Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 845390, 8 pages
doi:10.1155/2010/845390

Research Article

Multiplicative Concavity of the Integral of Multiplicatively Concave Functions

Yu-Ming Chu[1] and Xiao-Ming Zhang[2]

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou, Zhejiang 313000, China*
[2] *Haining College, Zhejiang TV University, Haining, Zhejiang 314400, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

Received 25 March 2010; Accepted 7 June 2010

Academic Editor: S. S. Dragomir

We prove that $G(x,y) = |\int_y^x f(t)dt|$ is multiplicatively concave on $[a,b] \times [a,b]$ if $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is continuous and multiplicatively concave.

1. Introduction

For convenience of the readers, we first recall some definitions and notations as follows.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A real-valued function $f : I \to \mathbb{R}$ is said to be convex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.1}$$

for all $x, y \in I$. And f is called concave if $-f$ is convex.

Definition 1.2. Let $I \subseteq (0,\infty)$ be an interval. A real-valued function $f : I \to (0,\infty)$ is said to be multiplicatively convex if

$$f(\sqrt{xy}) \leq \sqrt{f(x)f(y)} \tag{1.2}$$

for all $x, y \in I$. And f is called multiplicatively concave if $1/f$ is multiplicatively convex.

For $x = (x_1, x_2) \in \mathbb{R}_+^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ and $\alpha \geq 0$, we denote

$$\begin{aligned} \log x &= (\log x_1, \log x_2), \\ x^\alpha &= (x_1^\alpha, x_2^\alpha), \end{aligned} \tag{1.3}$$

$$e^x = (e^{x_1}, e^{x_2}). \tag{1.4}$$

For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we denote

$$xy = (x_1 y_1, x_2 y_2). \tag{1.5}$$

Definition 1.3. A set $E_1 \subseteq \mathbb{R}^2$ is said to be convex if $(x + y)/2 \in E_1$ whenever $x, y \in E_1$. And a set $E_2 \subseteq \mathbb{R}^2$ is said to be multiplicatively convex if $x^{1/2} y^{1/2} \in E_2$ whenever $x, y \in E_2$.

From Definition 1.3 we clearly see that $E_1 \subseteq \mathbb{R}_+^2$ is a multiplicatively convex set if and only if $\log E_1 = \{\log x : x \in E_1\}$ is a convex set, and $E_2 \subseteq \mathbb{R}^2$ is a convex set if and only if $e^{E_2} = \{e^x : x \in E_2\}$ is a multiplicatively convex set.

Definition 1.4. Let $E \subseteq \mathbb{R}^2$ be a convex set. A real-valued function $f : E \to \mathbb{R}$ is said to be convex if

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{1.6}$$

for all $x, y \in E$. And f is said to be concave if $-f$ is convex.

Definition 1.5. Let $E \subseteq \mathbb{R}_+^2$ be a multiplicatively convex set. A real-valued function $f : E \to (0, \infty)$ is said to be multiplicatively convex if

$$f\left(x^{1/2} y^{1/2}\right) \leq f^{1/2}(x) f^{1/2}(y) \tag{1.7}$$

for all $x, y \in E$. And f is called multiplicatively concave if $1/f$ is multiplicatively convex.

From Definitions 1.1 and 1.2, the following Theorem A is obvious.

Theorem A. *Suppose that I is a subinterval of $(0, \infty)$ and $f : I \to (0, \infty)$ is multiplicatively convex. Then*

$$F(x) = \log \circ f \circ \exp : \log(I) \longrightarrow \mathbb{R} \tag{1.8}$$

is convex. Conversely, if J is an interval and $F : J \to \mathbb{R}$ is convex, then

$$f = \exp \circ F \circ \log : \exp(J) \longrightarrow (0, \infty) \tag{1.9}$$

is multiplicatively convex.

Equivalently, f is a multiplicatively convex function if and only if $\log f(x)$ is a convex function of $\log x$. Modulo this characterization, the class of all multiplicatively convex functions was first considered by Motel [1], in a beautiful paper discussing the analogues of the notion of convex function in n variables. However, the roots of the research in this area can be traced long before him. In a long time, the subject of multiplicative convexity seems to be even forgotten, which is a pity because of its richness. Recently, the multiplicative convexity has been the subject of intensive research. In particular, many remarkable inequalities were found via the approach of multiplicative convexity (see [2–18]).

The main purpose of this paper is to prove Theorem 1.6.

Theorem 1.6. *If $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is continuous and multiplicatively concave, then $G(x,y) = |\int_x^y f(t)dt|$ is multiplicatively concave on $[a,b] \times [a,b]$.*

2. Lemmas and the Proof of Theorem 1.6

For the sake of readability, we first introduce and establish several lemmas which will be used to predigest the proof of Theorem 1.6.

Lemma 2.1 can be derived from Definitions 1.4 and 1.5.

Lemma 2.1. *If $E_1 \subset \mathbb{R}_+^2$ is a multiplicatively convex set, and $f : E_1 \to (0,\infty)$ is multiplicatively convex (or concave, resp.), then $F(x) = \log f(e^x)$ is convex (or concave, resp.) on $\log E_1 = \{\log x : x \in E_1\}$. Conversely, if $E_2 \subset \mathbb{R}^2$ is a convex set, and $F : E_2 \to \mathbb{R}$ is convex (or concave, resp.), then $f(x) = e^{F(\log x)}$ is multiplicatively convex (or concave, resp.) on $e^{E_2} = \{e^x : x \in E_2\}$.*

Lemma 2.2 (see [19]). *If $E \subset \mathbb{R}^2$ is a convex set, and $f : E \to \mathbb{R}$ is second-order differentiable, then f is convex (or concave, resp.) if and only if $L(x)$ is a positive (or negative, resp.) semidefinite matrix for all $x = (x_1, x_2) \in E$. Here*

$$L(x) = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix}, \tag{2.1}$$

and $f''_{ij} = \partial^2 f(x_1,x_2)/\partial x_i \partial x_j$, $i,j = 1,2$.

Making use of Lemmas 2.1 and 2.2 we get the following Lemma 2.3.

Lemma 2.3. *If $E \subset \mathbb{R}_+^2$ is a multiplicatively convex set, and $f : E \to (0,\infty)$ is second-order differentiable, then f is multiplicatively convex (or concave, resp.) if and only if $J(x)$ is a positive (or negative, resp.) semidefinite matrix for all $x = (x_1,x_2) \in E$. Here*

$$J(x) = \begin{pmatrix} f f''_{11} + \dfrac{f}{x_1} f'_1 - f'^2_1 & f f''_{12} - f'_1 f'_2 \\ f f''_{21} - f'_1 f'_2 & f f''_{22} + \dfrac{f}{x_2} f'_2 - f'^2_2 \end{pmatrix}, \tag{2.2}$$

$f''_{ij} = \partial f(x_1,x_2)/\partial x_i \partial x_j$, and $f'_i = \partial f(x_1,x_2)/\partial x_i$, $i,j = 1,2$.

Lemma 2.4 (see [2]). *If $I \subset (0,\infty)$ is an interval and $f : I \to (0,\infty)$ is differentiable, then f is multiplicatively convex (or concave, resp.) if and only if $xf'(x)/f(x)$ is increasing (or decreasing,* *resp.) on I. If moreover f is second-order differentiable, then f is multiplicaively convex (or concave, resp.) if and only if*

$$x\left[f(x)f''(x)-f'^{2}(x)\right]+f(x)f'(x)\geq (or\leq, resp.)0 \tag{2.3}$$

for all $x\in I$.

Lemma 2.5. *Suppose that $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a second-order differentiable multiplicatively concave function. If $g(x)=\int_a^x f(t)dt$, then g is also multiplicatively concave on $[a,b]$.*

Proof. For $x\in[a,b]$, from the expression of $g(x)$ we get

$$x\left[g(x)g''(x)-g'^{2}(x)\right]+g(x)g'(x)=\left[xf'(x)+f(x)\right]\int_a^x f(t)dt-xf^2(x). \tag{2.4}$$

According to Lemma 2.4, to prove that $g(x)$ is multiplicatively concave on $[a,b]$, it is sufficient to prove that

$$\left[xf'(x)+f(x)\right]\int_a^x f(t)dt-xf^2(x)\leq 0 \tag{2.5}$$

for all $x\in[a,b]$.

Next, set

$$\begin{aligned} E&=\left\{x\in[a,b]:xf'(x)+f(x)\leq 0\right\}\\ &=\left\{x\in[a,b]:\frac{xf'(x)}{f(x)}\leq -1\right\}. \end{aligned} \tag{2.6}$$

From Lemma 2.4 we know that $xf'(x)/f(x)$ is decreasing; the following three cases will complete the proof of inequality (2.5).

Case 1. $a\in E$. Then $E=[a,b]$, and $xf'(x)+f(x)\leq 0$ for all $x\in[a,b]$; hence (2.5) is true for all $x\in[a,b]$.

Case 2. $b\notin E$. Then $E=\phi$, that is, $xf'(x)+f(x)>0$ for all $x\in[a,b]$.

Let

$$h(x)=\int_a^x f(t)dt-\frac{xf^2(x)}{xf'(x)+f(x)}. \tag{2.7}$$

Then from the multiplicative concavity of f we clearly see that

$$h'(x) = xf(x)\frac{x\left[f(x)f''(x) - f'^2(x)\right] + f(x)f'(x)}{\left[xf'(x) + f(x)\right]^2} \leq 0 \tag{2.8}$$

for all $x \in [a, b]$.

From (2.7) and (2.8) we get

$$h(x) \leq h(a) = -\frac{af^2(a)}{af'(a) + f(a)} \leq 0 \tag{2.9}$$

for all $x \in [a, b]$. Therefore, inequality (2.5) follows from (2.7) and (2.9).

Case 3. $a \notin E$ and $b \in E$. Then there exists a unique $x_0 \in (a, b]$ such that $E = [x_0, b]$ and $xf'(x) + f(x) > 0$ for $x \in [a, x_0)$. Making use of the similar argument as in Case 2 we know that inequality (2.5) holds for $x \in [a, x_0)$; this result and $E = [x_0, b]$ imply that (2.5) holds for all $x \in [a, b]$. □

Lemma 2.6. *If* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *is a second-order differentiable multiplicatively concave function, then*

$$\left(f(a) + af'(a)\right)\left(f(b) + bf'(b)\right)\int_a^b f(t)dt \leq bf^2(b)\left(f(a) + af'(a)\right) - af^2(a)\left(f(b) + bf'(b)\right). \tag{2.10}$$

Proof. We divide the proof into five cases.

Case 1. $f(a) + af'(a) = 0$. Then from Lemma 2.4 we know that $xf'(x)/f(x)$ is decreasing on $[a, b]$; hence we get $f(b) + bf'(b) \leq 0$. It is obvious that inequality (2.10) holds in this case.

Case 2. $f(b) + bf'(b) = 0$. Then (2.10) follows from $f(a) + af'(a) \geq 0$.

Case 3. $f(a) + af'(a) < 0$. Then $f(x) + xf'(x) < 0$ for all $x \in [a, b]$. From (2.7) and (2.8) we get

$$h(b) = \int_a^b f(t)dt - \frac{bf^2(b)}{bf'(b) + f(b)} \leq -\frac{af^2(a)}{af'(a) + f(a)} = h(a). \tag{2.11}$$

Therefore, inequality (2.10) follows from inequality (2.11) and $f(x) + xf'(x) < 0$.

Case 4. $f(b) + bf'(b) > 0$. Then $f(x) + xf'(x) > 0$ for all $x \in [a, b]$; hence inequality (2.10) follows from (2.11) and $f(x) + xf'(x) > 0$.

Case 5. $f(a) + af'(a) > 0, f(b) + bf'(b) < 0$. Then we clearly see that (2.10) is true. □

Lemma 2.7. *If* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *is a second-order differentiable multiplicatively concave function, then* $G(x, y) = |\int_x^y f(t)dt|$ *is multiplicatively concave on* $[a, b] \times [a, b]$.

Proof. For $(x,y) \in [a,b]\times[a,b]$, without loss of generality, we assume that $y \le x$. Then simple computations lead to

$$GG''_{11} + \frac{G}{x}G'_1 - G'^2_1 = f'(x)\int_y^x f(t)dt + \frac{f(x)}{x}\int_y^x f(t)dt - f^2(x), \tag{2.12}$$

$$GG''_{22} + \frac{G}{y}G'_2 - G'^2_2 = -f'(y)\int_y^x f(t)dt - \frac{f(y)}{y}\int_y^x f(t)dt - f^2(y), \tag{2.13}$$

$$GG''_{12} - G'_1G'_2 = GG''_{21} - G'_1G'_2 = f(x)f(y). \tag{2.14}$$

From Lemma 2.5 we know that $F(x) = \int_y^x f(t)dt$ is multiplicatively concave; then Lemma 2.4 leads to

$$x\left[F(x)F''(x) - F'^2(x)\right] + F(x)F'(x) = \left[xf'(x) + f(x)\right]\int_y^x f(t)dt - xf^2(x) \le 0. \tag{2.15}$$

Combining (2.12) and (2.15) we get

$$GG''_{11} + \frac{G}{x}G'_1 - G'^2_1 \le 0. \tag{2.16}$$

Equations (2.12)–(2.14) and Lemma 2.6 yield

$$\begin{aligned}
&\left(GG''_{11} + \frac{G}{x}G'_1 - G'^2_1\right)\left(GG''_{22} + \frac{G}{y}G'_2 - G'^2_2\right) - \left(GG''_{12} - G'_1G'_2\right) \times \left(GG''_{21} - G'_2G'_1\right) \\
&\quad = \frac{\int_y^x f(t)dt}{xy}\Bigg[xf^2(x)\left(f(y) + yf'(y)\right) - yf^2(y)\left(f(x) + xf'(x)\right) \\
&\qquad\qquad - \left(f(x) + xf'(x)\right)\left(f(y) + yf'(y)\right)\int_y^x f(t)dt\Bigg] \\
&\quad \ge 0.
\end{aligned} \tag{2.17}$$

Therefore, Lemma 2.7 follows from (2.16) and (2.17) together with Lemma 2.3. □

Lemma 2.8 (see [20]). *For each continuous convex function $f : [a,b] \to \mathbb{R}$, there exists a sequence of infinitely differentiable convex functions $f_n : [a,b] \to \mathbb{R}$, $n = 1,2,3,\ldots$, such that $\{f_n\}$ converges uniformly to f on $[a,b]$.*

From Definitions 1.1 and 1.2, Theorem A, and Lemma 2.8 we can get Lemma 2.9 immediately.

Lemma 2.9. *For each continuous multiplicatively convex (or concave, resp.) function $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$, there exists a sequence of infinitely differentiable multiplicatively convex (or concave, resp.) functions $f_n : [a,b] \to (0,\infty)$, $n = 1,2,3,\ldots$, such that $\{f_n\}$ converges uniformly to f on $[a,b]$.*

Proof of Theorem 1.6. Since $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is a continuous multiplicatively concave function, from Lemma 2.9 we know that there exists a sequence of infinitely differentiable multiplicatively concave function $f_n : [a,b] \to (0,\infty)$, $n = 1,2,3,\ldots$, such that $\{f_n\}$ converges uniformly to f on $[a,b]$.

For $(x,y) \in [a,b] \times [a,b]$, taking $G_n(x,y) = |\int_x^y f_n(t)dt|$, $n = 1,2,3,\ldots$, then by Lemma 2.7 we clearly see that $G_n(x,y)$ is multiplicatively concave on $[a,b] \times [a,b]$ and

$$\lim_{n\to\infty} G_n(x,y) = \left|\int_x^y f(t)dt\right| = G(x,y). \tag{2.18}$$

Therefore, Theorem 1.6 follows from Definition 1.5 and (2.18). □

Acknowledgments

The research was supported by the Natural Science Foundation of China under Grant 60850005 and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

References

[1] P. Montel, "Sur les functions convexes et les fonctions sousharmoniques," *Journal de Mathématiques*, vol. 7, no. 9, pp. 29–60, 1928.
[2] C. P. Niculescu, "Convexity according to the geometric mean," *Mathematical Inequalities & Applications*, vol. 3, no. 2, pp. 155–167, 2000.
[3] C. P. Niculescu, "Convexity according to means," *Mathematical Inequalities & Applications*, vol. 6, no. 4, pp. 551–579, 2003.
[4] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer, New York, NY, USA, 2006.
[5] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Generalized convexity and inequalities," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1294–1308, 2007.
[6] C. E. Finol and M. Wójtowicz, "Multiplicative properties of real functions with applications to classical functions," *Aequationes Mathematicae*, vol. 59, no. 1-2, pp. 134–149, 2000.
[7] T. Trif, "Convexity of the gamma function with respect to Hölder means," in *Inequality Theory and Applications. Vol. 3*, pp. 189–195, Nova Science Publishers, Hauppauge, NY, USA, 2003.
[8] K. Z. Guan, "A class of symmetric functions for multiplicatively convex function," *Mathematical Inequalities & Applications*, vol. 10, no. 4, pp. 745–753, 2007.
[9] X.-M. Zhang and Z.-H. Yang, "Differential criterion of n-dimensional geometrically convex functions," *Journal of Applied Analysis*, vol. 13, no. 2, pp. 197–208, 2007.
[10] Y. Chu, X. M. Zhang, and G.-D. Wang, "The Schur geometrical convexity of the extended mean values," *Journal of Convex Analysis*, vol. 15, no. 4, pp. 707–718, 2008.
[11] Y. M. Chu, W. F. Xia, and T. H. Zhao, "Schur convexity for a class of symmetric functions," *Science China Mathematics*, vol. 53, no. 2, pp. 465–474, 2010.
[12] W.-F. Xia and Y.-M. Chu, "Schur-convexity for a class of symmetric functions and its applications," *Journal of Inequalities and Applications*, vol. 2009, Article ID 493759, 15 pages, 2009.
[13] X.-M. Zhang and Y.-M. Chu, "A double inequality for gamma function," *Journal of Inequalities and Applications*, vol. 2009, Article ID 503782, 7 pages, 2009.
[14] Y.-M. Chu and Y.-P. Lv, "The Schur harmonic convexity of the Hamy symmetric function and its applications," *Journal of Inequalities and Applications*, vol. 2009, Article ID 838529, 10 pages, 2009.
[15] T.-H. Zhao, Y.-M. Chu, and Y.-P. Jiang, "Monotonic and logarithmically convex properties of a function involving gamma functions," *Journal of Inequalities and Applications*, vol. 2009, Article ID 728612, 13 pages, 2009.

[16] X.-H. Zhang, G.-D. Wang, and Y.-M. Chu, “Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions,” *Journal of Mathematical Analysis and Applications*, vol. 353, no. 1, pp. 256–259, 2009.

[17] X.-M. Zhang and Y.-M. Chu, “The geometrical convexity and concavity of integral for convex and concave functions,” *International Journal of Modern Mathematics*, vol. 3, no. 3, pp. 345–350, 2008.

[18] X.-M. Zhang and Y.-M. Chu, “A new method to study analytic inequalities,” *Journal of Inequalities and Applications*, vol. 2010, Article ID 698012, 13 pages, 2010.

[19] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.

[20] J. J. Koliha, “Approximation of convex functions,” *Real Analysis Exchange*, vol. 29, no. 1, pp. 465–471, 2003-2004.