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Research Article

Approximation of Analytic Functions by Kummer Functions

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We solve the inhomogeneous Kummer differential equation of the form $xy'' + (\beta - x)y' - \alpha y = \sum_{m=0}^{\infty} a_m x^m$ and apply this result to the proof of a local Hyers-Ulam stability of the Kummer differential equation in a special class of analytic functions.

1. Introduction

Assume that X and Y are a topological vector space and a normed space, respectively, and that I is an open subset of X. If for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon \tag{1.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0 \tag{1.2}$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ depends on ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space X). We may apply this terminology for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1–6].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [7, 8]). Here, we will introduce a result of Alsina and Ger (see [9]). If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality

$|y'(x) - y(x)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \le 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al.. They proved in [10] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [11]).

Using the conventional power series method, the author [12] investigated the general solution of the inhomogeneous Legendre differential equation of the form

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + p(p+1)y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{1.3}$$

under some specific conditions, where p is a real number and the convergence radius of the power series is positive. Moreover, he applied this result to prove that every analytic function can be approximated in a neighborhood of 0 by the Legendre function with an error bound expressed by $C(x^2/(1-x^2))$ (see [13–16]).

In Section 2 of this paper, employing power series method, we will determine the general solution of the inhomogeneous Kummer (differential) equation

$$xy''(x) + (\beta - x)y'(x) - \alpha y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.4}$$

where α and β are constants and the coefficients a_m of the power series are given such that the radius of convergence is $\rho > 0$, whose value is in general permitted to be infinite. Moreover, using the idea from [12, 13, 15], we will prove the Hyers-Ulam stability of the Kummer's equation in a class of special analytic functions (see the class $\mathcal{C}_K$ in Section 3).

In this paper, $\mathbb{N}_0$ and $\mathbb{Z}$ denote the set of all nonnegative integers and the set of all integers, respectively. For each real number α, we use the notation $\lceil \alpha \rceil$ to denote the ceiling of α, that is, the least integer not less than α.

2. General Solution of (1.4)

The Kummer (differential) equation

$$xy''(x) + (\beta - x)y'(x) - \alpha y(x) = 0, \tag{2.1}$$

which is also called the confluent hypergeometric differential equation, appears frequently in practical problems and applications. The Kummer's equation (2.1) has a regular singularity at $x = 0$ and an irregular singularity at ∞. A power series solution of (2.1) is given by

$$M(\alpha, \beta, x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!(\beta)_m} x^m, \tag{2.2}$$

where $(\alpha)_m$ is the factorial function defined by $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)$ for all $m \in \mathbb{N}$. The above power series solution is called the Kummer function or the confluent hypergeometric function. We know that if neither α nor β is a nonpositive integer, then the power series for $M(\alpha,\beta,x)$ converges for all values of x.

Let us define

$$U(\alpha,\beta,x) = \frac{\pi}{\sin\beta\pi}\left[\frac{M(\alpha,\beta,x)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - x^{1-\beta}\frac{M(1+\alpha-\beta,2-\beta,x)}{\Gamma(\alpha)\Gamma(2-\beta)}\right]. \tag{2.3}$$

We know that if $\beta\neq 1$ then $M(\alpha,\beta,x)$ and $U(\alpha,\beta,x)$ are independent solutions of the Kummer's equation (2.1). When $\beta > 1$, $U(\alpha,\beta,x)$ is not defined at $x = 0$ because of the factor $x^{1-\beta}$ in the above definition of $U(\alpha,\beta,x)$.

By considering this fact, we define

$$I_\rho = \begin{cases} (-\rho,\rho), & (\text{for } \beta < 1),\\ (-\rho,0)\cup(0,\rho), & (\text{for } \beta > 1), \end{cases} \tag{2.4}$$

for any $0 < \rho \leq \infty$. It should be remarked that if $\beta\notin\mathbb{Z}$ and both α and $1+\alpha-\beta$ are not nonpositive integers, then $M(\alpha,\beta,x)$ and $U(\alpha,\beta,x)$ converge for all $x\in I_\infty$ (see [17, Section 13.1.3]).

Theorem 2.1. *Let α and β be real constants such that $\beta\notin\mathbb{Z}$ and neither α nor $1+\alpha-\beta$ is a nonpositive integer. Assume that the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$ and that there exists a real number $\mu\geq 0$ with*

$$\left|\frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| \leq \mu\left|\sum_{i=0}^{m-1}\frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right| \tag{2.5}$$

for all sufficiently large integers m. Let us define $\rho_0 = \min\{\rho, 1/\mu\}$ and $1/0 = \infty$. Then, every solution $y : I_{\rho_0} \to \mathbb{C}$ of the inhomogeneous Kummer's equation (1.4) *can be represented by*

$$y(x) = y_h(x) + \sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\frac{i!(\alpha)_m(\beta)_i a_i}{m!(\alpha)_{i+1}(\beta)_m}x^m, \tag{2.6}$$

where $y_h(x)$ is a solution of the Kummer's equation (2.1).

Proof. Assume that a function $y : I_{\rho_0} \to \mathbb{C}$ is given by (2.6). We first prove that the function $y_p(x)$, defined by $y(x) - y_h(x)$, satisfies the inhomogeneous Kummer's equation (1.4). Since

$$\begin{aligned} y_p'(x) &= \sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\frac{i!(\alpha)_m(\beta)_i a_i}{(m-1)!(\alpha)_{i+1}(\beta)_m}x^{m-1} = \sum_{m=0}^{\infty}\sum_{i=0}^{m}\frac{i!(\alpha)_{m+1}(\beta)_i a_i}{m!(\alpha)_{i+1}(\beta)_{m+1}}x^m,\\ y_p''(x) &= \sum_{m=1}^{\infty}\sum_{i=0}^{m}\frac{i!(\alpha)_{m+1}(\beta)_i a_i}{(m-1)!(\alpha)_{i+1}(\beta)_{m+1}}x^{m-1}, \end{aligned} \tag{2.7}$$

we have

$$
\begin{aligned}
xy_p''(x) + (\beta - x)y_p'(x) - \alpha y_p(x) &= a_0 + \sum_{m=1}^{\infty}\sum_{i=0}^{m}\frac{i!(\alpha)_{m+1}(\beta)_i(m+\beta)a_i}{m!(\alpha)_{i+1}(\beta)_{m+1}}x^m \\
&\quad - \sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\frac{i!(\alpha)_m(\beta)_i(m+\alpha)a_i}{m!(\alpha)_{i+1}(\beta)_m}x^m \\
&= a_0 + \sum_{m=1}^{\infty}a_m x^m,
\end{aligned}
\tag{2.8}
$$

which proves that $y_p(x)$ is a particular solution of the inhomogeneous Kummer's equation (1.4).

We now apply the ratio test to the power series expression of $y_p(x)$ as follows:

$$
y_p(x) = \sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\frac{i!(\alpha)_m(\beta)_i a_i}{m!(\alpha)_{i+1}(\beta)_m}x^m = \sum_{m=1}^{\infty}c_m x^m. \tag{2.9}
$$

Then, it follows from (2.5) that

$$
\begin{aligned}
\lim_{m\to\infty}\left|\frac{c_{m+1}}{c_m}\right| &\le \lim_{m\to\infty}\left|\frac{\alpha+m}{\beta+m}\right|\left[\frac{1}{m+1}+\frac{m}{m+1}\left|\frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right|\left|\sum_{i=0}^{m-1}\frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right|^{-1}\right] \\
&\le \mu.
\end{aligned}
\tag{2.10}
$$

Therefore, the power series expression of $y_p(x)$ converges for all $x \in I_{1/\mu}$. Moreover, the convergence region of the power series for $y_p(x)$ is the same as those of power series for $y_p'(x)$ and $y_p''(x)$. In this paper, the convergence region will denote the maximum open set where the relevant power series converges. Hence, the power series expression for $xy_p''(x) + (\beta - x)y_p'(x) - \alpha y_p(x)$ has the same convergence region as that of $y_p(x)$. This implies that $y_p(x)$ is well defined on I_{ρ_0} and so does for $y(x)$ in (2.6) because $y_h(x)$ converges for all $x \in I_\infty$ under our hypotheses for α and β (see above Theorem 2.1).

Since every solution to (1.4) can be expressed as a sum of a solution $y_h(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, every solution of (1.4) is certainly in the form of (2.6). □

Remark 2.2. We fix $\alpha = 1$ and $\beta = 10/3$, and we define

$$
a_0 = \frac{10}{3}, \qquad a_m = 1 + \frac{4m^2 - 6m - 3}{3m^2(m+1)} \tag{2.11}
$$

for every $m \in \mathbb{N}$. Then, since $\lim_{m\to\infty} a_m/a_{m-1} = 1$, there exists a real number $\mu > 1$ such that

$$\begin{aligned}
\left|\frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| &= \frac{10\cdot 13\cdot 16\cdots(3m+4)}{m3^{m-1}} a_{m-1}\cdot\frac{3m+7}{3m}\cdot\frac{a_m}{a_{m-1}}\cdot\frac{m}{m+1} \\
&= \frac{(m-1)!(\beta)_{m-1}a_{m-1}}{(\alpha)_m}\cdot\frac{3m+7}{3m}\cdot\frac{a_m}{a_{m-1}}\cdot\frac{m}{m+1} \\
&\le \mu\frac{(m-1)!(\beta)_{m-1}a_{m-1}}{(\alpha)_m} \\
&\le \mu\left|\sum_{i=0}^{m-1}\frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right|
\end{aligned} \tag{2.12}$$

for all sufficiently large integers m. Hence, the sequence $\{a_m\}$ satisfies condition (2.5) for all sufficiently large integers m.

3. Hyers-Ulam Stability of (2.1)

In this section, let α and β be real constants and assume that ρ is a constant with $0 < \rho \le \infty$. For a given $K \ge 0$, let us denote $\mathcal{C}_K$ the set of all functions $y : I_\rho \to \mathbb{C}$ with the properties (a) and (b):

(a) $y(x)$ is represented by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(b) it holds true that $\sum_{m=0}^{\infty} |a_m x^m| \le K|\sum_{m=0}^{\infty} a_m x^m|$ for all $x \in I_\rho$, where $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$ for each $m \in \mathbb{N}_0$.

It should be remarked that the power series $\sum_{m=0}^{\infty} a_m x^m$ in (b) has the same radius of convergence as that of $\sum_{m=0}^{\infty} b_m x^m$ given in (a).

In the following theorem, we will prove a local Hyers-Ulam stability of the Kummer's equation under some additional conditions. More precisely, if an analytic function satisfies some conditions given in the following theorem, then it can be approximated by a "combination" of Kummer functions such as $M(\alpha,\beta,x)$ and $M(1+\alpha-\beta, 2-\beta, x)$ (see the first part of Section 2).

Theorem 3.1. *Let α and β be real constants such that $\beta \notin \mathbb{Z}$ and neither α nor $1+\alpha-\beta$ is a nonpositive integer. Suppose a function $y : I_\rho \to \mathbb{C}$ is representable by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least $\rho > 0$. Assume that there exist nonnegative constants $\mu \ne 0$ and ν satisfying the condition*

$$\left|\frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| \le \mu\left|\sum_{i=0}^{m-1}\frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right| \le \nu\left|\frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| \tag{3.1}$$

for all $m \in \mathbb{N}_0$, *where* $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$. *Indeed, it is sufficient for the first inequality in* (3.1) *to hold true for all sufficiently large integers* m. *Let us define* $\rho_0 = \min\{\rho, 1/\mu\}$. *If* $y \in \mathcal{C}_K$ *and it satisfies the differential inequality*

$$\left|xy''(x) + (\beta - x)y'(x) - \alpha y(x)\right| \le \varepsilon \tag{3.2}$$

for all $x \in I_{\rho_0}$ *and for some* $\varepsilon \ge 0$, *then there exists a solution* $y_h : I_\infty \to \mathbb{C}$ *of the Kummer's equation* (2.1) *such that*

$$\left|y(x) - y_h(x)\right| \le \begin{cases} \dfrac{\nu}{\mu} \cdot \dfrac{2\alpha - 1}{\alpha} K\varepsilon & (\text{for } \alpha > 1), \\ \dfrac{\nu}{\mu}\left[\displaystyle\sum_{m=0}^{m_0-1}\left|\left|\frac{m+1}{m+\alpha}\right| - \left|\frac{m+2}{m+1+\alpha}\right|\right| + \frac{m_0+1}{m_0+\alpha}\right]K\varepsilon & (\text{for } \alpha \le 1), \end{cases} \tag{3.3}$$

for any $x \in I_{\rho_0}$, *where* $m_0 = \max\{0, \lceil -\alpha \rceil\}$.

Proof. By the definition of a_m, we have

$$\begin{aligned} &xy''(x) + (\beta - x)y'(x) - \alpha y(x) \\ &\quad = \sum_{m=0}^{\infty}\left[(m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m\right]x^m \\ &\quad = \sum_{m=0}^{\infty} a_m x^m \end{aligned} \tag{3.4}$$

for all $x \in I_\rho$. So by (3.2) we have

$$\left|\sum_{m=0}^{\infty} a_m x^m\right| \le \varepsilon \tag{3.5}$$

for any $x \in I_{\rho_0}$. Since $y \in \mathcal{C}_K$, this inequality together with (b) yields

$$\sum_{m=0}^{\infty}|a_m x^m| \le K\left|\sum_{m=0}^{\infty} a_m x^m\right| \le K\varepsilon \tag{3.6}$$

for each $x \in I_{\rho_0}$.

By Abel's formula (see [18, Theorem 6.30]), we have

$$\begin{aligned} &\sum_{m=0}^{n}|a_m x^m|\left|\frac{m+1}{m+\alpha}\right| \\ &\quad = \left(\sum_{i=0}^{n}\left|a_i x^i\right|\right)\left|\frac{n+2}{n+1+\alpha}\right| + \sum_{m=0}^{n}\left(\sum_{i=0}^{m}\left|a_i x^i\right|\right)\left(\left|\frac{m+1}{m+\alpha}\right| - \left|\frac{m+2}{m+1+\alpha}\right|\right) \end{aligned} \tag{3.7}$$

for any $x \in I_{\rho_0}$ and $n \in \mathbb{N}$. With $m_0 = \max\{0, \lceil -\alpha \rceil\}$ ($\lceil -\alpha \rceil$ is the ceiling of $-\alpha$), we know that

$$\begin{aligned} &\text{if } \alpha > 1, \text{ then } \frac{m+1}{m+\alpha} < \frac{m+2}{m+1+\alpha} \quad \text{for } m \geq 0; \\ &\text{if } \alpha \leq 1, \text{ then } \frac{m+1}{m+\alpha} \geq \frac{m+2}{m+1+\alpha} \quad \text{for } m \geq m_0. \end{aligned} \tag{3.8}$$

Due to (3.4), it follows from Theorem 2.1 and (2.6) that there exists a solution $y_h(x)$ of the Kummer's equation (2.1) such that

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} \sum_{i=0}^{m-1} \frac{i!(\alpha)_m (\beta)_i a_i}{m!(\alpha)_{i+1} (\beta)_m} x^m \tag{3.9}$$

for all $x \in I_{\rho_0}$. By using (3.1), (3.6), (3.7), and (3.8), we can estimate

$$\begin{aligned} |y(x) - y_h(x)| &\leq \sum_{m=0}^{\infty} \left| a_m x^m \frac{m+1}{m+\alpha} \right| \left| \frac{(\alpha)_{m+1}}{(m+1)!(\beta)_m a_m} \right| \left| \sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}} \right| \\ &\leq \frac{\nu}{\mu} \lim_{n\to\infty} \sum_{m=0}^{n} |a_m x^m| \left| \frac{m+1}{m+\alpha} \right| \\ &\leq \begin{cases} \dfrac{\nu}{\mu} \lim\limits_{n\to\infty} \left[K\varepsilon \left| \dfrac{n+2}{n+1+\alpha} \right| + \displaystyle\sum_{m=0}^{n} K\varepsilon \left(\dfrac{m+2}{m+1+\alpha} - \dfrac{m+1}{m+\alpha} \right) \right] & (\text{for } \alpha > 1), \\ \dfrac{\nu}{\mu} \lim\limits_{n\to\infty} \left[K\varepsilon \left| \dfrac{n+2}{n+1+\alpha} \right| + \displaystyle\sum_{m=0}^{m_0-1} K\varepsilon \left| \left| \dfrac{m+1}{m+\alpha} \right| - \left| \dfrac{m+2}{m+1+\alpha} \right| \right| \right. & \\ \qquad \left. + \displaystyle\sum_{m=m_0}^{n} K\varepsilon \left(\dfrac{m+1}{m+\alpha} - \dfrac{m+2}{m+1+\alpha} \right) \right] & (\text{for } \alpha \leq 1) \end{cases} \\ &= \begin{cases} \dfrac{\nu}{\mu} \cdot \dfrac{2\alpha - 1}{\alpha} K\varepsilon & (\text{for } \alpha > 1), \\ \dfrac{\nu}{\mu} \left[\displaystyle\sum_{m=0}^{m_0-1} \left| \left| \dfrac{m+1}{m+\alpha} \right| - \left| \dfrac{m+2}{m+1+\alpha} \right| \right| + \dfrac{m_0+1}{m_0+\alpha} \right] K\varepsilon & (\text{for } \alpha \leq 1) \end{cases} \end{aligned} \tag{3.10}$$

for all $x \in I_{\rho_0}$. □

We now assume a stronger condition, in comparison with (3.1), to approximate the given function $y(x)$ by a solution $y_h(x)$ of the Kummer's equation on a larger (punctured) interval.

Corollary 3.2. *Let α and β be real constants such that $\beta \notin \mathbb{Z}$ and neither α nor $1+\alpha-\beta$ is a nonpositive integer. Suppose a function $y : I_\infty \to \mathbb{C}$ is representable by a power series $\sum_{m=0}^{\infty} b_m x^m$ which* *converges for all $x \in I_\infty$. For every $m \in \mathbb{N}_0$, let us define $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$. Moreover, assume that*

$$\lim_{m\to\infty} \frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}} = 0, \qquad 0 < \left| \sum_{i=0}^{\infty} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}} \right| < \infty \tag{3.11}$$

and there exists a nonnegative constant ν satisfying

$$\left| \sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}} \right| \le \nu \left| \frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}} \right| \tag{3.12}$$

for all $m \in \mathbb{N}_0$. If $y \in C_K$ and it satisfies the differential inequality (3.2) for all $x \in I_\infty$ and for some $\varepsilon \ge 0$, then there exists a solution $y_n : I_\infty \to \mathbb{C}$ of the Kummer's equation (2.1) such that

$$|y(x) - y_n(x)| \le \begin{cases} \nu \cdot \dfrac{2\alpha - 1}{\alpha} K\varepsilon & (\text{for } \alpha > 1), \\ \nu \left[\displaystyle\sum_{m=0}^{m_0-1} \left| \left| \frac{m+1}{m+\alpha} \right| - \left| \frac{m+2}{m+1+\alpha} \right| \right| + \frac{m_0+1}{m_0+\alpha} \right] K\varepsilon & (\text{for } \alpha \le 1) \end{cases} \tag{3.13}$$

for any $x \in I_n$, where $m_0 = \max\{0, \lceil -\alpha \rceil\}$ and n is a sufficiently large integer.

Proof. In view of (3.11) and (3.12), we can choose a sufficiently large integer n with

$$\left| \frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}} \right| \le \frac{1}{n} \left| \sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}} \right| \le \frac{\nu}{n} \left| \frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}} \right|, \tag{3.14}$$

where the first inequality holds true for all sufficiently large m, and the second one holds true for all $m \in \mathbb{N}_0$.

If we define $\rho_0 = n$, then Theorem 3.1 implies that there exists a solution $y_n : I_\infty \to \mathbb{C}$ of the Kummer's equation such that the inequality given for $|y(x) - y_n(x)|$ holds true for any $x \in I_n$. □

4. An Example

We fix $\alpha = 1$, $\beta = 10/3$, $\varepsilon > 0$, and $0 < \rho < 1$. And we define

$$b_0 = 0, \qquad b_m = \frac{\varepsilon}{s} \cdot \frac{1}{m^2} \tag{4.1}$$

for all $m \in \mathbb{N}$, where we set $s = (5/3)(2-\rho)/(1-\rho)$. We further define

$$y(x) = \sum_{m=0}^{\infty} b_m x^m \tag{4.2}$$

for any $x \in I_\rho$.

Then, we set $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$, that is,

$$a_0 = \frac{10}{3} \cdot \frac{\varepsilon}{s}, \qquad a_m = \left(1 + \frac{4m^2 - 6m - 3}{3m^2(m+1)}\right)\frac{\varepsilon}{s} \le \frac{5}{3} \cdot \frac{\varepsilon}{s} \tag{4.3}$$

for every $m \in \mathbb{N}$. Obviously, all a_ms are positive, and the sequence $\{a_m\}$ is strictly monotone decreasing, from the 4th term on, to ε/s. More precisely, $a_0 > a_1 < a_2 < a_3 < a_4 > a_5 > a_6 > \cdots$.

Since

$$a_0 = \frac{10}{3} \cdot \frac{\varepsilon}{s} > \frac{1}{6} \cdot \frac{\varepsilon}{s} + \frac{41}{36} \cdot \frac{\varepsilon}{s} = a_1 + a_3, \tag{4.4}$$

we get

$$\begin{aligned} \left|\sum_{m=0}^{\infty} a_m x^m\right| &= \left|a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \left(a_4 x^4 + a_5 x^5\right) + \left(a_6 x^6 + a_7 x^7\right) + \cdots\right| \\ &\ge \left|a_0 + a_1 x + a_2 x^2 + a_3 x^3\right| \\ &\ge a_0 - a_1 - a_3 \\ &= \frac{73}{36} \cdot \frac{\varepsilon}{s} \end{aligned} \tag{4.5}$$

for each $x \in I_\rho$ and

$$\sum_{m=0}^{\infty} |a_m x^m| \le \sum_{m=0}^{\infty} a_m \rho^m \le \left(\frac{10}{3} + \sum_{m=1}^{\infty} \frac{5}{3}\rho^m\right)\frac{\varepsilon}{s} = \varepsilon \tag{4.6}$$

for all $x \in I_\rho$. Hence, we obtain

$$\sum_{m=0}^{\infty} |a_m x^m| \le K \left|\sum_{m=0}^{\infty} a_m x^m\right| \tag{4.7}$$

for any $x \in I_\rho$, where $K = (60/73) \cdot (2-\rho)/(1-\rho)$, implying that $y \in \mathcal{C}_K$.

We will now show that $\{a_m\}$ satisfies condition (3.1). For any $m \in \mathbb{N}$, we have

$$\begin{aligned}\left|\sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right| &= a_0 + \sum_{i=1}^{m-1} \frac{10 \cdot 13 \cdot 16 \cdots (3i+7)}{(i+1)3^i} a_i \\ &\leq \left[\frac{10}{3} + \sum_{i=1}^{m-1} \frac{10 \cdot 13 \cdot 16 \cdots (3i+7)}{(i+1)3^i} \cdot \frac{5}{3}\right] \frac{\varepsilon}{s}, \\ \left|\frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| &\geq \frac{10 \cdot 13 \cdot 16 \cdots (3m+7)}{3^m} \cdot \frac{1}{6} \cdot \frac{\varepsilon}{s},\end{aligned} \tag{4.8}$$

since $\lim_{m \to \infty} a_m = \varepsilon/s$.

It follows from (4.8) that

$$\begin{aligned}\left|\sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right| &\leq 10\left[\frac{1}{3} + \sum_{i=1}^{m-1} \frac{10 \cdot 13 \cdot 16 \cdots (3i+7)}{(i+1)3^i} \cdot \frac{1}{6}\right] \frac{\varepsilon}{s} \\ &= 10\left[\frac{1}{3} + \frac{10 \cdot 13 \cdots (3m+7)}{3^m} \sum_{i=1}^{m-1} \frac{3^{m-i}}{(3i+10)\cdots(3m+7)} \cdot \frac{1}{i+1} \cdot \frac{1}{6}\right] \frac{\varepsilon}{s} \\ &\leq 10\left[\frac{1}{3} + \frac{10 \cdot 13 \cdot 16 \cdots (3m+7)}{3^m} \sum_{i=1}^{m-1} \frac{1}{(i+1)^2} \cdot \frac{1}{6}\right] \frac{\varepsilon}{s} \\ &\leq 10 \frac{10 \cdot 13 \cdot 16 \cdots (3m+7)}{3^m}\left[\frac{1}{10} + \frac{1}{6}(\zeta(2)-1)\right] \frac{\varepsilon}{s} \\ &= \frac{5\pi^2 - 12}{3} \cdot \frac{10 \cdot 13 \cdot 16 \cdots (3m+7)}{3^m} \cdot \frac{1}{6} \cdot \frac{\varepsilon}{s} \\ &\leq \frac{5\pi^2 - 12}{3}\left|\frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right|.\end{aligned} \tag{4.9}$$

We know that the inequality (4.9) is also true for $m = 0$.

On the other hand, in view of Remark 2.2, there exists a constant $\mu > 1$ such that inequality (2.12) holds true for all sufficiently large integers m. By (2.12) and (4.9), we conclude that $\{a_m\}$ satisfies condition (3.1) with $\nu = (5\pi^2 - 12)\mu/3$.

Finally, it follows from (4.6) that

$$\left|xy''(x) + (\beta - x)y'(x) - \alpha y(x)\right| = \left|\sum_{m=0}^{\infty} a_m x^m\right| \leq \sum_{m=0}^{\infty} |a_m x^m| \leq \varepsilon \tag{4.10}$$

for all $x \in I_{\rho_0}$ with $\rho_0 = \min\{\rho, 1/\mu\}$.

According to Theorem 3.1, there exists a solution $y_h : I_\infty \to \mathbb{C}$ of the Kummer's equation (2.1) such that

$$\left|y(x) - y_h(x)\right| \leq \frac{100\pi^2 - 240}{73} \cdot \frac{2-\rho}{1-\rho}\varepsilon \tag{4.11}$$

for all $x \in I_{\rho_0}$.

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