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Research Article

Global Existence and Asymptotic Behavior of Solutions for Some Nonlinear Hyperbolic Equation

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The initial boundary value problem for a class of hyperbolic equation with nonlinear dissipative term $u_{tt} - \sum_{i=1}^{n} (\partial/\partial x_i)(|\partial u/\partial x_i|^{p-2}(\partial u/\partial x_i)) + a|u_t|^{q-2}u_t = b|u|^{r-2}u$ in a bounded domain is studied. The existence of global solutions for this problem is proved by constructing a stable set in $W_0^{1,p}(\Omega)$ and show the asymptotic behavior of the global solutions through the use of an important lemma of Komornik.

1. Introduction

We are concerned with the global solvability and asymptotic stability for the following hyperbolic equation in a bounded domain

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + a|u_t|^{q-2}u_t = b|u|^{r-2}u, \quad x \in \Omega,\ t > 0 \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{1.2}$$

and boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $a, b > 0$ and $q, r > 2$ are real numbers, and $\Delta_p = -\sum_{i=1}^{n}(\partial/\partial x_i)(|\partial/\partial x_i|^{p-2}(\partial/\partial x_i))$ is a divergence operator (degenerate Laplace operator) with $p > 2$, which is called a p-Laplace operator.

Equations of type (1.1) are used to describe longitudinal motion in viscoelasticity mechanics and can also be seen as field equations governing the longitudinal motion of a viscoelastic configuration obeying the nonlinear Voight model [1–4].

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data [4–6]. For $a = 0$, the source term causes finite time blow-up of solutions with negative initial energy if $r > p$ [7].

The interaction between the damping and the source terms was first considered by Levine [8, 9] in the case $p = q = 2$. He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [10] extended Levine's result to the nonlinear damping case $q > 2$. In their work, the authors considered (1.1)–(1.3) with $p = 2$ and introduced a method different from the one known as the concavity method. They determined suitable relations between q and r, for which there is global existence or alternatively finite time blow-up. Precisely, they showed that solutions with negative energy continue to exist globally in time t if $q \geq r$ and blow up in finite time if $q < r$ and the initial energy is sufficiently negative. Vitillaro [11] extended these results to situations where the damping is nonlinear and the solution has positive initial energy. For the Cauchy problem of (1.1), Todorova [12] has also established similar results.

Zhijian in [13–15] studied the problem (1.1)–(1.3) and obtained global existence results under the growth assumptions on the nonlinear terms and initial data. These global existence results have been improved by Liu and Zhao [16] by using a new method. As for the nonexistence of global solutions, Yang [17] obtained the blow-up properties for the problem (1.1)–(1.3) with the following restriction on the initial energy $E(0) < \min\{-((rk_1 + pk_2)/(r - p))^{1/\delta}, -1\}$, where $r > p$ and k_1, k_2, and δ are some positive constants.

Because the p-Laplace operator Δ_p is nonlinear operator, the reasoning of proof and computation is greatly different from the Laplace operator $\Delta = \sum_{i=1}^{n}\partial^2/\partial x_i^2$. By mean of the Galerkin method and compactness criteria and a difference inequality introduced by Nakao [18], the author [19, 20] has proved the existence and decay estimate of global solutions for the problem (1.1)–(1.3) with inhomogeneous term $f(x,t)$ and $p \geq r$.

In this paper we are going to investigate the global existence for the problem (1.1)–(1.3) by applying the potential well theory introduced by Sattinger [21], and we show the asymptotic behavior of global solutions through the use of the lemma of Komornik [22].

We adopt the usual notation and convention. Let $W^{k,p}(\Omega)$ denote the Sobolev space with the norm $\|u\|_{W^{k,p}(\Omega)} = (\sum_{|\alpha|\leq k}\|D^\alpha u\|_{L^p(\Omega)}^p)^{1/p}$, and let $W_0^{k,p}(\Omega)$ denote the closure in $W^{k,p}(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, and $\|\cdot\|$ denotes $L^2(\Omega)$ norm and write equivalent norm $\|\nabla\cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants and it may be different at each appearance.

2. Main Results

In order to state and study our main results, we first define the following functionals:

$$K(u) = \|\nabla u\|_p^p - b\|u\|_r^r, \qquad J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r \tag{2.1}$$

for $u \in W_0^{1,p}(\Omega)$. Then we define the stable set H by

$$H = \left\{u \in W_0^{1,p}(\Omega),\ K(u) > 0\right\} \cup \{0\}. \tag{2.2}$$

We denote the total energy associated with (1.1)–(1.3) by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r = \frac{1}{2}\|u_t\|^2 + J(u) \tag{2.3}$$

for $u \in W_0^{1,p}(\Omega)$, $t \geq 0$, and $E(0) = (1/2)\|u_1\|^2 + J(u_0)$ is the total energy of the initial data.

For latter applications, we list up some lemmas.

Lemma 2.1. *Let $u \in W_0^{1,p}(\Omega)$, then $u \in L^r(\Omega)$ and the inequality $\|u\|_r \leq C\|u\|_{W_0^{1,p}(\Omega)}$ holds with a constant $C > 0$ depending on Ω, p, and r, provided that* (i) *$2 \leq r < +\infty$ if $2 \leq n \leq p$;* (ii) *$2 \leq r \leq np/(n-p)$, $2 < p < n$.*

Lemma 2.2 (see [22]). *Let $y(t) : R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $\beta \geq 1$ and $A > 0$ such that*

$$\int_s^{+\infty} y(t)^{(\beta+1)/2} dt \leq Ay(s), \quad 0 \leq s < +\infty, \tag{2.4}$$

then $y(t) \leq Cy(0)(1+t)^{-2/(\beta-1)}$, for all $t \geq 0$, if $\beta > 1$, and $y(t) \leq Cy(0)e^{-\omega t}$, for all $t \geq 0$, if $\beta = 1$, where C and ω are positive constants independent of $y(0)$.

Lemma 2.3. *Let $u(t,x)$ be a solutions to problem* (1.1)–(1.3). *Then $E(t)$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -a\|u_t(t)\|_q^q. \tag{2.5}$$

Proof. By multiplying (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt}E(u(t)) = -a\|u_t(t)\|_q^q \leq 0. \tag{2.6}$$

Therefore, $E(t)$ is a nonincreasing function on t. □

We need the following local existence result, which is known as a standard one (see [13–15]).

Theorem 2.4. *Suppose that $2 < p < r < np/(n-p)$, $n > p$ and $2 < p < r < \infty$, $n \leq p$. If $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$, then there exists $T > 0$ such that the problem* (1.1)–(1.3) *has a unique local solution $u(t)$ in the class*

$$u \in L^\infty\left([0,T); W_0^{1,p}(\Omega)\right), \qquad u_t \in L^\infty\left([0,T); L^2(\Omega)\right) \cap L^q([0,T); L^q(\Omega)). \tag{2.7}$$

Lemma 2.5. *Assume that the hypotheses in Theorem 2.4 hold, then*

$$\frac{r-p}{rp}\|\nabla u\|_p^p \leq J(u), \tag{2.8}$$

for $u \in H$.

Proof. By the definition of $K(u)$ and $J(u)$, we have the following identity:

$$rJ(u) = K(u) + \frac{r-p}{p}\|\nabla u\|_p^p. \tag{2.9}$$

Since $u \in H$, so we have $K(u) \geq 0$. Therefore, we obtain from (2.9) that

$$\frac{r-p}{rp}\|\nabla u\|_p^p \leq J(u). \tag{2.10}$$

□

Lemma 2.6. *Suppose that* $2 < p < r < np/(n-p)$, $n > p$ *and* $2 < p < r < \infty$, $n \leq p$. *If* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *such that*

$$\theta = bC^r\left(\frac{rp}{r-p}E(0)\right)^{(r-p)/p} < 1, \tag{2.11}$$

then $u(t) \in H$, *for each* $t \in [0,T)$.

Proof. Since $u_0 \in H$, so $K(u_0) > 0$. Then there exists $t_m \leq T$ such that $K(u(t)) \geq 0$ for all $t \in [0, t_m)$. Thus, we get from (2.3) and (2.8) that

$$\frac{r-p}{rp}\|\nabla u\|_p^p \leq J(u) \leq E(t), \tag{2.12}$$

and it follows from Lemma 2.3 that

$$\|\nabla u\|_p^p \leq \frac{rp}{r-p}E(0). \tag{2.13}$$

Next, we easily arrive at from Lemma 2.1, (2.11), and (2.13) that

$$\begin{aligned} b\|u\|_r^r &\leq bC^r\|\nabla u\|_p^r = bC^r\|\nabla u\|_p^{r-p}\|\nabla u\|_p^p \\ &\leq bC^r\left(\frac{rp}{r-p}E(0)\right)^{(r-p)/p}\|\nabla u\|_p^p \\ &= \theta\|\nabla u\|_p^p < \|\nabla u\|_p^p, \quad \forall t \in [0, t_m). \end{aligned} \tag{2.14}$$

Hence

$$\|\nabla u\|_p^p - b\|u\|_r^r > 0, \quad \forall t \in [0, t_m), \tag{2.15}$$

which implies that $u(t) \in H$, for all $t \in [0, t_m)$. By noting that

$$bC^r\left(\frac{rp}{r-p}E(t_m)\right)^{(r-p)/p} < bC^r\left(\frac{rp}{r-p}E(0)\right)^{(r-p)/p} < 1, \tag{2.16}$$

we repeat the steps (2.12)–(2.14) to extend t_m to $2t_m$. By continuing the procedure, the assertion of Lemma 2.6 is proved. □

Theorem 2.7. *Assume that* $2 < p < r < np/(n-p)$, $n > p$ *and* $2 < p < r < \infty$, $n \leq p$. $u(t)$ *is a local solution of problem* (1.1)–(1.3) *on* $[0, T)$. *If* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *satisfy* (2.11), *then the solution* $u(t)$ *is a global solution of the problem* (1.1)–(1.3).

Proof. It suffices to show that $\|u_t\|^2 + \|\nabla u\|_p^p$ is bounded independently of t.

Under the hypotheses in Theorem 2.7, we get from Lemma 2.6 that $u(t) \in H$ on $[0, T)$. So the formula (2.8) in Lemma 2.5 holds on $[0, T)$. Therefore, we have from (2.8) and Lemma 2.3 that

$$\frac{1}{2}\|u_t\|^2 + \frac{r-p}{rp}\|\nabla u\|_p^p \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0). \tag{2.17}$$

Hence, we get

$$\|u_t\|^2 + \|\nabla u\|_p^p \leq \max\left(2, \frac{rp}{r-p}\right)E(0) < +\infty. \tag{2.18}$$

The above inequality and the continuation principle lead to the global existence of the solution, that is, $T = +\infty$. Thus, the solution $u(t)$ is a global solution of the problem (1.1)–(1.3). □

The following theorem shows the asymptotic behavior of global solutions of problem (1.1)–(1.3).

Theorem 2.8. *If the hypotheses in Theorem 2.7 are valid, and* $2 < q < np/(n-p)$, $n > p$ *and* $2 < q < \infty$, $n \leq p$, *then the global solutions of problem* (1.1)–(1.3) *have the following asymptotic behavior:*

$$\lim_{t \to +\infty}\|u_t(t)\| = 0, \qquad \lim_{t \to +\infty}\|\nabla u(t)\|_p = 0. \tag{2.19}$$

Proof. Multiplying by $E(t)^{(q-2)/2}u$ on both sides of (1.1) and integrating over $\Omega \times [S,T]$, we obtain that

$$0 = \int_S^T \int_\Omega E(t)^{(q-2)/2} u \left[u_{tt} + \Delta_p u + a|u_t|^{q-2}u_t - bu|u|^{r-2} \right] dx\, dt, \tag{2.20}$$

where $0 \leq S < T < +\infty$.

Since

$$\begin{aligned} \int_S^T \int_\Omega E(t)^{(q-2)/2} u u_{tt} dx\, dt = {} & \int_\Omega E(t)^{(q-2)/2} u u_t dx \Big|_S^T - \int_S^T \int_\Omega E(t)^{(q-2)/2} |u_t|^2 dx\, dt \\ & - \frac{q-2}{2} \int_S^T \int_\Omega E(t)^{(q-4)/2} E'(t) u u_t dx\, dt, \end{aligned} \tag{2.21}$$

so, substituting the formula (2.21) into the right-hand side of (2.20), we get that

$$\begin{aligned} 0 = {} & \int_S^T \int_\Omega E(t)^{(q-2)/2} \left(|u_t|^2 + \frac{2}{p} |\nabla u|_p^p - \frac{2b}{r} |u|^r \right) dx\, dt \\ & - \int_S^T \int_\Omega E(t)^{(q-2)/2} \left[2|u_t|^2 - a|u_t|^{q-2} u_t u \right] dx\, dt \\ & - \frac{q-2}{2} \int_S^T \int_\Omega E(t)^{(q-4)/2} E'(t) u u_t dx\, dt + \int_\Omega E(t)^{(q-2)/2} u u_t dx \Big|_S^T \\ & + b\left(\frac{2}{r} - 1 \right) \int_S^T E(t)^{(q-2)/2} \|u\|_r^r dt + \frac{p-2}{p} \int_S^T E(t)^{(q-2)/2} \|\nabla u\|_p^p dt. \end{aligned} \tag{2.22}$$

We obtain from (2.14) and (2.12) that

$$\begin{aligned} b\left(1 - \frac{2}{r} \right) \int_S^T E(t)^{(q-2)/2} \|u\|_r^r dt & \leq \theta \frac{r-2}{r} \int_S^T E(t)^{(q-2)/2} \|\nabla u\|_p^p dt \\ & \leq \frac{p(r-2)}{r-p} \theta \int_S^T E(t)^{q/2} dt, \end{aligned} \tag{2.23}$$

$$\frac{p-2}{p} \int_S^T E(t)^{(q-2)/2} \|\nabla u\|_p^p dx\, dt \leq \frac{r(p-2)}{r-p} \int_S^T E(t)^{q/2} dt. \tag{2.24}$$

It follows from (2.22), (2.23), and (2.24) that

$$\begin{aligned}
&\frac{4r-p[(r-2)\theta+r+2]}{r-p}\int_S^T E(t)^{q/2}dt \\
&\quad\le \int_S^T\int_\Omega E(t)^{(q-2)/2}\Big[2|u_t|^2 - a|u_t|^{q-2}u_t u\Big]dx\,dt \\
&\qquad+\frac{q-2}{2}\int_S^T\int_\Omega E(t)^{(q-4)/2}E'(t)uu_t dx\,dt - \int_\Omega E(t)^{(q-2)/2}uu_t dx\Big|_S^T.
\end{aligned}\tag{2.25}$$

We have from Hölder inequality, Lemma 2.1, and (2.17) that

$$\begin{aligned}
&\left|\frac{q-2}{2}\int_S^T\int_\Omega E(t)^{(q-4)/2}E'(t)uu_t dx\,dt\right| \\
&\quad\le \frac{q-2}{2}\int_S^T E(t)^{(q-4)/2}|E'(t)|\left(\frac{C^p rp}{r-p}\cdot\frac{r-p}{rp}\|\nabla u\|_p^p + \frac{1}{2}\|u_t\|^2\right)dt \\
&\quad\le -\frac{q-2}{2}\max\left(\frac{C^p rp}{r-p},1\right)\int_S^T E(t)^{(q-2)/2}E'(t)dt \\
&\quad= -\frac{q-2}{q}\max\left(\frac{C^p rp}{r-p},1\right)E(t)^{q/2}\Big|_S^T \le ME(S)^{q/2},
\end{aligned}\tag{2.26}$$

and similarly, we have

$$\left|-\int_\Omega E(t)^{(q-2)/2}uu_t dx\Big|_S^T\right| \le \max\left(\frac{C^p rp}{r-p},1\right)E(t)^{q/2}\Big|_S^T \le ME(S)^{q/2}.\tag{2.27}$$

Substituting the estimates (2.26) and (2.27) into (2.25), we conclude that

$$\begin{aligned}
&\frac{4r-p[(r-2)\theta+r+2]}{r-p}\int_S^T E(t)^{q/2}dt \\
&\quad\le \int_S^T\int_\Omega E(t)^{(q-2)/2}\Big[2|u_t|^2 - a|u_t|^{q-2}u_t u\Big]dx\,dt + ME(S)^{q/2}.
\end{aligned}\tag{2.28}$$

It follows from $0<\theta<1$ that $(4r-p[(r-2)\theta+r+2])/(r-p)>0$.

We get from Young inequality and Lemma 2.3 that

$$\begin{aligned}2\int_S^T\int_\Omega E(t)^{(q-2)/2}|u_t|^2dx\,dt &\le \int_S^T\int_\Omega\left(\varepsilon_1 E(t)^{q/2}+M(\varepsilon_1)|u_t|^q\right)dx\,dt\\ &\le M\varepsilon_1\int_S^T E(t)^{q/2}dt+M(\varepsilon_1)\int_S^T\|u_t\|_q^q dt\\ &= M\varepsilon_1\int_S^T E(t)^{q/2}dt-\frac{M(\varepsilon_1)}{a}(E(T)-E(S))\\ &\le M\varepsilon_1\int_S^T E(t)^{q/2}dt+ME(S).\end{aligned}\tag{2.29}$$

From Young inequality, Lemmas 2.1 and 2.3, and (2.17), We receive that

$$\begin{aligned}&-a\int_S^T\int_\Omega E(t)^{(q-2)/2}uu_t|u_t|^{q-2}dx\,dt\\ &\quad\le a\int_S^T E(t)^{(q-2)/2}\left(\varepsilon_2\|u\|_q^q+M(\varepsilon_2)\|u_t\|_q^q\right)dt\\ &\quad\le aC^q\varepsilon_2E(0)^{(q-2)/2}\int_S^T\|\nabla u\|_p^q dt+aM(\varepsilon_2)E(S)^{(q-2)/2}\int_S^T\|u_t\|_q^q dt\\ &\quad\le aC^q\varepsilon_2E(0)^{(q-2)/2}\left(\frac{rp}{r-p}\right)^{q/p}\int_S^T E(t)^{q/2}dt+M(\varepsilon_2)E(S)^{q/2}.\end{aligned}\tag{2.30}$$

Choosing small enough ε_1 and ε_2, such that

$$M\varepsilon_1+aC^qE(0)^{(q-2)/2}\left(\frac{rp}{r-p}\right)^{q/p}\varepsilon_2<\frac{4r-p[(r-2)\theta+r+2]}{r-p},\tag{2.31}$$

then, substituting (2.29) and (2.30) into (2.28), we get

$$\int_S^T E(t)^{q/2}dt\le ME(S)+ME(S)^{q/2}\le M(1+E(0))^{(q-2)/2}E(S).\tag{2.32}$$

Therefore, we have from Lemma 2.2 that

$$E(t)\le M(E(0))(1+t)^{-(q-2)/2},\quad t\in[0,+\infty),\tag{2.33}$$

where $M(E(0))$ is a positive constant depending on $E(0)$.

We conclude from (2.17) and (2.33) that $\lim_{t\to+\infty}\|u_t(t)\|=0$ and $\lim_{t\to+\infty}\|\nabla u(t)\|_p=0$. The proof of Theorem 2.8 is thus finished. □

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