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Research Article

On the Superstability of the Pexider Type Trigonometric Functional Equation

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We will investigate the superstability of the (hyperbolic) trigonometric functional equation from the following functional equations: $f(x+y) \pm g(x-y) = \lambda f(x)g(y)$ and $f(x+y) \pm g(x-y) = \lambda g(x)f(y)$, which can be considered the mixed functional equations of the sine function and cosine function, the hyperbolic sine function and hyperbolic cosine function, and the exponential functions, respectively.

1. Introduction

Baker et al. in [1] stated the following: if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is frequently referred to as superstability.

The superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

and the sine functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2, \tag{S}$$

were investigated by Baker [2] and Cholewa [3], respectively. Their results were improved by Badora [4], Badora and Ger [5], Găvruţa [6], and Kim (see [7, 8]).

The superstability of the Wilson equation

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{C_{fg}}$$

was investigated by Kannappan and Kim [9].

The superstability of the trigonometric functional equation concerned with the sine and the cosine equations

$$f(x+y) - f(x-y) = 2f(x)f(y), \tag{T}$$

$$f(x+y) - f(x-y) = 2f(x)g(y), \tag{T_{fg}}$$

$$f(x+y) - f(x-y) = 2g(x)f(y), \tag{T_{gf}}$$

$$f(x+y) - f(x-y) = 2g(x)h(y), \tag{T_{gh}}$$

was investigated by Kim [10, 11], Kim and Lee [12].

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions also satisfy the above mentioned equations; thus they can be called by the *hyperbolic* cosine (sine, trigonometric, exponential) functional equations, respectively.

For example,

$$\begin{gathered}
\cosh(x+y) + \cosh(x-y) = 2\cosh(x)\cosh(y), \\
\sinh(x+y) + \sinh(x-y) = 2\sinh(x)\cosh(y), \\
\sinh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) = \sinh(x)\sinh(y), \\
ca^{x+y} + ca^{x-y} = 2\frac{ca^x}{2}\left(a^y + a^{-y}\right) = 2ce^x\frac{a^y + a^{-y}}{2}, \\
e^{x+y} + e^{x-y} = 2\frac{e^x}{2}\left(e^y + e^{-y}\right) = 2e^x\cosh(y), \\
(n(x+y)+c) + (n(x-y)+c) = 2(nx+c) \ : \ \text{Jensen equation, for } f(x) = nx + c,
\end{gathered} \tag{1.1}$$

where a and c are constants.

The aim of this paper is to investigate the superstability of the (hyperbolic) sine functional equation (S) from the following functional equations:

$$f(x+y) + g(x-y) = \lambda f(x)g(y), \tag{C_{fgfg}}$$

$$f(x+y) + g(x-y) = \lambda g(x)f(y), \tag{C_{fggf}}$$

$$f(x+y) - g(x-y) = \lambda f(x)g(y), \tag{T_{fgfg}}$$

$$f(x+y) - g(x-y) = \lambda g(x)f(y), \tag{T_{fggf}}$$

on the abelian group. As corollaries, we obtain the superstability of (S) from the following functional equations:

$$f(x+y)+g(x-y)=\lambda f(x)f(y), \tag{C_{fgff}}$$

$$f(x+y)+g(x-y)=\lambda g(x)g(y), \tag{C_{fggg}}$$

$$f(x+y)-g(x-y)=\lambda f(x)f(y), \tag{T_{fgff}}$$

$$f(x+y)-g(x-y)=\lambda g(x)g(y). \tag{T_{fggg}}$$

Furthermore, the obtained results can be extended to the Banach space.

In this paper, let $(G,+)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with (C), $(G,+)$ only needs Abelian which is not 2-divisibility.

We may assume that f and g are nonzero functions, λ, ε is a nonnegative real constant, and $\varphi : G \to \mathbb{R}$ is a mapping. For simplicity, we will form the notations of the equation as follows:

$$f(x+y)+f(x-y)=\lambda f(x)f(y), \tag{C^{λ}}$$

$$f(x+y)+f(x-y)=\lambda f(x)g(y), \tag{C^{λ}_{fg}}$$

$$g(x+y)+g(x-y)=\lambda g(x)g(y), \tag{C^{λ}_{g}}$$

$$g(x+y)+g(x-y)=\lambda g(x)f(y). \tag{C^{λ}_{gf}}$$

2. Superstability of the Functional Equations

In this section, we will investigate the superstability of the (hyperbolic) sine functional equation (S) from the functional equations (C_{fgfg}), (C_{fggf}), (T_{fgfg}), and (T_{fggf}) under the conditions from which the differences of each equation are bounded by $\varphi(x)$ and $\varphi(y)$.

Theorem 2.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y)+g(x-y)-\lambda f(x)g(y)\right| \le \varphi(x) \quad \forall x, y \in G. \tag{2.1}$$

Then, either g with $f(0)=0$ is bounded or f satisfies (S). *Particularly, if g satisfies (C^{λ}), then f and g are the solutions of the Wilson type equation (C^{λ}_{fg}).*

Proof. Let g be the unbounded solution of the inequality (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0 \ne |g(y_n)| \to \infty$ as $n \to \infty$.

Taking $y=y_n$ in the inequality (2.1), dividing both sides by $|\lambda g(y_n)|$, and passing to the limit as $n \to \infty$, we obtain the following:

$$f(x)=\lim_{n\to\infty}\frac{f(x+y_n)+g(x-y_n)}{\lambda g(y_n)}, \quad x \in G. \tag{2.2}$$

Using (2.1), we have

$$\begin{aligned}&\big|f(x+(y+y_n))+g(x-(y+y_n))-\lambda f(x)g(y+y_n)\\&f(x+(-y+y_n))+g(x-(-y+y_n))-\lambda f(x)g(-y+y_n)\big|\leq 2\varphi(x),\end{aligned}\tag{2.3}$$

and thus,

$$\begin{aligned}&\left|\frac{f((x+y)+y_n)+g((x+y)-y_n)}{\lambda g(y_n)}\right.\\&\left.+\frac{f((x-y)+y_n)+g((x-y)-y_n)}{\lambda g(y_n)}-\lambda f(x)\cdot\frac{g(y+y_n)+g(-y+y_n)}{\lambda g(y_n)}\right|\\&\leq\frac{2\varphi(x)}{|\lambda\|g(y_n)|}\end{aligned}\tag{2.4}$$

for all $x, y \in G$.

We conclude that, for every $y \in G$, there exists a limit function

$$k_1(y) := \lim_{n\to\infty}\frac{g(y+y_n)+g(-y+y_n)}{\lambda g(y_n)},\tag{2.5}$$

where the function $k_1 : G \to \mathbb{C}$ satisfies the equation

$$f(x+y)+f(x-y)=\lambda f(x)k_1(y)\quad\forall x,y\in G.\tag{2.6}$$

Applying the case $f(0) = 0$ in (2.6), which implies that f is odd and keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned}f(x+y)^2-f(x-y)^2&=\lambda f(x)k_1(y)[f(x+y)-f(x-y)]\\&=f(x)[f(x+2y)-f(x-2y)]\\&=f(x)[f(2y+x)+f(2y-x)]\\&=\lambda f(x)f(2y)k_1(x).\end{aligned}\tag{2.7}$$

Putting $y = x$ in (2.6), we obtain the equation

$$f(2x)=\lambda f(x)k_1(x),\quad x\in G.\tag{2.8}$$

This, in return, leads to the equation

$$f(x+y)^2-f(x-y)^2=f(2x)f(2y)\tag{2.9}$$

being valid for all $x, y \in G$, which, in the light of the unique 2-divisibility of G, states nothing else but (S).

Particularly, if g satisfies (C^λ), the limit k_1 states nothing else but g; thus, (2.6) validates the required equation (C^λ_{fg}). □

Corollary 2.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x+y) + g(x-y) - \lambda f(x) f(y) \right| \le \varphi(x) \quad \forall x, y \in G. \tag{2.10}$$

Then, either f with $f(0) = 0$ is bounded or f satisfies (S).

Proof. Replacing $g(y)$ by $f(y)$ in (2.1) of Theorem 2.1, an obvious slight change in the proof steps applied in Theorem 2.1 allows us to show that f satisfies (S).

Namely, for f be unbounded, there exists a sequence $\{y_n\}$ in G such that $0 \ne |f(y_n)| \to \infty$ as $n \to \infty$. Taking $y = y_n$ in the inequality (2.1), dividing both sides by $|\lambda f(y_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$f(x) = \lim_{n \to \infty} \frac{f(x + y_n) + g(x - y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.11}$$

A similar procedure to that applied after formula (2.2) yields the required result by using of (2.6). □

Theorem 2.3. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x+y) + g(x-y) - \lambda g(x) f(y) \right| \le \varphi(x) \quad \forall x, y \in G. \tag{2.12}$$

Then, either f with $g(0) = 0$ is bounded or g satisfies (S). *Particularly, if f satisfies (C^λ), then g and f are the solutions of equation (C^λ_{gf}).*

Proof. For the unbounded f, we can choose a sequence $\{y_n\}$ in G such that $0 \ne |f(y_n)| \to \infty$ as $n \to \infty$.

The same reasoning to the proof applied in Theorem 2.1 for (2.12) with $y = y_n$ gives

$$g(x) = \lim_{n \to \infty} \frac{f(x + y_n) + g(x - y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.13}$$

Substituting $y + y_n$ and $-y + y_n$ for y in (2.12), and dividing by $|\lambda f(y_n)|$, then it gives us the existence of a limit function

$$k_2(y) := \lim_{n \to \infty} \frac{f(y + y_n) + f(-y + y_n)}{\lambda f(y_n)}, \tag{2.14}$$

where the function $k_2 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y) + g(x-y) = \lambda g(x)k_2(y) \quad \forall x, y \in G. \tag{2.15}$$

Applying the case $g(0) = 0$ in (2.15), it implies that g is odd.

A similar procedure to that applied after formula (2.6) allows us to show that g satisfies (S).

Particularly, if f satisfies (C^λ), the limit k_2 states nothing else but f; thus, the required equation (C^λ_{gf}) holds from (2.15). □

Corollary 2.4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda g(x)g(y)| \le \varphi(x) \quad \forall x, y \in G. \tag{2.16}$$

Then, either g with $g(0) = 0$ is bounded or g satisfies (S).

Proof. Substituting $g(y)$ for $f(y)$ in (2.12) of Theorem 2.3, the next of the proof runs along that of the above theorem. □

Theorem 2.5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda f(x)g(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{2.17}$$

Then, either f with $g(0) = 0$ is bounded or g satisfies (S). *Particularly, if f satisfies* (C^λ), *then g and f are solutions of the Wilson type equation* (C^λ_{gf}).

Proof. For the unbounded f of the inequality (2.17), we can choose a sequence $\{x_n\}$ in G such that $0 \ne |f(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in the inequality (2.17), dividing both sides by $|\lambda f(x_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$g(y) = \lim_{n\to\infty} \frac{f(x_n+y) + g(x_n-y)}{\lambda f(x_n)}, \quad x \in G. \tag{2.18}$$

In (2.17), replacing x by $x_n + y$ and $x_n - y$, replacing y by x, and dividing by $|\lambda f(x_n)|$, it then gives us the existence of a limit function

$$k_3(x) := \lim_{n\to\infty} \frac{f(x_n+y) + f(x_n-y)}{\lambda f(x_n)}, \tag{2.19}$$

where the function $k_3 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y) + g(x-y) = \lambda g(x)k_3(y) \quad \forall x, y \in G. \tag{2.20}$$

Applying the case $g(0) = 0$ in (2.20), it implies that g is odd. Since (2.20) equals to (2.6), an obvious slight change in the proof steps applied after formula (2.15) allows us to see that g satisfies (S). Particularly, if f satisfies (C^{λ}), then the limit k_3 states nothing else but f, thus, the required equation (C^{λ}_{fg}) holds from (2.20). □

Corollary 2.6. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda f(x)f(y)| \le \varphi(y), \quad \forall x, y \in G. \tag{2.21}$$

Then, either f with $f(0) = 0$ is bounded or f satisfies (S).

Proof. Substituting $f(y)$ for $g(y)$ in (2.17) of Theorem 2.5, as Corollary 2.4, we then obtain the required result from the above theorem. □

Theorem 2.7. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda g(x)f(y)| \le \varphi(y), \quad \forall x, y \in G. \tag{2.22}$$

Then, either g with $f(0) = 0$ is bounded or f satisfies (S). *Particularly, if g satisfies (C^{λ}), then f and g are the solutions of equation (C^{λ}_{fg}).*

Proof. For the unbounded g, we can choose a sequence $\{x_n\}$ in G such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

For (2.22) with $x = x_n$, the same reasoning as the proof applied in Theorem 2.1 gives us

$$f(y) = \lim_{n\to\infty} \frac{f(x_n+y) + g(x_n-y)}{\lambda g(x_n)}, \quad x \in G. \tag{2.23}$$

In (2.22), replacing x by $x_n + y$ and $x_n - y$, replacing x by y, and dividing by $|\lambda g(x_n)|$, it then gives us the existence of a limit function

$$k_4(y) := \lim_{n\to\infty} \frac{g(x_n+y) + g(x_n-y)}{\lambda g(x_n)}, \tag{2.24}$$

where the function $k_4 : G \to \mathbb{C}$ satisfies the equation

$$f(x+y) + f(x-y) = \lambda f(x)k_4(y), \quad \forall x, y \in G. \tag{2.25}$$

Since (2.25) is the same as (2.6), the next proof runs along that of Theorem 2.1. □

Corollary 2.8. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda g(x)g(y)| \le \varphi(y), \quad \forall x, y \in G. \tag{2.26}$$

Then, either g with $g(0) = 0$ is bounded or g satisfies (S).

Proof. Substituting $g(y)$ for $f(y)$ in (2.22) of Theorem 2.7, the next proof runs along that of the above theorem. □

The cases $\varphi(x)$, $\varphi(y)$ in the following result follow the procedure applied in Theorems 2.1 and 2.5, respectively. In the obtained result, applying the cases $\lambda = 2$ and $\varphi(x) = \varphi(y) = \varepsilon$, then they are founded in [2, 4–6].

Corollary 2.9. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - \lambda f(x)f(y)| \leq \begin{cases} \varphi(x), \\ \varphi(y), \end{cases} \quad \forall x, y \in G. \tag{2.27}$$

Then, either f is bounded or f satisfies (C^{λ}).

Proof. For f being unbounded, we can choose two sequences $\{x_n\}$ and $\{y_n\}$ in G such that $0 \neq |f(x_n)|$ and $|f(y_n)| \to \infty$ as $n \to \infty$.

(i) case $\varphi(x)$

Taking $y = y_n$ in inequality (2.27), dividing it by $|\lambda f(y_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$f(x) = \lim_{n \to \infty} \frac{f(x+y_n) + f(x-y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.28}$$

In (2.27), replacing y by $y + y_n$ and $-y + y_n$, and dividing by $|\lambda f(y_n)|$, it then gives, with the application of (2.28), that f satisfies (C^{λ}).

(ii) case $\varphi(y)$

For the chosen sequence $\{x_n\}$, the procedure as (i) implies

$$f(y) = \lim_{n \to \infty} \frac{f(x_n+y) + f(x_n-y)}{\lambda f(x_n)}, \quad x \in G. \tag{2.29}$$

In (2.27), replacing x by $x_n + y$ and $x_n - y$ and replacing y by x, the other procedure is the same as (i). □

Since the proofs of the following results (Theorems 2.10–2.16 and Corollaries 2.11–2.17) for the functional equations (T_{fgfg}), (T_{fggf}), (T_{fgff}), and (T_{fggg}) are, respectively, the same processes those of Theorems 2.1–2.7 and Corollaries 2.2–2.8, as we will skip their proofs.

Theorem 2.10. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda f(x)g(y)| \leq \varphi(x) \quad \forall x, y \in G. \tag{2.30}$$

Then, either g *with* $f(0) = 0$ *is bounded or* f *satisfies* (S). *Particularly, if* g *satisfies* (C^{λ}), *then* f *and* g *are solutions of the Wilson type equation* (C^{λ}_{fg}).

Corollary 2.11. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda f(x)f(y)| \le \varphi(x) \quad \forall x, y \in G. \tag{2.31}$$

Then, either f *with* $f(0) = 0$ *is bounded or* f *satisfies* (S).

Theorem 2.12. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda g(x)f(y)| \le \varphi(x) \quad \forall x, y \in G. \tag{2.32}$$

Then, either f *with* $g(0) = 0$ *is bounded or* g *satisfies* (S). *Particularly, if* f *satisfies* (C^{λ}), *then* g *and* f *are solutions of the Wilson type equation* (C^{λ}_{gf}).

Corollary 2.13. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda g(x)g(y)| \le \varphi(x) \quad \forall x, y \in G. \tag{2.33}$$

Then, either g *with* $g(0) = 0$ *is bounded or* g *satisfies* (S).

Theorem 2.14. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda f(x)g(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{2.34}$$

Then, either f *with* $g(0) = 0$ *is bounded or* g *satisfies* (S). *Particularly, if* f *satisfies* (C^{λ}), *then* g *and* f *are solutions of the Wilson type equation* (C^{λ}_{gf}).

Corollary 2.15. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda f(x)f(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{2.35}$$

Then, either f *with* $f(0) = 0$ *is bounded or* f *satisfies* (S).

Theorem 2.16. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda g(x)f(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{2.36}$$

Then, either g *with* $f(0) = 0$ *is bounded or* f *satisfies* (S). *Particularly, if* g *satisfies* (C^{λ}), *then* f *and* g *are the solutions of equation* (C^{λ}_{fg}).

Corollary 2.17. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda g(x)g(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{2.37}$$

Then, either g with $g(0) = 0$ is bounded or g satisfies (S).

The cases $\varphi(x)$ and $\varphi(y)$ in the following result follow the procedure applied in Theorems 2.10 and 2.14, respectively. In the obtained result, applying the cases $\lambda = 2$ and $\varphi(x) = \varphi(y) = \varepsilon$, then they are founded in [10–12].

Corollary 2.18. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - \lambda f(x)f(y)| \le \begin{cases} \varphi(x), \\ \varphi(y), \end{cases} \quad \forall x, y \in G. \tag{2.38}$$

Then f is bounded.

Proof. For f being unbounded, we can choose two sequences $\{x_n\}$ and $\{y_n\}$ in G such that $0 \neq |f(x_n)|$ and $|f(y_n)| \to \infty$ as $n \to \infty$.

(i) case $\varphi(x)$

First, going though the same process of the case (i) of Corollary 2.9, then we obtain that f satisfies (C^{λ}).

Secondly, from the chosen sequence $\{y_n\}$, we obtain

$$f(x) = \lim_{n \to \infty} \frac{f(x+y_n) - f(x-y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.39}$$

Replacing x by $x + y_n$ and $x - y_n$ in (2.38), then we obtain, from their difference, the inequality

$$\begin{aligned} &\left| \frac{f((x+y)+y_n) - f((x+y)-y_n)}{\lambda f(y_n)} \right. \\ &\quad \left. - \frac{f((x-y)+y_n) - f((x-y)-y_n)}{\lambda f(y_n)} - \lambda \cdot \frac{f(x+y_n) - f(x-y_n)}{\lambda f(y_n)} \cdot f(y) \right| \\ &\quad \le \frac{2\varphi(x)}{|\lambda||f(y_n)|}, \end{aligned} \tag{2.40}$$

which gives, with the application of (2.39), the equation

$$f(x+y) - f(x-y) = \lambda f(x)f(y). \tag{T^{λ}}$$

Thus, since the function f satisfies two equations (C^{λ}) and (T^{λ}), the function f states nothing else but zero. It is a contradiction assuming that f is nonzero. Thus f is bounded.

(ii) case $\varphi(y)$

For the chosen sequence $\{x_n\}$, the same procedure as in the above case (i) gives us the required result. □

Remark 2.19. (a) Substituting f for g of the second term of the stability inequalities in all the results of Section 2, then we obtain the same number of corollaries, which are the stability of the (hyperbolic) cosine type functional equations (C^{λ}_{fg}), (C^{λ}_{gf}), and the (hyperbolic) trigonometric type functional equations $(f(x+y) - f(x-y) = \lambda f(x)g(y),\ f(x+y) - f(x-y) = \lambda g(x)f(y))$.

(b) Applying the case $\lambda = 2$ in all the results of Section 2 and (a)'s application, then we obtain the same number of corollaries. Some of their stabilities were founded in papers [6, 7, 9–11].

(c) Applying $\varphi(x) = \varphi(y) = \varepsilon$ in all the results of Section 2 and (a)'s application, then we obtain the same number of corollaries. Some of their stabilities were founded in papers [6, 7, 9–12].

(d) Applying $\lambda = 2$ and $\varphi(x) = \varphi(y) = \varepsilon$ in all the results of Section 2, (a)'s, (b)'s, and (c)'s applications, then we obtain the same number of corollaries. Some of their stabilities were founded in papers [5–7, 9–12].

3. Extension to the Banach Space

In all the results presented in Section 2, the range of functions on the abelian group can be extended to the Banach space. For simplicity, we will only prove the plus case of (3.1) of Theorem 3.1. The other cases are similar to this, thus their proofs will be omitted.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g : G \to E$ satisfy one of each inequalities*

$$\|f(x+y) \pm g(x-y) - \lambda f(x)g(y)\| \le \varphi(x), \tag{3.1}$$

$$\|f(x+y) \pm g(x-y) - \lambda g(x)f(y)\| \le \varphi(x) \tag{3.2}$$

for all $x, y \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$, then,

(i) *case* (3.1), *either $x^* \circ g$ with $f(0) = 0$ is bounded or f satisfies* (S). *Particularly, if g satisfies (C^{λ}), then f and g are the solutions of the Wilson type equation (C^{λ}_{fg}).*

(ii) *case* (3.2), *either f with $g(0) = 0$ is bounded or g satisfies* (S). *Particularly, if f satisfies (C^{λ}), then g and f are the solutions of the Wilson type equation (C^{λ}_{gf}).*

Proof. (i) As + and − have the same procedure, we will only show the plus case in (3.1).

Assume that (3.1) holds and arbitrarily fixes a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned} \varphi(x) &\geq \|f(x+y) + g(x-y) - \lambda f(x)g(y)\| \\ &= \sup_{\|y^*\|=1} |y^*(f(x+y) + g(x-y) - \lambda f(x)g(y))| \\ &\geq |x^*(f(x+y)) + x^*(g(x-y)) - 2x^*(f(x))x^*(g(y))|, \end{aligned} \tag{3.3}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of inequality (2.1). Since, by assumption, the superposition $x^* \circ g$ with $f(0) = 0$ is unbounded, an appeal to Theorem 2.1 shows that the two results hold.

First, the function $x^* \circ f$ solves (S). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $\mathfrak{D}(x, y) : G \times G \to \mathbb{C}$ defined by

$$\mathfrak{D}S(x,y) := f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y), \tag{3.4}$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathfrak{D}S(x,y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\} \tag{3.5}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathfrak{D}S(x,y) = 0, \quad \forall x, y \in G, \tag{3.6}$$

as claimed.

Second, in particular, if $x^* \circ g$ satisfies (C^λ), then $x^* \circ f$ and $x^* \circ g$ are solutions of the Wilson type equation (C^λ_{fg}). This means that

$$\mathfrak{D}C^\lambda_{fg}(x,y) := f(x+y) + f(x-y) - \lambda f(x)g(y), \tag{3.7}$$

falls into the kernel of x^*. Through the above process, we obtain

$$\mathfrak{D}C^\lambda_{fg}(x,y) = 0, \quad \forall x, y \in G, \tag{3.8}$$

as claimed. The minus case is also similar.

(ii) Case (3.2) also runs along the proof of case (3.1) . □

Corollary 3.2. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach space. Assume that* $f, g : G \to E$ *satisfy one of each inequalities*

$$\|f(x+y) \pm g(x-y) - \lambda f(x)f(y)\| \leq \varphi(x), \tag{3.9}$$

$$\|f(x+y) \pm g(x-y) - \lambda g(x)g(y)\| \leq \varphi(x) \tag{3.10}$$

for all $x, y \in G$*. For an arbitrary linear multiplicative functional* $x^* \in E^*$*,*

(i) *in case* (3.9)*, either* $x^* \circ f$ *with* $f(0) = 0$ *is bounded or* f *satisfies* (S).

(ii)*in case* (3.10)*, either* $x^* \circ g$ *with* $g(0) = 0$ *is bounded or* g *satisfies* (S).

Theorem 3.3. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach space. Assume that* $f, g : G \to E$ *satisfy one of each inequalities*

$$\|f(x+y) \pm g(x-y) - \lambda f(x)g(y)\| \leq \varphi(y), \tag{3.11}$$

$$\|f(x+y) \pm g(x-y) - \lambda g(x)f(y)\| \leq \varphi(y) \tag{3.12}$$

for all $x, y \in G$*. For an arbitrary linear multiplicative functional* $x^* \in E^*$*,*
then,

(i) *in case* (3.11)*, either* $x^* \circ f$ *with* $g(0) = 0$ *is bounded or* g *satisfies* (S)*. Particularly, if* f *satisfies* (C^λ)*, then* g *and* f *are solutions of the Wilson type equation* (C^λ_{gf}).

(ii) *in case* (3.12)*, either* g *with* $f(0) = 0$ *is bounded or* f *satisfies* (S)*. Particularly, if* g *satisfies* (C^λ)*, then* f *and* g *are solutions of the Wilson type equation* (C^λ_{fg}).

Corollary 3.4. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach space. Assume that* $f, g : G \to E$ *satisfy one of each inequalities*

$$\|f(x+y) \pm g(x-y) - \lambda f(x)f(y)\| \leq \varphi(y), \tag{3.13}$$

$$\|f(x+y) \pm g(x-y) - \lambda g(x)g(y)\| \leq \varphi(y) \tag{3.14}$$

for all $x, y \in G$*. For an arbitrary linear multiplicative functional* $x^* \in E^*$*,*

(i) *in case* (3.13)*, either* $x^* \circ f$ *with* $f(0) = 0$ *is bounded or* f *satisfies* (S).

(ii) *in case* (3.14)*, either* $x^* \circ g$ *with* $g(0) = 0$ *is bounded or* g *satisfies* (S).

Remark 3.5. We obtain the same number of corollaries on the Banach space for all the theorems mentioned in Section 2 and all the results obtained by applying of (a), (b), (c), and (d) in Remark 2.19, which are founded in papers [4–7, 10–12].

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