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Research Article

Some New Results on Determinantal Inequalities and Applications

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Some new upper and lower bounds on determinants are presented for diagonally dominant matrices and general H-matrices by using different methods. These bounds are some improvements of results given by Ostrowski (1952) and (1937), Price (1951), Wang and Zhang (2002), Huang and Liu (2005), and so forth. In addition, these bounds are also used to localize some numerical characters (e.g., the minimum eigenvalues, singular values and condition numbers) of certain matrices.

1. Introduction

As it is well known, the determinant has a long history of application, which can be traced back to Leibniz (1646–1716), and its properties were developed by Vandermonde (1735–1796), Laplace (1749–1827), Cauchy (1789–1857) Jacobi (1804–1851), and so forth; see [1]. So it has hitherto great influence on every branch of mathematics (see, e.g., [1–4]).

Throughout the paper, let $\mathbb{C}^{m\times n}(\mathbb{R}^{m\times n})$ denote the set of all $m \times n$ complex (real) matrices and $N \triangleq \{1,2,\dots,n\}$. For $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ and any $i\in N$, we define

$$R_i(A)=\sum_{j\neq i}\left|a_{ij}\right|,\qquad r_i(A)=\sum_{j=i+1}^{n}\left|a_{ij}\right|,\qquad l_i(A)=\sum_{j=1}^{i-1}\left|a_{ij}\right|,\qquad \rho_i(A)=\frac{R_i(A)}{|a_{ii}|}. \tag{1.1}$$

According to [5, 6], a matrix $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ is called a diagonally dominant($\mathfrak{D}\mathfrak{D}$) one, if for any $i\in N$, $|a_{ii}|\geq R_i(A)$, and a square matrix A is strictly (row) diagonally dominant($\mathcal{S}\mathfrak{D}\mathfrak{D}$) if $|a_{ii}|>R_i(A)$ for each $i\in N$. A nonsingular M-matrix [7] is a Z-matrix (i.e., all its off-diagonal elements are nonpositive) with nonnegative inverse, and a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is an H-matrix ($\mathscr{H}$) if and only if its comparison matrix $\mathscr{M}(A) = [m_{ij}]$ is a nonsingular M-matrix, where

$$m_{ij} = \begin{cases} |a_{ii}|, & \text{for } i = j, \\ -|a_{ij}|, & \text{for } i \neq j. \end{cases} \tag{1.2}$$

M-matrices and H-matrices have an important role in many fields; see, for example, [5–7].

In addition, there are various generalizations of $\mathcal{SDD}$ class. Recall that a doubly strictly diagonally dominant($\mathcal{DSDD}$) [8] is a matrix such that for all $1 \leqslant i, j \leqslant n (n \geq 2), i \neq j$, one has

$$|a_{ii}||a_{jj}| > R_i(A)R_j(A). \tag{1.3}$$

If there exists a positive diagonal matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ such that $AX \in \mathcal{SDD}$, then A is generalized strictly diagonally dominant($\mathcal{GSDD}$). A $\mathcal{GSDD}$ matrix is nothing but an H-matrix (see [7, page 185]).

The estimation for determinants ($\det A$) is an attractive topic in matrix theory and numerical analysis, especially in mathematical physics, since computers are not very valid for analysis of matrices with parameters, which plays an essential role in various applications (see, [1–3]). Therefore, this problem has been discussed by many articles and some elegant and useful results were obtained as follows.

First, Ostrowski [9] proved, under the hypothesis $A = [a_{ij}] \in \mathcal{SDD}$, that

$$|\det A| \geq \prod_{i=1}^{n} [|a_{ii}| - R_i(A)]. \tag{1.4}$$

Subsequently, Price [10] suggested another new expression as

$$\prod_{i=1}^{n} (|a_{ii}| + r_i(A)) \geq |\det A| \geq \prod_{i=1}^{n} (|a_{ii}| - r_i(A)). \tag{1.5}$$

In [11], the above inequalities (1.4)-(1.5) are improved in such a way, for an arbitrary index $k \in N$, that

$$|a_{kk}| \prod_{i=1}^{k-1} (|a_{ii}| + \rho r_i) \prod_{i=k+1}^{n} (|a_{ii}| + \rho l_i) \geq |\det A| \geq |a_{kk}| \prod_{i=1}^{k-1} (|a_{ii}| - \rho r_i) \prod_{i=k+1}^{n} (|a_{ii}| - \rho l_i), \tag{1.6}$$

where $\rho = \max_i\{\rho_i(A)\}$ with r_i and l_i representing $r_i(A)$ and $l_i(A)$, respectively.

Recently, Huang and Liu [12] presented the following result, for $A \in \mathcal{SDD}$, that

$$\prod_{i=1}^{n} (|a_{ii}| + m_i r_i(A)) \geq |\det A| \geq \prod_{i=1}^{n} (|a_{ii}| - m_i r_i(A)), \quad (a_{n,n+1} = 0), \tag{1.7}$$

where

$$m_i = \max_{i+1 \le p \le n} \frac{|a_{pi}|}{|a_{pp}| - \sum_{j=i+1, \neq p}^{n} |a_{pj}|} < 1, \quad i = 1, 2, \ldots, n-1, \ m_n = 0. \tag{1.8}$$

In 2007, Kolotilina [13] also obtained some interesting results for a subclass, referred to as PBDD$(n1, n2)$, of the class of nonsingular H-matrices.

Inspired by these works, we will exhibit some new upper and lower bounds for determinants with principal diagonal dominant and general H-matrices by using different methods, which improve on the above inequalities (1.4)–(1.7). Finally, these bounds are used to localize some numerical characters, for example, the minimum eigenvalues, singular values, the condition number of matrix, and so forth.

This paper is organized as follows. In Section 2, we present some notations and preliminary results for certain determinants by using different methods. Subsequently, we apply them to estimate for some bounds of some numerical characters of matrices in Section 3.

2. Estimations for Matrix Determinants

First, let us consider the problem on the signs of determinants.

Lemma 2.1 (see [14]). *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and let B be an M-matrix; if $\mathcal{M}(A) \ge B$, then $A \in \mathscr{H}$ and $B^{-1} \ge |A^{-1}|$.*

Theorem 2.2. *For any nonsingular H-matrix, $A = [a_{ij}] \in \mathbb{C}^{n\times n}$; if its determinant is a real number, then* $\det A > 0$ (< 0) *if and only if* $\prod_{i=1}^{n} a_{ii} > 0$ (< 0).

Proof. For any $x \in [0,1]$, set $A(x) = [a_{ij}(x)] \in \mathbb{C}^{n\times n}$, where

$$a_{ij}(x) = \begin{cases} a_{ii}, & \text{for } i = 1, \ldots, n, \\ x a_{ij}, & \text{for } i \neq j, \ i, j = 1, \ldots, n. \end{cases} \tag{2.1}$$

Since A is a nonsingular H-matrix and $x \in [0,1]$, then by Lemma 2.1, $A(x) \in \mathscr{H}$ and nonsingular.

Note that $A(x)$ is a continuous function in $x \in [0,1]$; if

$$\det A(0) \det A(1) = \prod_{i=1}^{n} a_{ii} \det A < 0, \tag{2.2}$$

then there exists a real number $\xi \in (0,1)$ such that $\det A(\xi) = 0$, which is contrary to the fact that $A(\xi)$ is nonsingular. Therefore, we have that

$$\det A(0) \det A(1) = \prod_{i=1}^{n} a_{ii} \det A > 0. \tag{2.3}$$

Thus, the proof is completed. □

Remark 2.3. For these results in this paper, one can obtain much sharper estimates on determinants by computing the signs of determinants using Theorem 2.2.

Next, we establish some new bounds of determinants by using different techniques.

2.1. Determinants and Inverses of Matrices

In this section, we firstly give some lemmas, involving about some inequalities for the entries of matrix A^{-1}. They will be useful in the following proofs.

In addition, for convenience, we will denote by $A^{m,n}$ the principal submatrix of A formed from all rows and all columns with indices between m and n inclusively, for example, $A^{2,n}$ is the submatrix of A obtained by deleting the first row and the first column of A.

Lemma 2.4 (see [15]). *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$; if $A \in \mathcal{SDD}$, then $A^{-1} = [b_{ij}]$ exists and*

$$|b_{ii}| \leq \frac{1}{a_{ii} - \sum_{j\neq i}|a_{ij}|s_{ji}(A)}, \quad \textit{for any } i \in N, \tag{2.4}$$

where

$$s_{ji}(A) = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|\rho_k(A)}{a_{jj}} \leq \rho_j(A), \quad 1 \leq j \neq i \leq n. \tag{2.5}$$

Theorem 2.5. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$; if $A = [a_{ij}] \in \mathcal{SDD}$, then*

$$\prod_{i=1}^{n}\left(|a_{ii}| - \sum_{j=i+1}^{n}|a_{ij}|s_{ji}\left(A^{i,n}\right)\right) \leq |\det A| \leq \prod_{i=1}^{n}\left(|a_{ii}| + \sum_{j=i+1}^{n}|a_{ij}|s_{ji}\left(A^{i,n}\right)\right), \quad (a_{n,n+1} = 0). \tag{2.6}$$

Proof. Our method is very simple. Note that

$$|b_{ii}| = \frac{|\det A_{ii}|}{|\det A|}, \quad \text{for any } i \in N, \tag{2.7}$$

where $A_{ii} \in \mathbb{C}^{(n-1)\times(n-1)}$ denotes the submatrix of A obtained by deleting row i and column i. So, we obtain by Lemma 2.4, for $i = 1$,

$$|\det A| \geq \left(|a_{11}| - \sum_{j\neq 1}|a_{1j}|s_{j1}(A)\right)\left|\det A^{2,n}\right|. \tag{2.8}$$

Since $A \in \mathcal{SDD}$, then $A^{2,n} \in \mathcal{SDD}$. Thus if one applies the induction with respect to k $(k \geq 2)$ to $A^{k,n}$, by using (2.8), then it is not difficult to get the left inequality of (2.6). Similarly, the right inequality of (2.6) can be also proved. □

Note that, for each i, the row dominance factor $\rho_i(A^{k,n})$ for $A^{k,n}$ does not exceed the corresponding factor $\rho_i(A)$ for A (assuming that the original row indices of A remain "attached" to the rows in $A^{k,n}$). Hence the following corollary is obvious.

Corollary 2.6. *If $A = [a_{ij}] \in \mathcal{SDD}$, then*

$$\prod_{i=1}^{n}\left(|a_{ii}| - \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right) \le |\det A| \le \prod_{i=1}^{n}\left(|a_{ii}| + \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right), \quad (a_{n,n+1} = 0). \tag{2.9}$$

Remark 2.7. Obviously, the above results improve the inequalities (1.4)–(1.7). In fact, if $A \in \mathfrak{DD}$ and is nonsingular, then $A + \varepsilon I \in \mathcal{SDD}$ (for any $\varepsilon > 0$). By continuity, one knows that Theorem 2.5 and Corollary 2.6 hold for any nonsingular $\mathfrak{DD}$, too.

In addition, if A is a nonsingular $\mathfrak{DD}$ M-matrix, then we can obtain the sharper bounds.

Theorem 2.8. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a nonsingular $\mathfrak{DD}$ M-matrix, then*

$$\prod_{i=1}^{n}\left(|a_{ii}| - s_1\left(A^{i,n}\right)\right) \le \det A \le \prod_{i=1}^{n}\left(|a_{ii}| + s_1\left(A^{i,n}\right)\right). \tag{2.10}$$

Especially, one has that

$$\prod_{i=1}^{n}(|a_{ii}| - s_i(A) + l_i(A)) \le \det A \le \prod_{i=1}^{n}(|a_{ii}| + s_i(A) - l_i(A)), \tag{2.11}$$

where $s_i(A)$ is defined by the following recursive equations:

$$\begin{gathered} s_n(A) = R_n(A), \\ s_k(A) = \sum_{i=1}^{k-1}|a_{ki}| + \sum_{i=k+1}^{n}|a_{ki}|\frac{s_i(A)}{|a_{ii}|}, \quad k = n-1,\ldots,1. \end{gathered} \tag{2.12}$$

Proof. First, by Theorem 2.2, one knows that $\det A > 0$. Second, by [16, Lemma 2.3], we have that

$$\frac{1}{|a_{11}| + s_1(A)} \le \frac{\det\left(A^{2,n}\right)}{\det A} \le \frac{1}{|a_{11}| - s_1(A)}. \tag{2.13}$$

Similar to the proof of Theorem 2.5, one may deduce inequality (2.10).

Finally, by the definition of $s_i(A)$, it is easy to see that

$$s_1\left(A^{k,n}\right) \leq s_k(A) - l_k(A) - \sum_{i=k+1}^{n} \frac{|a_{ki}| \sum_{j=1}^{k-1} |a_{ij}|}{|a_{ii}|} \leq s_k(A) - l_k(A). \tag{2.14}$$

Therefore, inequality (2.11) is obvious. The proof is completed. □

In addition, it is worthy to mention that there exist some other choices for the number $s_{ji}(A)$ in Lemma 2.4 and Theorem 2.5, which may be better than $s_{ji}(A)$ for some matrices. But they seem complicated for the computation. For example, the number $m_{ji}(A)$ in [15, 17] is given as

$$m_{ji}(A) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| t_k(A)}{a_{jj}}, \quad 1 \leq j \neq i \leq n, \tag{2.15}$$

where

$$t_k(A) = \max_{l \neq k} \left\{ \frac{|a_{lk}|}{|a_{ll}| - \sum_{j \neq l,k} |a_{lj}|} \right\} \leq \max_{l \neq k} \{\rho_l(A)\}. \tag{2.16}$$

2.2. Determinants and the Max-Norm

Now let us consider relationships between determinants and the max-norm of matrices.

Here, for a vector $x = (x_1, x_2, \ldots, x_n)^T$ and a matrix A, $||x||_\infty$ and $||A||_\infty$ mean $||x||_\infty = \max_i\{|x_i|\}$ and $||A||_\infty = \sup_{||x||_\infty=1} ||Ax||_\infty$, respectively.

Lemma 2.9. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an $\mathfrak{DSDD}$ matrix and $B = [b_{ij}] \in \mathbb{C}^{n\times m}$, then*

$$\left\| A^{-1}B \right\|_\infty \leq \max_{i \neq j} \left\{ \frac{R_i(A)\Lambda_j(B) + |a_{jj}|\Lambda_i(B)}{|a_{ii}a_{jj}| - R_i(A)R_j(A)} \right\}, \tag{2.17}$$

where $\Lambda_i(B) = \sum_{j=1}^{m} |b_{ij}|$.

Proof. By the definition of the matrix norm, there exists an m-dimensional vector $x = (x_1, \ldots, x_m)^T$ with $||x||_\infty = 1$ such that

$$\left\| A^{-1}B \right\|_\infty = \left\| A^{-1}Bx \right\|_\infty = \|y\|_\infty = y_i, \tag{2.18}$$

where $A^{-1}Bx = y$ and $y = (y_1, \ldots, y_n)^T$. Now denote that $j \in \{k \mid \max_{k \neq i} |y_k|\}$, then

$$\sum_{l=1}^{m} b_{il} x_l = a_{ii} y_i + \sum_{l \neq i,j} a_{il} y_l + a_{ij} y_j, \tag{2.19a}$$

$$\sum_{l=1}^{m} b_{jl} x_l = a_{jj} y_j + \sum_{l \neq i,j} a_{jl} y_l + a_{ji} y_i, \tag{2.19b}$$

that is,

$$\sum_{l=1}^{m} |b_{il}| \geq |a_{ii}| |y_i| - \sum_{l \neq i} |a_{il}| |y_j|, \tag{2.20a}$$

$$\sum_{l=1}^{m} |b_{jl}| \geq |a_{jj}| |y_j| - \sum_{l \neq j} |a_{jl}| |y_i|. \tag{2.20b}$$

Then substituting (2.20a) in (2.20b), we have that

$$|y_i| \leq \frac{R_i(A)\Lambda_j(B) + |a_{jj}|\Lambda_i(B)}{|a_{ii}a_{jj}| - R_i(A)R_j(A)}. \tag{2.21}$$

The proof is completed. □

Corollary 2.10. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is an $\mathfrak{D}\mathcal{S}\mathfrak{D}\mathfrak{D}$ matrix, then*

$$\left\| A^{-1} \right\|_\infty \leq \max_{i \neq j} \left\{ \frac{|a_{jj}| + R_i(A)}{|a_{ii}a_{jj}| - R_i(A)R_j(A)} \right\}. \tag{2.22}$$

Remark 2.11. In [18], Corollary 2.10 has been proved in the $\mathcal{S}\mathfrak{D}\mathfrak{D}$ case. In fact, one can easily prove that the bound (2.22) is better than the following classical Ahlberg-Nilson-Varah [19] bound for any $\mathcal{S}\mathfrak{D}\mathfrak{D}$ matrix:

$$\left\| A^{-1} \right\|_\infty \leq \max_{i} \left\{ \frac{1}{|a_{ii}| - R_i(A)} \right\}. \tag{2.23}$$

The following theorem is analogous to (1.7).

Theorem 2.12. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an $\mathfrak{D}\mathcal{S}\mathfrak{D}\mathfrak{D}$ matrix, then*

$$\prod_{i=1}^{n} \left(|a_{ii}| + s_i\left(A^{i,n}\right) r_i(A) \right) \geq |\det A| \geq \prod_{i=1}^{n} \left(|a_{ii}| - s_i\left(A^{i,n}\right) r_i(A) \right), \tag{2.24}$$

where

$$s_m(A) = \max_{m+1\le i\ne j\le n}\left\{\frac{|a_{jj}||a_{im}| + R_i(A)|a_{jm}|}{|a_{ii}a_{jj}| - R_i(A)R_j(A)}\right\},\quad m = 1,2,\dots,n-1,\ s_n(A) = 0. \tag{2.25}$$

Proof. Let

$$A = \begin{bmatrix} a_{11} & \alpha^T \\ \beta & A^{2,n} \end{bmatrix}, \tag{2.26}$$

where $\alpha^T = (a_{12},\dots,a_{1n})$, $\beta = (a_{21},\dots,a_{n1})^T$. By Schur's Theorem (see [5]), we have that

$$\det A = \det A^{2,n}\cdot\left(a_{11} - \alpha^T\left(A^{2,n}\right)^{-1}\beta\right). \tag{2.27}$$

Then

$$\left(|a_{11}| - \left|\alpha^T\left(A^{2,n}\right)^{-1}\beta\right|\right)\left|\det A^{2,n}\right| \le |\det A| \le \left(|a_{11}| + \left|\alpha^T\left(A^{2,n}\right)^{-1}\beta\right|\right)\left|\det A^{2,n}\right|. \tag{2.28}$$

In addition, by Lemma 2.9, we get that

$$\begin{aligned}\left|\alpha^T\left(A^{2,n}\right)^{-1}\right| &\le \|\alpha\|_1\left\|(A^{2,n})^{-1}\beta\right\|_\infty \\ &= \sum_{j=2}^{n}|a_{1j}|\left\|(A^{2,n})^{-1}\beta\right\|_\infty \\ &\le \sum_{j=2}^{n}|a_{1j}|\cdot\max_{2\le i\ne j\le n}\frac{R_i(A)|a_{j1}| + |a_{jj}||a_{i1}|}{|a_{ii}a_{jj}| - R_i(A)R_j(A)} \\ &= s_1(A)\sum_{j=2}^{n}|a_{1j}|.\end{aligned} \tag{2.29}$$

Thus substituting (2.29) in (2.28) and applying (2.28) to $A^{2,n}$, one can get inequality (2.24) by induction. The proof is completed. □

In fact, the problem of bounding $\|A^{-1}\|_\infty$ satisfying certain assumptions was considered in some literature; see [13, 16, 18, 20]. Recently, Kolotilina [21] also obtained some interesting results for the so-called PM- and PH-matrices, which form a subclass of nonsingular M- and H-matrices, respectively.

Let $A = [a_{ij}] \in \mathbb{C}^{m\times m}$, $(m \ge 1)$, and let

$$\langle m\rangle = \bigcup_{i=1}^{n} M_i,\quad 1 \le n \le m \tag{2.30}$$

be a partitioning of the index set $\langle m\rangle = \{1,\ldots,m\}$ into disjoint nonempty subsets. Denote that $A_{ij} = A[M_i, M_j],\ i,j = 1,\ldots,n$ and represent A in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}. \tag{2.31}$$

Then the following result was obtained in [21].

Lemma 2.13 (see [21]). *For the block matrix A defined by* (2.31)*, let A_{ii} be nonsingular for all $i = 1,\ldots,n$. Then*

$$\left\|A^{-1}\right\|_\infty \le \left\|\widetilde{N}(A)^{-1}\right\|_\infty, \tag{2.32}$$

where

$$\widetilde{N}(A) = \begin{bmatrix} \|A_{11}^{-1}\|_\infty^{-1} & -\|A_{12}\|_\infty & \cdots & -\|A_{1n}\|_\infty \\ -\|A_{21}\|_\infty & \|A_{22}^{-1}\|_\infty^{-1} & \cdots & -\|A_{2n}\|_\infty \\ \cdots & \cdots & \cdots & \cdots \\ -\|A_{n1}\|_\infty & -\|A_{n2}\|_\infty & \cdots & \|A_{nn}^{-1}\|_\infty^{-1} \end{bmatrix}. \tag{2.33}$$

Obviously, by Lemma 2.13, many of results on $\|A^{-1}\|_\infty$ can be generalized to the block case. Note that one usually needs to compute many good inverses of submatrices $A_{ii}(1 \le i \le n)$ for a large matrix. However, the result (2.32) can not be improved to $\|A^{-1}\|_\infty \le \|N(A)^{-1}\|_\infty$, since we have that $\|N(A)^{-1}\|_\infty \le \|\widetilde{N}(A)^{-1}\|_\infty$, where

$$N(A) = \begin{bmatrix} \|A_{11}\|_\infty & -\|A_{12}\|_\infty & \cdots & -\|A_{1n}\|_\infty \\ -\|A_{21}\|_\infty & \|A_{22}\|_\infty & \cdots & -\|A_{2n}\|_\infty \\ \cdots & \cdots & \cdots & \cdots \\ -\|A_{n1}\|_\infty & -\|A_{n2}\|_\infty & \cdots & \|A_{nn}\|_\infty \end{bmatrix}. \tag{2.34}$$

For example, let us consider the following block-partitioned matrix:

$$A = \begin{bmatrix} 1 & -1 & \vdots & 0 \\ 0 & 0.5 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{2.35}$$

It is easy to compute that $\|A^{-1}\|_\infty = 2 \nleq \|N(A)^{-1}\|_\infty = 1$, but $\|A^{-1}\|_\infty = 2 \le \|\widetilde{N}(A)^{-1}\|_\infty = 2$. In the same time, it shows that the bound (2.32) is sharp.

2.3. Determinants of M- and H-Matrices

First, according to the proof of Theorem 2.12, one may further obtain the following conclusion for general M-matrices.

Theorem 2.14. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a nonsingular M-matrix, then*

$$\det A \le \prod_{i=1}^{n}\left(a_{ii} - \sum_{j=i+1}^{n} a_{ij} z_j\left(A^{i,n}\right)\right) \le \prod_{i=1}^{n} a_{ii}, \tag{2.36}$$

where

$$\begin{gathered} z_n\left(A^{i,n}\right) = \frac{a_{ni}}{a_{nn}}, \\ z_k\left(A^{i,n}\right) = \frac{1}{a_{kk}}\left(a_{ki} + \sum_{j=k+1}^{n} |a_{kj}| z_j\left(A^{i,n}\right)\right), \quad k = n-1, \dots, i+1. \end{gathered} \tag{2.37}$$

Proof. Let A be partitioned into

$$A = \begin{bmatrix} a_{11} & x^T \\ y & A^{2,n} \end{bmatrix}, \tag{2.38}$$

then

$$\det A = \left(a_{11} - x^T\left(A^{2,n}\right)^{-1} y\right)\det A^{2,n}. \tag{2.39}$$

Let $A^{2,n} = D - U - L$, where D, $-U$, and $-L$ are diagonal, strict upper and strict lower triangular parts of $A^{2,n}$, respectively. Since $A^{2,n} \le D - U$ and $A^{2,n}$ is also nonsingular M-matrix, then, by Lemma 2.1, $(A^{2,n})^{-1} \ge (D-U)^{-1}$. So

$$x^T\left(A^{2,n}\right)^{-1} y \ge x^T (D-U)^{-1} y. \tag{2.40}$$

Denote that $(D-U)^{-1}y = (z_2, \dots, z_n)^T \triangleq z$, then $Dz = y + Uz$, that is,

$$\begin{gathered} z_n = \frac{a_{n1}}{a_{nn}}, \\ z_k = \frac{1}{a_{kk}}\left(a_{k1} + \sum_{j=k+1}^{n} |a_{kj}| z_j\right), \quad k = n-1, \dots, 2. \end{gathered} \tag{2.41}$$

So

$$a_{11} - x^T \left(A^{2,n}\right)^{-1} y \le a_{11} - \sum_{j=2}^{n} a_{1j} z_j, \tag{2.42}$$

thus,

$$\det A \le \left(a_{11} - \sum_{j=2}^{n} \left|a_{1j}\right| z_j \right) \det A^{2,n}. \tag{2.43}$$

Applying the induction with respect to k, $(k \ge 2)$ to $A^{k,n}$, the result (2.36) is obtained and the proof is completed. □

In the above proof, if we replace $(A^{2,n})^{-1} \ge (D - U)^{-1}$ with $(A^{2,n})^{-1} \ge D^{-1}$, then the following result can be obtained.

Corollary 2.15. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a nonsingular M-matrix, then*

$$\det A \le a_{11} \prod_{i=2}^{n} \left(a_{ii} - \sum_{j=1}^{i-1} \frac{a_{ji} a_{ij}}{a_{jj}} \right) \le \prod_{i=1}^{n} a_{ii}, \tag{2.44}$$

or

$$\det A \le a_{nn} \prod_{i=1}^{n-1} \left(a_{ii} - \sum_{j=i+1}^{n} \frac{a_{ij} a_{ji}}{a_{jj}} \right) \le \prod_{i=1}^{n} a_{ii}. \tag{2.45}$$

Corollary 2.16. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a nonsingular M-matrix, then $A^{-1} = [b_{ij}]$ exists and*

$$b_{ii} \ge \frac{1}{a_{ii} - \sum_{k=1,\neq i}^{n} a_{ik} a_{ki} / a_{kk}} \ge \frac{1}{a_{ii}}, \quad i \in N. \tag{2.46}$$

Next, let us consider some H-matrices. For a general H-matrix A, as it is well known, there exists a positive diagonal matrix X such that $X^{-1}AX \in \mathcal{SDD}$. For example, the following S-strictly diagonally dominant matrices ($\mathcal{S}$-$\mathcal{SD}$) are illustrated.

Definition 2.17 (see [6, 22]). Given any nonempty subset S of N, let $\overline{S}$ denote its complement in N. If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \ge 2)$ satisfies

(1) $|a_{ii}| > R_i^S$, for all $i \in S$,

(2) $(|a_{ii}| - R_i^S)(|a_{jj}| - R_j^{\overline{S}}) > R_i^{\overline{S}} R_j^S$, for all $i \in S$, $j \in \overline{S}$,

then A is said to be an $\mathcal{S}$-$\mathcal{SD}$ matrix and nonsingular, where

$$R_i^S := \sum_{j\in S\setminus\{i\}} \left|a_{ij}\right|, \qquad R_i^{\overline{S}} := \sum_{j\in \overline{S}\setminus\{i\}} \left|a_{ij}\right|. \tag{2.47}$$

For $\mathcal{S}$-$\mathcal{SD}$ matrices, we may let $X = \operatorname{diag}(x_1, x_2, \dots, x_n)$, where

$$x_i := \begin{cases} \gamma & \text{when } i \in S, \\ 1 & \text{when } i \in \overline{S}. \end{cases} \tag{2.48}$$

$$\max_{i \in S} \left\{ \frac{R_i^{\overline{S}}}{|a_{ii}| - R_i^S} \right\} < \gamma < \min_{j \in \overline{S}} \left\{ \frac{|a_{jj}| - R_j^{\overline{S}}}{R_j^S} \right\}, \tag{2.49}$$

then $X^{-1}AX \in \mathcal{SDD}$. In addition, analogous to [12], one may choose the permutation matrix P such that

$$Y = P^T XP = \operatorname{diag}(y_1, \dots, y_n), \tag{2.50}$$

where the entries y_i are monotonically ordered as $y_1 \geq y_2 \geq \cdots \geq y_n$. Thus, all results on $\mathcal{SDD}$ also hold for the general H-matrix A, since $\det A = \det(XP)^{-1}A(XP)$. For example, we apply Corollary 2.6 to $C = (XP)^{-1}A(XP)$, then we obtain the following result.

Theorem 2.18. *For an H-matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, it holds that*

$$\prod_{i=1}^{n} \left(|a_{ii}| - \sum_{j=i+1}^{n} |b_{ij}| \mu_j \right) \leq |\det A| \leq \prod_{i=1}^{n} \left(|a_{ii}| + \sum_{j=i+1}^{n} |b_{ij}| \mu_j \right), \quad (b_{n,n+1} = 0), \tag{2.51}$$

where $\mu_j = R_j(B)/|a_{jj}|$, $j \in N$, and $B = P^T AP = [b_{ij}]$; P is as in (2.50).

Finally, it is mentioned that, for many matrices which are not diagonally dominant, specially when the off-diagonal entries of each row have close values, one may make use of B-matrices to obtain better lower bounds of determinant.

Definition 2.19 (see [23, 24]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, then one may write it as $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} a_{11} - r_1^+ & a_{12} - r_1^+ & \cdots & a_{1n} - r_1^+ \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} - r_n^+ & a_{n2} - r_n^+ & \cdots & a_{nn} - r_n^+ \end{bmatrix}, \qquad C = \begin{bmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & & \vdots \\ r_n^+ & \cdots & r_n^+ \end{bmatrix}. \tag{2.52}$$

If $B^+ \in \mathcal{SDD}$, then A is called a B-matrix (denoted by $A \in \mathcal{B}$) and $\det A \geq \det B^+$, where $r_i^+ = \max\{0, a_{ij} \mid \text{for all } j \neq i\}$, $i \in N$.

For example, let us consider the following two matrices:

$$A_1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}. \tag{2.53}$$

Obviously, they are not strictly diagonally dominant matrices neither are they H-matrices. Now by Definition 2.19, we, respectively, have that

$$B_1^+ = \begin{bmatrix} 0.5 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad B_2^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.54}$$

that is, A_1 and A_2 are both B-matrices. According to Corollary 2.6 (or other results) and Definition 2.19, we further have that

$$\det A_1 \geq \det B_1^+ = 0.5, \qquad \det A_2 \geq \det B_2^+ = 1.0. \tag{2.55}$$

3. Applications to Some Estimations for Numerical Characters of Matrices

In this section, we will apply some results in Section 2 to get some simple and interesting estimates for some numerical characters of matrices. Regarding other applications such as the stability of finite and infinite dimensional systems and the solutions of nonlinear equations of mathematical physics, we refer readers to [1, 2, 25] for full details.

For convenience, we will use the following notations and definitions. For $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, we denote by $\|A\|_F$, λ_i and σ_i, for all $i \in N$, Frobenius norm, its eigenvalues and singular values, respectively. And we assume that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|, \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n. \tag{3.1}$$

Set $S = \min_{i\in N}\{i \mid |a_{ii}| + R_i(A)\}$, for any $k \in S$, and define $T_i = |a_{ii}| + R_i(A)$ for all $i \in N \setminus \{k\}$. Clearly, the value of $|\lambda_n|$ or σ_n can serve as a kind of measure for the nonsingularity of A. Especially, the smallest eigenvalue can characterize certain properties of corresponding physical systems. For example, it represents decay rates of signals in linear electrical circuits [26]. In this section, we find their lower bounds.

Theorem 3.1. *If $A = [a_{ij}] \in \mathcal{SDD}$, then $A^{-1} = [c_{ij}]$ exists and*

$$|\lambda_n| \geq \frac{1}{n\max_{1\leq i\leq n}\{|c_{ii}|\}} \geq \frac{1}{n}\min_{1\leq i\leq n}\left\{ |a_{ii}| - \sum_{j\neq i}|a_{ij}|s_{ji} \right\}, \tag{3.2}$$

where s_{ji} is as in Lemma 2.4. If $A \in \mathcal{H}$, then

$$|\lambda_n| \geq \frac{\prod_{i=1}^{n}\left(|a_{ii}| - \sum_{j=i+1}^{n}|b_{ij}|\mu_j\right)}{\prod_{i=1}^{n-1}T_i}, \tag{3.3}$$

where μ_j, $j \in N$, and $B = [b_{ij}]$ are as in Theorem 2.18.

Proof. For (3.2), by [27, Theorem 10], we have that

$$|\lambda_n| \geq \frac{|\det A|}{n \max_{1\leq i\leq n}\{|\det A_{ii}|\}}. \tag{3.4}$$

Note that $|c_{ii}| = |\det A_{ii}|/|\det A|$. According to Lemma 2.4 or Theorem 2.5, the conclusion (3.2) can be easily followed.

Similarly, by [27, Theorem 8],

$$|\lambda_n| \geq \frac{|\det A|}{\prod_{i=1}^{n-1} T_i}, \tag{3.5}$$

and by Theorem 2.18, the result (3.3) also holds. □

In addition, we may apply the above results to investigate the estimation of singular values and the condition number $\operatorname{cond}(A) \triangleq \sigma_1/\sigma_n$, since, for any matrix $A = [a_{ij}]$,

$$|\det(A)| = \sigma_1\sigma_1 \cdots \sigma_n, \tag{3.6}$$

$$\max\left\{1, \frac{\|A\|_F}{n|\det A|^{1/n}}\right\} \leq \operatorname{cond}(A) \leq \frac{\|A\|_F^n}{|\det A|} \tag{3.7}$$

(see [5]). For example, applying Corollary 2.6 to (3.6) and (3.7), respectively, we have the following results.

Theorem 3.2. *If $A = [a_{ij}] \in \mathcal{S}\mathfrak{D}\mathfrak{D}$, then*

$$\begin{aligned}
\sigma_1 &\geq \left(\prod_{i=1}^{n}\left(|a_{ii}| - \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right)\right)^{1/n}, \\
\sigma_n &\leq \left(\prod_{i=1}^{n}\left(|a_{ii}| + \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right)\right)^{1/n}, \quad (a_{n,n+1} = 0),
\end{aligned} \tag{3.8}$$

$$\max\left\{1, \frac{\|A\|_F}{n\prod_{i=1}^{n}\left(|a_{ii}| + \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right)^{1/n}}\right\} \leq \operatorname{cond}(A) \leq \frac{\|A\|_F^n}{\prod_{i=1}^{n}\left(|a_{ii}| - \sum_{j=i+1}^{n}|a_{ij}|\rho_j(A)\right)}. \tag{3.9}$$

Finally, let us recall another result on the estimate of the smallest singular values. In 2002, an interesting result for a nonsingular complex matrix A of order n as a function of $\det A$, $\|A\|_F$, and k singular values is due to Piazza and Politi [28]:

$$\operatorname{cond}(A) \leq \frac{2^k}{\prod_{i=2}^{k}\sigma_i}\frac{1}{|\det A|}\left(\frac{\|A\|_F}{\sqrt{n+k-1}}\right)^{n+k-1}, \quad \text{for } 1 \leq k \leq n-1, \tag{3.10}$$

where $\prod_{i=2}^{k=1}\sigma_i = 1$.

So, by (3.10), let $k = n - 1$ and note that (3.6), then

$$\frac{1}{\sigma_n^2} \leq \frac{2^{n-1}}{|\det A|^2} \left(\frac{\|A\|_F}{\sqrt{2n-2}} \right)^{2n-2}, \tag{3.11}$$

which implies that

$$\sigma_n \geq |\det A| \left(\frac{\sqrt{n-1}}{\|A\|_F} \right)^{n-1}. \tag{3.12}$$

Applying those inequalities on determinants in Section 2 to (3.12), one may further obtain many more interesting conclusions.

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References

[1] R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*, vol. 134 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1999.

[2] M. I. Gil', *Operator Functions and Localization of Spectra*, vol. 1830 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2003.

[3] M. I. Gil' and S. S. Cheng, "Solution estimates for semilinear difference-delay equations with continuous time," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 82027, 8 pages, 2007.

[4] C. Krattenthaler, "Advanced determinant calculus: a complement," *Linear Algebra and Its Applications*, vol. 411, pp. 68–166, 2005.

[5] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[6] R. S. Varga, *Geršgorin and His Circles*, vol. 36 of *Springer Series in Computational Mathematics*, Springer, Berlin, Germany, 2004.

[7] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, vol. 9 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[8] B. Li and M. J. Tsatsomeros, "Doubly diagonally dominant matrices," *Linear Algebra and Its Applications*, vol. 261, pp. 221–235, 1997.

[9] A. M. Ostrowski, "Sur la determination des borns inferieures pour une class des determinants," *Bulletin des Sciences Mathmatiques*, vol. 61, no. 2, pp. 19–32, 1937.

[10] G. B. Price, "Bounds for determinants with dominant principal diagonal," *Proceedings of the American Mathematical Society*, vol. 2, pp. 497–502, 1951.

[11] A. M. Ostrowski, "Note on bounds for determinants with dominant principal diagonal," *Proceedings of the American Mathematical Society*, vol. 3, pp. 26–30, 1952.

[12] T.-Z. Huang and X.-P. Liu, "Estimations for certain determinants," *Computers & Mathematics with Applications*, vol. 50, no. 10–12, pp. 1677–1684, 2005.

[13] L. Yu. Kolotilina, "Bounds for the determinants and inverses of some H-matrices," *Zapiski Nauchnykh Seminarov (POMI)*, vol. 346, no. 1, pp. 81–102, 2007.

[14] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1991.

[15] H.-B. Li, T.-Z. Huang, S.-Q. Shen, and H. Li, “Lower bounds for the minimum eigenvalue of Hadamard product of an M-matrix and its inverse,” *Linear Algebra and Its Applications*, vol. 420, no. 1, pp. 235–247, 2007.

[16] W. Li, “The infinity norm bound for the inverse of nonsingular diagonal dominant matrices,” *Applied Mathematics Letters*, vol. 21, no. 3, pp. 258–263, 2008.

[17] Y.-T. Li, F.-B. Chen, and D.-F. Wang, “New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse,” *Linear Algebra and Its Applications*, vol. 430, no. 4, pp. 1423–1431, 2009.

[18] C.-L. Wang and G.-J. Zhang, “Some simple estimates for the singular values of matrices,” *Acta Mathematicae Applicatae Sinica*, vol. 18, no. 1, pp. 117–122, 2002.

[19] J. M. Varah, “A lower bound for the smallest singular value of a matrix,” *Linear Algebra and Its Applications*, vol. 11, pp. 3–5, 1975.

[20] T.-Z. Huang, “Estimation of $\|A^{-1}\|_\infty$ and the smallest singular value,” *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1075–1080, 2008.

[21] L. Yu. Kolotilina, “Bounds for the infinity norm of the inverse for certain M- and H-matrices,” *Linear Algebra and Its Applications*, vol. 430, no. 2-3, pp. 692–702, 2009.

[22] N. Morača, “Upper bounds for the infinity norm of the inverse of SDD and $\mathcal{S}$-SDD matrices,” *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 666–678, 2007.

[23] J. M. Peña, “A class of P-matrices with applications to the localization of the eigenvalues of a real matrix,” *SIAM Journal on Matrix Analysis and Applications*, vol. 22, no. 4, pp. 1027–1037, 2001.

[24] H.-B. Li, T.-Z. Huang, and H. Li, “On some subclasses of P-matrices,” *Numerical Linear Algebra with Applications*, vol. 14, no. 5, pp. 391–405, 2007.

[25] J. Z. Liu, J. Zhang, and Y. Liu, “Trace inequalities for matrix products and trace bounds for the solution of the algebraic Riccati equations,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 101085, 17 pages, 2009.

[26] P. N. Shivakumar, J. J. Williams, Q. Ye, and C. A. Marinov, “On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics,” *SIAM Journal on Matrix Analysis and Applications*, vol. 17, no. 2, pp. 298–312, 1996.

[27] Z. C. Shi and B. Y. Wang, “Several numerical characters of certain classes of matrices,” *Acta Mathematica Sinica*, vol. 15, no. 3, pp. 326–341, 1965 (Chinese).

[28] G. Piazza and T. Politi, “An upper bound for the condition number of a matrix in spectral norm,” *Journal of Computational and Applied Mathematics*, vol. 143, no. 1, pp. 141–144, 2002.