Research Article

Proof of One Optimal Inequality for Generalized Logarithmic, Arithmetic, and Geometric Means

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Two open problems were posed in the work of Long and Chu (2010). In this paper, we give the solutions of these problems.

1. Introduction

The arithmetic $A(a, b)$ and geometric $G(a, b)$ means of two positive numbers a and b are defined by $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$, respectively. If p is a real number, then the generalized logarithmic mean $L_p(a, b)$ with parameter p of two positive numbers a, b is defined by

$$L_p(a, b) = \begin{cases} a, & a = b, \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p}, & p \neq 0, p \neq -1, a \neq b, \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, & p = 0, a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p = -1, a \neq b. \end{cases} \quad (1.1)$$

In the paper [1], Long and Chu propose the two following open problems:

Open Problem 1. What is the least value p such that the inequality

$$\alpha A(a, b) + (1 - \alpha)G(a, b) < L_p(a, b) \quad (1.2)$$

holds for $\alpha \in (0, 1/2)$ and all $a, b > 0$ with $a \neq b$?

Open Problem 2. What is the greatest value q such that the inequality

$$\alpha A(a, b) + (1 - \alpha)G(a, b) > L_q(a, b) \quad (1.3)$$

holds for $\alpha \in (1/2, 1)$ and all $a, b > 0$ with $a \neq b$?

For information on the history, background, properties, and applications of inequalities for generalized logarithmic, arithmetic, and geometric means, please refer to [1–19] and related references there in.

The aim of this article is to prove the following Theorem 2.1.

2. Main Result

Theorem 2.1. Let $\alpha \in (0, 1/2) \cup (1/2, 1)$, $a \neq b$, $a > 0$, $b > 0$. Let $p(\alpha)$ be a solution of

$$\frac{1}{p} \ln(1 + p) + \ln\left(\frac{\alpha}{2}\right) = 0 \quad \text{in } (-1, 1). \quad (2.1)$$

Then,

$$\text{if } \alpha \in \left(0, \frac{1}{2}\right), \text{ then } \alpha A(a, b) + (1 - \alpha)G(a, b) < L_p(a, b) \text{ for } p \geq p(\alpha) \quad (2.2)$$

and $p(\alpha)$ is the best constant,

$$\text{if } \alpha \in \left(\frac{1}{2}, 1\right), \text{ then } \alpha A(a, b) + (1 - \alpha)G(a, b) > L_p(a, b) \text{ for } p \leq p(\alpha) \quad (2.3)$$

and $p(\alpha)$ is the best constant.

3. Proof of Theorem 2.1

Because $L_p(a, b)$ is increasing with respect to $p \in \mathbb{R}$ for fixed a and b , it suffices to prove that for any $\alpha \in (0, 1/2)$ (resp., $\alpha \in (1/2, 1)$) there exists $p(\alpha)$ such that $\alpha A(a, b) + (1 - \alpha)G(a, b) < L_{p(\alpha)}(a, b)$ (resp., $\alpha A(a, b) + (1 - \alpha)G(a, b) > L_{p(\alpha)}(a, b)$), and $p(\alpha)$ is the best constant.

Without loss of generality, we assume that $a > b > 0$. Let $p \neq 0$, $p \neq -1$. Equations (2.2), (2.3) are equivalent to

$$\alpha \left(\frac{a+b}{2} \right) + (1-\alpha) \sqrt{ab} \leq \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}. \quad (3.1)$$

On putting $t = \sqrt{b/a}$, we obtain (3.1) is equivalent to

$$\frac{1}{p} \ln \left(\frac{1-t^{2p+2}}{(p+1)(1-t^2)} \right) - \ln \left(\frac{\alpha}{2} (1+t^2) + (1-\alpha)t \right) \geq 0, \quad t \in (0,1). \quad (3.2)$$

Introduce the function $H : (0,1) \times (0,1) \times (-1,1) \rightarrow \mathbb{R}$ by

$$\begin{aligned} H(t, \alpha, p) &= \frac{1}{p} \ln \left(\frac{1-t^{2p+2}}{(p+1)(1-t^2)} \right) - \ln \left(\frac{\alpha}{2} (1+t^2) + (1-\alpha)t \right), \quad p \neq 0, \\ H(t, \alpha, 0) &= \lim_{p \rightarrow 0} H(t, \alpha, p). \end{aligned} \quad (3.3)$$

Simple computations yield for $p \neq 0$

$$\begin{aligned} \frac{\partial H(t, \alpha, p)}{\partial t} &= \frac{2}{p} \left(\frac{pt^{2p+3} - (p+1)t^{2p+1} + t}{(1-t^2)(1-t^{2p+2})} \right) - 2 \left(\frac{\alpha t + 1 - \alpha}{\alpha(1+t^2) + 2(1-\alpha)t} \right), \\ \frac{\partial H(t, \alpha, 0)}{\partial t} &= \lim_{p \rightarrow 0} \frac{\partial H(t, \alpha, p)}{\partial t}. \end{aligned} \quad (3.4)$$

Let $\alpha \in (0, 1/2) \cup (1/2, 1)$ and $p(\alpha)$ the unique solution to

$$\frac{1}{p} \ln(1+p) + \ln \left(\frac{\alpha}{2} \right) = 0. \quad (3.5)$$

To see that $p(\alpha)$ is optimal in both cases (2.2), (2.3), note that $\lim_{t \rightarrow 0^+} H(t, \alpha, p(\alpha)) = 0$. Thus, if the constant is decreased (resp., increased), then the desired bound for H would not hold for small t . This follows from the fact that for a fixed α , the function

$$H(0^+, \alpha, p) = - \left(\frac{1}{p} \right) \ln(p+1) - \ln \left(\frac{\alpha}{2} \right) \quad (3.6)$$

is nondecreasing.

From now on, let $p = p(\alpha)$ for $\alpha \in (0, 1/2) \cup (1/2, 1)$. To show the estimates for this p , we start from observing that $H(0+, \alpha, p) = H(1-, \alpha, p) = 0$. Furthermore, one easily checks that

$$\begin{aligned} H'_t(0+, \alpha, p) &= \infty \quad \text{for } \alpha < \frac{1}{2}, \\ H'_t(0+, \alpha, p) &= \frac{2(\alpha - 1)}{\alpha} < 0 \quad \text{for } \alpha > \frac{1}{2}. \end{aligned} \quad (3.7)$$

Thus, it suffices to verify that $H'_t(\cdot, \alpha, p)$ has exactly one zero inside the interval $(0, 1)$. It follows from the mean value theorem. After some computations, this is equivalent to saying that the function R given by

$$\begin{aligned} R(t, \alpha, p) &= \ln \left(\frac{\alpha(p+1)t^3 + (1-\alpha)(p+2)t^2 + \alpha(1-p)t - p(1-\alpha)}{-p(1-\alpha)t^3 + \alpha(1-p)t^2 + (1-\alpha)(p+2)t + \alpha(p+1)} \right) \\ &\quad - (2p+1) \ln t = \ln \frac{s_1(t)}{s_2(t)} - (2p+1) \ln t \end{aligned} \quad (3.8)$$

has exactly one root in $(0, 1)$. Here, the expression under the logarithm may be nonpositive, so we define R on a maximal interval, contained in $(0, 1)$. It is easy to see that this interval must be of the form $(t_0, 1)$, for some $t_0 \in (0, 1)$. This follows from the fact that s_2 is strictly positive on $(0, 1)$ and s_1 is strictly increasing on this interval.

Since $R(1-) = 0$ and $R(t_0+) = \pm\infty$, we will be done if we show that R' has exactly one root in $(0, 1)$. After some computations, we obtain that the equation $R'(t) = 0$ is equivalent to

$$g(t) = \alpha(1-\alpha)(2p+1)(1+t^2) + 2\left(p(2\alpha^2 - 2\alpha + 1) + \alpha^2 - 4\alpha + 2\right)t = 0. \quad (3.9)$$

Because g is a quadratic polynomial in the variable t , all that remains is to show that

$$g(0)g(1) = \alpha(1-\alpha)(2p+1)(p-3\alpha+2) < 0 \quad (3.10)$$

or, in virtue of the definition of $p = p(\alpha)$,

$$(2p+1) \left(p+2 - \frac{6}{(p+1)^{1/p}} \right) < 0. \quad (3.11)$$

This can be easily established by some elementary calculations. It completes the proof.

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