int_a^b r(x)\mathcal{B}(x)^s dx}\right)^{1/(1-s)},
\end{aligned}
\tag{2.53}
$$

where $\mathcal{A}(x) = v_1(x)/v_2(x)$ and $\mathcal{B}(x) = u_1(x)/u_2(x)$.

Remark 2.19. In the case of Riemann-Liouville fractional integral of order $\alpha > 0$, we well use the notation $\overline{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ with $u_i(x)$, $u_i(x)$ with $I_a^\alpha u_i(x)$, and $q(x)$ with $Q_I(x)$.

Remark 2.20. In the case of Caputo fractional derivative of order $\alpha > 0$, we well use the notation $\widetilde{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ with $u_i^{(n)}(x)$, $u_i(x)$ with $D_{*a}^{\alpha}u_i(x)$, and $q(x)$ with $Q_D(x)$.

Remark 2.21. In the case of L_∞ fractional derivative, we will use the notation $\widehat{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ with $D_a^{\beta}u_i(x), u_i(x)$ with $D_a^{\alpha}u_i(x)$, and $q(x)$ with $Q_L(x)$.

3. Exponential Convexity

Lemma 3.1. *Let $s \in \mathbb{R}$, and let $\varphi_s : \mathbb{R}^+ \to \mathbb{R}$ be a function defined as*

$$\varphi_s(x) := \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{3.1}$$

Then φ_s is strictly convex on $\mathbb{R}^+$ for each $s \in \mathbb{R}$.

Proof. Since $\varphi_s''(x) = x^{s-2} > 0$ for all $x \in \mathbb{R}^+$, $s \in \mathbb{R}$, therefore, φ is strictly convex on $\mathbb{R}^+$ for each $s \in \mathbb{R}$. □

Theorem 3.2. *Let $u_i \in U(v, K)$ $(i = 1, 2)$, $u_i(x), v_i(x) > 0$ $(i = 1, 2)$, $r(x) \geq 0$ for all $x \in [a, b]$, let $q(x)$ be given in* (2.2), *and*

$$\bigwedge\nolimits_t = \int_a^b q(x)\varphi_t\left(\frac{v_1(x)}{v_2(x)}\right)dx - \int_a^b r(x)\varphi_t\left(\frac{u_1(x)}{u_2(x)}\right)dx. \tag{3.2}$$

Then the following statements are valid.

(a) *For $n \in N$ and $s_i \in \mathbb{R}$, $i = 1, \dots, n$, the matrix $[\bigwedge_{(s_i+s_j)/2}]_{i,j=1}^n$ is a positive semidefinite matrix. Particularly*

$$\det\left[\bigwedge\nolimits_{(s_i+s_j)/2}\right]_{i,j=1}^k \geq 0 \quad \text{for } k = 1, \dots n. \tag{3.3}$$

(b) *The function $s \mapsto \bigwedge_s$ is exponentially convex on $\mathbb{R}$.*

(c) *The function $s \mapsto \bigwedge_s$ is* log-convex *on $\mathbb{R}$, and the following inequality holds, for $-\infty < r < s < t < \infty$:*

$$\bigwedge\nolimits_s^{t-r} \leq \bigwedge\nolimits_r^{t-s} \bigwedge\nolimits_t^{s-r}. \tag{3.4}$$

Proof. (a) Here we define a new function μ,

$$\mu(x) = \sum_{i,j=1}^{k} a_i a_j \varphi_{s_{ij}}(x), \tag{3.5}$$

for $k = 1, \ldots, n$, $a_i \in \mathbb{R}$, $s_{ij} \in \mathbb{R}$, where $s_{ij} = (s_i + s_j)/2$,

$$\mu''(x) = \sum_{i,j=1}^{n} a_i a_j x^{s_{ij}-2} = \left(\sum_{i=1}^{n} a_i x^{(s_i/2)-1} \right)^2 \geq 0. \tag{3.6}$$

This shows that $\mu(x)$ is convex for $x \geq 0$. Using Theorem 2.1, we have

$$\sum_{i,j=1}^{k} a_i a_j \bigwedge\nolimits_{s_{ij}} \geq 0. \tag{3.7}$$

From the above result, it shows that the matrix $[\bigwedge_{(s_i+s_j)/2}]_{i,j=1}^{n}$ is a positive semidefinite matrix. Specially, we get

$$\det \left[\bigwedge\nolimits_{(s_i+s_j)/2} \right]_{i,j=1}^{k} \geq 0 \quad \forall k = 1, \ldots n. \tag{3.8}$$

(b) Since

$$\begin{aligned} \lim_{s \to 1} \bigwedge\nolimits_s &= \bigwedge\nolimits_1, \\ \lim_{s \to 0} \bigwedge\nolimits_s &= \bigwedge\nolimits_0, \end{aligned} \tag{3.9}$$

it follows that $\bigwedge_s$ is continuous for $s \in \mathbb{R}$. Then, by using Proposition 1.4, we get the exponential convexity of the function $s \mapsto \bigwedge_s$.

(c) Since $\bigwedge_s$ is continuous for $s \in \mathbb{R}$ and using Corollary 1.5, we get that $\bigwedge_s$ is log-convex. Now by Definition 1.2 with $f(t) = \log \bigwedge_t$ and $r, s, t \in \mathbb{R}$ such that $r < s < t$, we get

$$\log \bigwedge\nolimits_s^{t-r} \leq \log \bigwedge\nolimits_r^{t-s} + \log \bigwedge\nolimits_t^{s-r}, \tag{3.10}$$

which is equivalent to (3.4). □

Corollary 3.3. *Let $u_i \in C([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x)$, $I_a^\alpha u_1(x)/I_a^\alpha u_2(x) \in \mathbb{R}^+$, $u_1(x)$, $u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$, let $Q_I(t)$ be given in* (2.11), *and*

$$\overline{\bigwedge}_t = \int_a^b Q_I(x)\varphi_t\left(\frac{u_1(x)}{u_2(x)}\right)dx - \int_a^b r(x)\varphi_t\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right)dx. \tag{3.11}$$

Then the statement of Theorem 3.2 with $\overline{\bigwedge}_t$ instead of $\bigwedge_t$ is valid.

Corollary 3.4. *Let $u_i \in AC^n([a,b])$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t)$, $D_{*a}^\alpha u_1(x)/D_{*a}^\alpha u_2(x) \in \mathbb{R}^+$, $u_1(x)$, $u_2(x)$ have Caputo fractional derivative of order $\alpha > 0$, let $Q_D(t)$ be given in* (2.15), *and*

$$\widetilde{\bigwedge}_t = \int_a^b Q_D(x)\varphi_t\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right)dx - \int_a^b r(x)\varphi_t\left(\frac{D_{*a}^\alpha u_1(x)}{D_{*a}^\alpha u_2(x)}\right)dx. \tag{3.12}$$

Then the statement of Theorem 3.2 with $\widetilde{\bigwedge}_t$ instead of $\bigwedge_t$ is valid.

Corollary 3.5. *Let $\beta > \alpha \geq 0$, $u_i \in L_1\ (a,b)$ $(i = 1,2)$ has L_∞ fractional derivative, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $D_a^{\beta-k}u_i(a) = 0$ for $k = 1,\dots,[\beta]+1$ $(i = 1,2)$, $D_a^\alpha u_1(x)/D_a^\alpha u_2(x)$, $D_a^\beta u_1(x)/D_a^\beta u_2(x) \in \mathbb{R}^+$, let $Q_L(t)$ be given in* (2.22), *and*

$$\widehat{\bigwedge}_t = \int_a^b Q_L(x)\varphi_t\left(\frac{D_a^\beta u_1(x)}{D_a^\beta u_2(x)}\right)dx - \int_a^b r(x)\varphi_t\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right)dx. \tag{3.13}$$

Then the statement of Theorem 3.2 with $\widehat{\bigwedge}_t$ instead of $\bigwedge_t$ is valid.

In the following theorem, we prove the monotonicity property of $M_{s,t}$ defined in (2.52).

Theorem 3.6. *Let the assumption of Theorem 3.2 be satisfied, also let $\bigwedge_t$ be defined in* (3.2), *and $t, s, u, v \in \mathbb{R}$ such that $s \leq v, t \leq u$. Then the following inequality is true:*

$$M_{s,t} \leq M_{v,u}. \tag{3.14}$$

Proof. For a convex function φ, using the Definition 1.2, we get the following inequality:

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \leq \frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1} \tag{3.15}$$

with $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2$, and $y_1 \neq y_2$. Since by Theorem 3.2 we get that $\bigwedge_t$ is log-convex. We set $\varphi(t) = \log \bigwedge_t$, $x_1 = s$, $x_2 = t$, $y_1 = v$, $y_2 = u$, $s \neq t$, and $v \neq u$. Terefore, we get

$$\begin{gathered} \frac{\log \bigwedge_t - \log \bigwedge_s}{t-s} \leq \frac{\log \bigwedge_u - \log \bigwedge_v}{u-v}, \\ \log\left(\frac{\bigwedge_t}{\bigwedge_s}\right)^{1/(t-s)} \leq \log\left(\frac{\bigwedge_u}{\bigwedge_v}\right)^{1/(u-v)}, \end{gathered} \tag{3.16}$$

which is equivalent to (3.14) for $s \neq t$, $v \neq u$.

For $s = t$, $v = u$, we get the required result by taking limit in (3.16). □

Corollary 3.7. *Let $u_i \in C([a,b])$ $(i = 1,2)$, and let the assumption of Corollary 3.3 be satisfied, also let $\overline{\bigwedge}_t$ be defined by* (3.11)*. For $t,s,u,v \in \mathbb{R}$ such that $s \leq v$, $t \leq u$, then the following inequality holds:*

$$\overline{M}_{s,t} \leq \overline{M}_{v,u}. \tag{3.17}$$

Corollary 3.8. *Let $u_i \in AC^n([a,b])$ $(i = 1,2)$ and let the assumption of Corollary 3.4 be satisfied, also let $\widetilde{\bigwedge}_t$ be defined by* (3.12)*. For $t,s,u,v \in \mathbb{R}$ such that $s \leq v$, $t \leq u$, then the following inequality holds:*

$$\widetilde{M}_{s,t} \leq \widetilde{M}_{v,u}. \tag{3.18}$$

Corollary 3.9. *Let $\beta > \alpha \geq 0$, $u_i \in L_1(a,b)$ $(i = 1,2)$ and the assumption of Corollary 3.5 be satisfied, also let $\widehat{\bigwedge}_t$ be defined by* (3.13)*. For $t,s,u,v \in \mathbb{R}$ such that $s \leq v$, $t \leq u$. Then following inequality holds*

$$\widehat{M}_{s,t} \leq \widehat{M}_{v,u}. \tag{3.19}$$

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