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Research Article

Optimal Power Mean Bounds for the Weighted Geometric Mean of Classical Means

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For $p \in \mathbb{R}$, the power mean of order p of two positive numbers a and b is defined by $M_p(a,b) = ((a^p + b^p)/2)^{1/p}$, for $p \neq 0$, and $M_p(a,b) = \sqrt{ab}$, for $p = 0$. In this paper, we answer the question: what are the greatest value p and the least value q such that the double inequality $M_p(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_q(a,b)$ holds for all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$? Here $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ denote the classical arithmetic, geometric, and harmonic means, respectively.

1. Introduction

For $p \in \mathbb{R}$, the power mean of order p of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ can be found in literatures [1–12]. It is well known that $M_p(a,b)$ is continuous and increasing with respect to $p \in \mathbb{R}$ for fixed a and b.

Let $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the classical arithmetic, geometric, and harmonic means of two positive numbers a and b, respectively. Then

$$\begin{aligned} \min\{a,b\} &\leq H(a,b) = M_{-1}(a,b) \leq G(a,b) = M_0(a,b) \\ &\leq A(a,b) = M_1(a,b) \leq \max\{a,b\}. \end{aligned} \tag{1.2}$$

In [13], Alzer and Janous established the following sharp double inequality (see also [14, page 350]):

$$M_{\log 2/\log 3}(a,b) \leq \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leq M_{2/3}(a,b) \tag{1.3}$$

for all $a,b > 0$.

In [15], Mao proved

$$M_{1/3}(a,b) \leq \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \leq M_{1/2}(a,b) \tag{1.4}$$

for all $a,b > 0$, and $M_{1/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)A(a,b) + (2/3)G(a,b)$.

The following sharp bounds for $(2/3)G + (1/3)H$ and $(1/3)G + (2/3)H$ in terms of power mean are proved in [16]:

$$\begin{aligned} M_{-1/3}(a,b) &\leq \frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) \leq M_0(a,b), \\ M_{-2/3}(a,b) &\leq \frac{1}{3}G(a,b) + \frac{2}{3}H(a,b) \leq M_0(a,b) \end{aligned} \tag{1.5}$$

for all $a,b > 0$.

The purpose of this paper is to answer the question: what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_q(a,b) \tag{1.6}$$

holds for all $a,b > 0$ and $\alpha,\beta > 0$ with $\alpha + \beta < 1$?

2. Main Result

In order to establish our main results we need the following lemma.

Lemma 2.1. *If* $\lambda \in (-1,0) \bigcup (0,1)$, $t \geq 1$ *and* $f(t) = (1/\lambda)\log((t^{\lambda}+1)/2) - \lambda\log((t+1)/2) - ((1-\lambda)/2)\log t$, *then*

(1) $f(t) > 0$ *for* $\lambda \in (0,1)$ *and* $t > 1$;

(2) $f(t) < 0$ *for* $\lambda \in (-1,0)$ *and* $t > 1$.

Proof. Simple computations lead to

$$f(1) = 0, \tag{2.1}$$

$$f'(t) = \frac{g(t)}{t(t+1)(t^{\lambda}+1)}, \tag{2.2}$$

where $g(t) = ((1-\lambda)/2)t^{\lambda+1} + ((1+\lambda)/2)t^{\lambda} - ((1+\lambda)/2)t - ((1-\lambda)/2)$:

$$g(1) = 0, \tag{2.3}$$

$$g'(t) = \frac{(1-\lambda)(1+\lambda)}{2}t^{\lambda} + \frac{\lambda(1+\lambda)}{2}t^{\lambda-1} - \frac{1+\lambda}{2}, \tag{2.4}$$

$$g'(1) = 0, \tag{2.5}$$

$$g''(t) = \frac{\lambda(1-\lambda)(1+\lambda)}{2}(t-1)t^{\lambda-2}. \tag{2.6}$$

(1) If $\lambda \in (0,1)$ and $t > 1$, then (2.6) implies

$$g''(t) > 0. \tag{2.7}$$

Therefore, Lemma 2.1(1) follows from (2.1)–(2.3) and (2.5) together with (2.7).

(2) If $\lambda \in (-1,0)$ and $t > 1$, then (2.6) yields

$$g''(t) < 0. \tag{2.8}$$

Therefore, Lemma 2.1(2) follows from (2.1)–(2.3) and (2.5) together with (2.8). □

Theorem 2.2. *For all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$, one has*

(1) $M_{2\alpha+\beta-1}(a,b) = M_0(a,b) = A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ *for* $2\alpha + \beta = 1$;

(2) $M_{2\alpha+\beta-1}(a,b) \geq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \geq M_0(a,b)$ *for* $2\alpha + \beta > 1$, *and* $M_{2\alpha+\beta-1}(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_0(a,b)$ *for* $2\alpha + \beta < 1$, *each equality occurs if and only if* $a = b$, *and* $M_0(a,b)$ *and* $M_{2\alpha+\beta-1}(a,b)$ *are the best possible power mean bounds for the product* $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Proof. (1) If $2\alpha + \beta = 1$, then simple computations lead to

$$\begin{aligned} A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) &= \left(\frac{a+b}{2}\right)^{2\alpha+\beta-1}(ab)^{1-(\alpha+(\beta/2))} \\ &= \sqrt{ab} = M_0(a,b) = M_{2\alpha+\beta-1}(a,b). \end{aligned} \tag{2.9}$$

(2) If $2\alpha+\beta\neq 1$ and $a=b$, then we clearly see that

$$A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)=M_{2\alpha+\beta-1}(a,b)=M_0(a,b)=a. \tag{2.10}$$

If $2\alpha+\beta\neq 1$ and $a\neq b$, without loss of generality, we assume that $a>b$. Let $t=(a/b)>1$ and $\lambda=2\alpha+\beta-1$, then $\lambda\in(-1,0)\bigcup(0,1)$, and simple computations lead to

$$\begin{aligned}
&\log M_{2\alpha+\beta-1}(a,b)-\log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right]\\
&=\frac{1}{2\alpha+\beta-1}\log\frac{t^{2\alpha+\beta-1}+1}{2}-(2\alpha+\beta-1)\log\frac{1+t}{2}-\left(1-\alpha-\frac{\beta}{2}\right)\log t\\
&=\frac{1}{\lambda}\log\frac{t^{\lambda}+1}{2}-\lambda\log\frac{t+1}{2}-\frac{1-\lambda}{2}\log t,
\end{aligned} \tag{2.11}$$

$$\frac{A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)}{M_0(a,b)}=\left(\frac{\sqrt{t}+\left(1/\sqrt{t}\right)}{2}\right)^{2\alpha+\beta-1}. \tag{2.12}$$

Therefore, $M_{2\alpha+\beta-1}(a,b)>A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)>M_0(a,b)$ for $2\alpha+\beta>1$ follows from (2.11) and Lemma 2.1(1) together with (2.12), and $M_{2\alpha+\beta-1}(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)<M_0(a,b)$ for $2\alpha+\beta<1$ follows from (2.11) and Lemma 2.1(2) together with (2.12). □

Next, we prove that $M_0(a,b)$ and $M_{2\alpha+\beta-1}(a,b)$ are the best possible power mean bounds for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Firstly, we prove that $M_{2\alpha+\beta-1}(a,b)$ is the best possible upper power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta>1$.

For any $\epsilon\in(0,2\alpha+\beta-1)$ and $x>0$, one has

$$\begin{aligned}
&\left[M_{2\alpha+\beta-1-\epsilon}(1,1+x)\right]^{2\alpha+\beta-1-\epsilon}-\left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{2\alpha+\beta-1-\epsilon}\\
&=\frac{(1+x)^{2\alpha+\beta-1-\epsilon}+1}{2}-\left(1+\frac{x}{2}\right)^{(2\alpha+\beta-1)(2\alpha+\beta-1-\epsilon)}(1+x)^{(1-\alpha-(\beta/2))(2\alpha+\beta-1-\epsilon)}.
\end{aligned} \tag{2.13}$$

Let $x\to 0$, then the Taylor expansion leads to

$$\begin{aligned}
&\frac{(1+x)^{2\alpha+\beta-1-\epsilon}+1}{2}-\left(1+\frac{x}{2}\right)^{(2\alpha+\beta-1)(2\alpha+\beta-1-\epsilon)}(1+x)^{(1-\alpha-(\beta/2))(2\alpha+\beta-1-\epsilon)}\\
&\quad=-\frac{1}{8}\epsilon(2\alpha+\beta-1-\epsilon)x^2+o\left(x^2\right).
\end{aligned} \tag{2.14}$$

Equations (2.13) and (2.14) imply that if $2\alpha+\beta>1$, then for any $\epsilon\in(0,2\alpha+\beta-1)$ there exists $\delta_1=\delta_1(\epsilon,\alpha,\beta)>0$, such that $M_{2\alpha+\beta-1-\epsilon}(1,1+x)<A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x\in(0,\delta_1)$.

Secondly, we prove that $M_0(a,b)$ is the best possible lower power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta>1$.

For any $\epsilon>0$ and $t>1$, one has

$$\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{\epsilon}(t,1)}=\frac{\left((1+t^{-1})/2\right)^{2\alpha+\beta-1}}{\left((1+t^{-\epsilon})/2\right)^{1/\epsilon}}t^{\alpha+(\beta/2)-1}. \tag{2.15}$$

From (2.15) and $\alpha+(\beta/2)<1$, we clearly see that

$$\lim_{t\to+\infty}\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{\epsilon}(t,1)}=0. \tag{2.16}$$

Equation (2.16) implies that if $2\alpha+\beta>1$, then for any $\epsilon\in(0,2\alpha+\beta-1)$ there exists $T_1=T_1(\epsilon,\alpha,\beta)>1$, such that $A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)<M_{\epsilon}(t,1)$ for $t\in(T_1,+\infty)$.

Thirdly, we prove that $M_{2\alpha+\beta-1}(a,b)$ is the best possible lower power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta<1$.

For any $\epsilon\in(0,1-2\alpha-\beta)$ and $x>0$, one has

$$\begin{aligned}&\left[M_{2\alpha+\beta-1+\epsilon}(1,1+x)\right]^{1-2\alpha-\beta-\epsilon}-\left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{1-2\alpha-\beta-\epsilon}\\&\quad=\frac{g(x)}{\left[1+(1+x)^{1-2\alpha-\beta-\epsilon}\right](1+(x/2))^{(1-2\alpha-\beta)(1-2\alpha-\beta-\epsilon)}},\end{aligned} \tag{2.17}$$

where $g(x)=2(1+x)^{1-2\alpha-\beta-\epsilon}(1+(x/2))^{(1-2\alpha-\beta)(1-2\alpha-\beta-\epsilon)}-(1+x)^{(1-\alpha-(\beta/2))(1-2\alpha-\beta-\epsilon)}[1+(1+x)^{1-2\alpha-\beta-\epsilon}]$.

Let $x\to 0$, then the Taylor expansion leads to

$$g(x)=\frac{1}{4}\epsilon(1-2\alpha-\beta-\epsilon)x^2+o\left(x^2\right). \tag{2.18}$$

Equations (2.17) and (2.18) imply that if $2\alpha+\beta<1$, then for any $\epsilon\in(0,1-2\alpha-\beta)$ there exists $0<\delta_2=\delta_2(\epsilon,\alpha,\beta)<1$, such that $M_{2\alpha+\beta-1+\epsilon}(1,1+x)>A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x\in(0,\delta_2)$.

Finally, we prove that $M_0(a,b)$ is the best possible upper power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta<1$.

For any $\epsilon>0$ and $t>1$, one has

$$\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{-\epsilon}(t,1)}=\frac{\left((1+t^{-1})/2\right)^{2\alpha+\beta-1}}{\left((1+t^{-\epsilon})/2\right)^{-1/\epsilon}}t^{\alpha+(\beta/2)}. \tag{2.19}$$

From (2.19) and $\alpha+(\beta/2)>0$ we clearly see that

$$\lim_{t\to+\infty}\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{-\epsilon}(t,1)}=+\infty. \tag{2.20}$$

Equation (2.20) implies that if $2\alpha + \beta < 1$, then for any $\epsilon > 0$ there exists $T_2 = T_2(\epsilon, \alpha, \beta) > 1$, such that $A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1) > M_{-\epsilon}(t,1)$ for $t \in (T_2, +\infty)$.

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