

Research Article

Derivatives of Orthonormal Polynomials and Coefficients of Hermite-Fejér Interpolation Polynomials with Exponential-Type Weights

H. S. Jung¹ and R. Sakai²

¹ Department of Mathematics Education, Sungkyunkwan University, Seoul 110-745, South Korea

² Department of Mathematics, Meijo University, Nagoya 468-8502, Japan

Correspondence should be addressed to H. S. Jung, hsun90@skku.edu

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Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^2 : \mathbb{R} \rightarrow [0, \infty)$ be an even function. In this paper, we consider the exponential-type weights $w_\rho(x) = |x|^\rho \exp(-Q(x))$, $\rho > -1/2$, $x \in \mathbb{R}$, and the orthonormal polynomials $p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho(x)$. So, we obtain a certain differential equation of higher order with respect to $p_n(w_\rho^2; x)$ and we estimate the higher-order derivatives of $p_n(w_\rho^2; x)$ and the coefficients of the higher-order Hermite-Fejér interpolation polynomial based at the zeros of $p_n(w_\rho^2; x)$.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $Q \in C^2 : \mathbb{R} \rightarrow \mathbb{R}^+$ be an even function and let $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x) dx < \infty$ for all $n = 0, 1, 2, \dots$. For $\rho > -1/2$, we set

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}. \quad (1.1)$$

Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx = \delta_{mn} \text{ (Kronecker's delta)}, \quad (1.2)$$
$$p_{n,\rho}(x) = \gamma_n x^n + \dots, \quad \gamma_n = \gamma_{n,\rho} > 0.$$

We denote the zeros of $p_{n,\rho}(x)$ by

$$-\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \quad (1.3)$$

A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of n (or x) such that $b_n \leq Cc_n$ for n being large enough. We write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote the class of polynomials of degree at most n by \mathcal{P}_n .

Throughout C, C_1, C_2, \dots denote positive constants independent of n, x, t , and polynomials of degree at most n . The same symbol does not necessarily denote the same constant in different occurrences.

We shall be interested in the following subclass of weights from [1].

Definition 1.1. Let $Q : \mathbb{R} \rightarrow \mathbb{R}^+$ be even and satisfy the following properties.

- (a) $Q'(x)$ is continuous in \mathbb{R} , with $Q(0) = 0$.
- (b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
- (c) One has

$$\lim_{x \rightarrow \infty} Q(x) = \infty. \quad (1.4)$$

- (d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \quad (1.5)$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}. \quad (1.6)$$

- (e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \quad (1.7)$$

Then we write $w \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of \mathbb{R} and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \quad (1.8)$$

then we write $w \in \mathcal{F}(C^2+)$.

In the following we introduce useful notations.

- (a) Mhaskar-Rahmanov-Saff (MRS) numbers a_x is defined as the positive roots of the following equations:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du, \quad x > 0. \tag{1.9}$$

- (b) Let

$$\eta_x = (xT(a_x))^{-2/3}, \quad x > 0. \tag{1.10}$$

- (c) The function $\varphi_u(x)$ is defined as the following:

$$\varphi_u(x) = \begin{cases} \frac{a_{2u}^2 - x^2}{u[(a_u + x + a_u \eta_u)(a_u - x + a_u \eta_u)]^{1/2}}, & |x| \leq a_u, \\ \varphi_u(a_u), & a_u < |x|. \end{cases} \tag{1.11}$$

In [2, 3] we estimated the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ associated with the weight $w_\rho^2 = |x|^{2\rho} \exp(-2Q(x))$, $\rho > -1/2$ and obtained some results with respect to the derivatives of orthonormal polynomials $p_{n,\rho}(x)$. In this paper, we will obtain the higher derivatives of $p_{n,\rho}(x)$. To estimate of the higher derivatives of the orthonormal polynomials sequence, we need further assumptions for $Q(x)$ as follows.

Definition 1.2. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$ and let ν be a positive integer. Assume that $Q(x)$ is ν -times continuously differentiable on \mathbb{R} and satisfies the followings.

- (a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $0 \leq i \leq \nu + 1$ are positive for $x > 0$.
 (b) There exist positive constants $C_i > 0$ such that for $x \in \mathbb{R} \setminus \{0\}$

$$\left| Q^{(i+1)}(x) \right| \leq C_i \left| Q^{(i)}(x) \right| \frac{|Q'(x)|}{Q(x)}, \quad i = 1, \dots, \nu. \tag{1.12}$$

- (c) There exist constants $0 \leq \delta < 1$ and $c_1 > 0$ such that on $(0, c_1]$

$$Q^{(\nu+1)}(x) \leq C \left(\frac{1}{x} \right)^\delta. \tag{1.13}$$

Then we write $w(x) \in \mathcal{F}_\nu(C^2+)$. Furthermore, $w(x) \in \mathcal{F}_\nu(C^2+)$ and $Q(x)$ satisfies one of the following.

- (a) $Q'(x)/Q(x)$ is quasi-increasing on a certain positive interval $[c_2, \infty)$.
 (b) $Q^{(\nu+1)}(x)$ is nondecreasing on a certain positive interval $[c_2, \infty)$.
 (c) There exists a constant $0 \leq \delta < 1$ such that $Q^{(\nu+1)}(x) \leq C(1/x)^\delta$ on $[c_2, \infty)$.

Then we write $w(x) \in \tilde{\mathcal{F}}_\nu(C^2+)$.

Now, consider some typical examples of $\mathcal{F}(C^2+)$. Define for $\alpha > 1$ and $l \geq 1$,

$$Q_{l,\alpha}(x) := \exp_l(|x|^\alpha) - \exp_l(0). \quad (1.14)$$

More precisely, define for $\alpha + m > 1$, $m \geq 0$, $l \geq 1$ and $\alpha \geq 0$,

$$Q_{l,\alpha,m}(x) := |x|^m (\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)) \quad (1.15)$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$, and define

$$Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1. \quad (1.16)$$

In the following, we consider the exponential weights with the exponents $Q_{l,\alpha,m}(x)$. Then we have the following examples (see [4]).

Example 1.3. Let ν be a positive integer. Let $m + \alpha - \nu > 0$. Then one has the following.

- (a) $w(x) = \exp(-Q_{l,\alpha,m}(x))$ belongs to $\mathcal{F}_\nu(C^2+)$.
- (b) If $l \geq 2$ and $\alpha > 0$, then there exists a constant $c_1 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_1, ∞) .
- (c) When $l = 1$, if $\alpha \geq 1$, then there exists a constant $c_2 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_2, ∞) , and if $0 < \alpha < 1$, then $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasidecreasing on (c_2, ∞) .
- (d) When $l = 1$ and $0 < \alpha < 1$, $Q_{l,\alpha,m}^{(\nu+1)}(x)$ is nondecreasing on a certain positive interval (c_2, ∞) .

In this paper, we will consider the orthonormal polynomials $p_{n,\rho}(x)$ with respect to the weight class $\tilde{\mathcal{F}}_\nu(C^2+)$. Our main themes in this paper are to obtain a certain differential equation for $p_{n,\rho}(x)$ of higher-order and to estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ and the coefficients of the higher-order Hermite-Fejér interpolation polynomials based at the zeros of $p_{n,\rho}(x)$. More precisely, we will estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. These estimations will play an important role in investigating convergence or divergence of higher-order Hermite-Fejér interpolation polynomials (see [5–16]).

This paper is organized as follows. In Section 2, we will obtain the differential equations for $p_{n,\rho}(x)$ of higher-order. In Section 3, we will give estimations of higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ in a certain finite interval for two cases of an odd order and of an even order. In addition, we estimate the higher-order derivatives of $p_{n,\rho}(x)$ at all zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. Furthermore, we will estimate the coefficients of higher-order Hermite-Fejér interpolation polynomials based at the zeros of $p_{n,\rho}(x)$, in Section 4.

2. Higher-Order Differential Equation for Orthonormal Polynomials

In the rest of this paper we often denote $p_{n,\rho}(x)$ and $x_{k,n,\rho}$ simply by $p_n(x)$ and x_{kn} , respectively. Let $\rho_n = \rho$ if n is odd, $\rho_n = 0$ otherwise, and define the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$ as follows:

$$\begin{aligned} A_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n^2(u) \overline{Q(x,u)} w_\rho^2(u) du, \\ B_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n(u) p_{n-1}(u) \overline{Q(x,u)} w_\rho^2(u) du, \end{aligned} \quad (2.1)$$

where $\overline{Q(x,u)} = (Q'(x) - Q'(u))/(x-u)$ and $b_n = (\gamma_{n-1})/\gamma_n$. Then in [3, Theorem 4.1] we have a relation of the orthonormal polynomial $p_n(x)$ with respect to the weight $w_\rho^2(x)$:

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n \frac{p_n(x)}{x}. \quad (2.2)$$

Theorem 2.1 (cf. [6, Theorem 3.3]). *Let $\rho > -1/2$ and $w(x) \in \mathcal{F}(C^2)$. Then for $|x| > 0$ one has the second-order differential relation as follows:*

$$a(x)p_n''(x) + b(x)p_n'(x) + c(x)p_n(x) + D(x) + E(x) = 0. \quad (2.3)$$

Here, one knows that for any integer $n \geq 1$,

$$\begin{aligned} a(x) &= A_n(x), & b(x) &= -2Q'(x)A_n(x) - A_n'(x), \\ c(x) &= \frac{b_n A_n^2(x) A_{n-1}(x)}{b_{n-1}} + A_n(x) B_n(x) B_{n-1}(x) - \frac{x A_n(x) A_{n-1}(x) B_n(x)}{b_{n-1}} \\ &\quad + A_n(x) B_n'(x) - A_n'(x) B_n(x) - 2\rho_n \frac{A_n(x) A_{n-1}(x)}{b_{n-1}} \\ &=: c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x), \\ D(x) &= d(x) \frac{p_n(x)}{x}, & E(x) &= e_1(x) \frac{p_n'(x)}{x} + e_2(x) \frac{p_n(x)}{x^2}, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} d(x) &= 2\rho_n (A_n(x) B_n(x) - A_n'(x)) + 2\rho_{n-1} A_n(x) B_n(x), \\ e_1(x) &= 2(\rho_n + \rho_{n-1}) A_n(x), & e_2(x) &= -2\rho_n A_n(x). \end{aligned} \quad (2.5)$$

Especially, when n is odd, one has

$$a(x)p_n''(x) + b(x)p_n'(x) + c(x)p_n(x) + d(x)q_{n-1}(x) + 2\rho A_n(x)q_{n-1}'(x) = 0, \quad (2.6)$$

where $q_{n-1}(x)$ is the polynomial of degree $n-1$ with $p_n(x) = xq_{n-1}(x)$.

Proof. We may similarly repeat the calculation [6, Proof of Theorem 3.3], and then we obtain the results. We stand for $A_n := A_n(x), B_n := B_n(x)$ simply. Applying (2.2) to $p'_{n-1}(x)$ we also see

$$p'_{n-1}(x) = A_{n-1}p_{n-2}(x) - B_{n-1}p_{n-1}(x) - 2\rho_{n-1} \frac{p_{n-1}(x)}{x}, \quad (2.7)$$

and so if we use the recurrence formula

$$xp_{n-1}(x) = b_n p_n(x) + b_{n-1} p_{n-2}(x) \quad (2.8)$$

and use (2.2) too, then we obtain the following:

$$\begin{aligned} p'_{n-1}(x) &= \frac{1}{b_{n-1}A_n} \left\{ (xA_{n-1} - b_{n-1}B_{n-1})p'_n(x) \right. \\ &\quad + (xA_{n-1}B_n - b_{n-1}B_nB_{n-1} - b_nA_nA_{n-1})p_n(x) \\ &\quad \left. + \frac{2\rho_n}{x}(xA_{n-1} - b_{n-1}B_{n-1})p_n(x) - \frac{2\rho_{n-1}b_{n-1}}{x}(p'_n(x) + B_n p_n(x)) \right\}. \end{aligned} \quad (2.9)$$

We differentiate the left and right sides of (2.2) and substitute (2.2) and (2.9). Then consequently, we have, for $n \geq 1$,

$$\begin{aligned} p''_n(x) &= - \left\{ B_{n-1} + B_n - \frac{x A_{n-1}}{b_{n-1}} - \frac{A'_n}{A_n} \right\} p'_n(x) \\ &\quad - \left\{ \frac{b_n A_{n-1} A_n}{b_{n-1}} + B_{n-1} B_n - \frac{x A_{n-1} B_n}{b_{n-1}} + B'_n - \frac{A'_n B_n}{A_n} - 2\rho \frac{A_{n-1}}{b_{n-1}} \right\} p_n(x) \\ &\quad - 2\rho_n \left(B_n - \frac{A'_n}{A_n} \right) \frac{p_n(x)}{x} - 2\rho_n \frac{x p'_n(x) - p_n(x)}{x^2} - 2\rho_{n-1} \frac{p'_n(x) + B_n p_n(x)}{x}. \end{aligned} \quad (2.10)$$

Using the recurrence formula (2.8) and $u/(u-x) = 1 + x/(u-x)$, we have

$$\begin{aligned} B_n + B_{n-1} &= 2 \int_{-\infty}^{\infty} p_{n-1}(u) \{b_n p_n(u) + b_{n-1} p_{n-2}(u)\} \overline{Q(x,u)} w_\rho^2(u) du \\ &= 2 \int_{-\infty}^{\infty} p_{n-1}^2(u) Q'(u) w_\rho^2(u) du - 2Q'(x) + 2x \int_{-\infty}^{\infty} p_{n-1}^2(u) \overline{Q(x,u)} w_\rho^2(u) du \\ &= -2Q'(x) + \frac{x A_{n-1}}{b_{n-1}}, \end{aligned} \quad (2.11)$$

because $Q'(u)$ is an odd function. Therefore, we have

$$b(x) = -2Q'(x)A_n - A'_n. \quad (2.12)$$

When n is odd, since $x p'_n(x) - p_n(x) = x^2 q'_{n-1}(x)$, (2.6) is proved. \square

For the higher-order differential equation for orthonormal polynomials, we see that for $j = 0, 1, 2, \dots, \nu - 2$ and $|x| > 0$

$$\begin{aligned}
 D^{(j)}(x) &= \sum_{t=0}^j \left(\sum_{i=t}^j \frac{(-1)^{i-t} j!}{(j-i)! t!} d^{(j-i)}(x) x^{-(i-t+1)} \right) p_n^{(t)}(x), \\
 E^{(j)}(x) &= \sum_{t=0}^j \left(\sum_{i=t}^j \frac{(-1)^{i-t} j!}{(j-i)! t!} e_1^{(j-i)}(x) x^{-(i-t+1)} \right) p_n^{(t+1)}(x) \\
 &\quad + \sum_{t=0}^j \left(\sum_{i=t}^j \frac{(-1)^{i-t} j!(i-t+1)}{(j-i)! t!} e_2^{(j-i)}(x) x^{-(i-t+2)} \right) p_n^{(t)}(x).
 \end{aligned}
 \tag{2.13}$$

Let $\binom{j}{-1} = 0$ for nonnegative integer j . In the following theorem, we show the higher-order differential equation for orthonormal polynomials.

Theorem 2.2. *Let $\rho > -1/2$ and $w(x) \in \mathcal{F}(C^2)$. Let $\nu \geq 2$ and $j = 0, 1, \dots, \nu - 2$. Then one has the following equation for $|x| > 0$:*

$$B_{j+2}^{[j]}(x) p_n^{(j+2)}(x) + B_{j+1}^{[j]}(x) p_n^{(j+1)}(x) + \sum_{s=0}^j B_s^{[j]}(x) p_n^{(s)}(x) = 0,
 \tag{2.14}$$

where

$$B_{j+2}^{[j]}(x) = a(x), \quad B_{j+1}^{[j]}(x) = ja'(x) + b(x) + \frac{e_1(x)}{x},
 \tag{2.15}$$

and for $j \geq 1$ and $1 \leq s \leq j$

$$\begin{aligned}
 B_s^{[j]}(x) &= \binom{j}{s-2} a^{(j-s+2)}(x) + \binom{j}{s-1} b^{(j-s+1)}(x) + \binom{j}{s} c^{(j-s)}(x) \\
 &\quad + \sum_{i=s}^j \frac{(-1)^{i-s} j!}{(j-i)! s!} d^{(j-i)}(x) x^{-(i-s+1)} + \sum_{i=s-1}^j \frac{(-1)^{i-s+1} j!}{(j-i)! (s-1)!} e_1^{(j-i)}(x) x^{-(i-s+2)} \\
 &\quad + \sum_{i=s}^j \frac{(-1)^{i-s} j!(i-s+1)}{(j-i)! s!} e_2^{(j-i)}(x) x^{-(i-s+2)},
 \end{aligned}
 \tag{2.16}$$

and for $j \geq 0$

$$B_0^{[j]}(x) = c^{(j)}(x) + \sum_{i=0}^j \frac{(-1)^i j!}{(j-i)!} d^{(j-i)}(x) x^{-(i+1)} + \sum_{i=0}^j \frac{(-1)^i j!(i+1)}{(j-i)!} e_2^{(j-i)}(x) x^{-(i+2)}.
 \tag{2.17}$$

Proof. It comes from Theorem 2.1 and (2.13). □

Corollary 2.3. *Under the same assumptions as Theorem 2.1, if n is odd, then*

$$\begin{aligned} C_{j+2}^{[j]}(0)p_n^{(j+2)}(0) + C_{j+1}^{[j]}(0)p_n^{(j+1)}(0) + \sum_{s=1}^j C_s^{[j]}(0)p_n^{(s)}(0) &= 0, \quad j \geq 1, \\ C_2^{[0]}(0)p_n''(0) + C_1^{[0]}(0)p_n'(0) &= 0, \quad j = 0, \end{aligned} \quad (2.18)$$

where $C_{j+2}^{[j]}(x) = A_n(0) + (2\rho/(j+2))A_n(0)$ and for $1 \leq s \leq j+1$

$$\begin{aligned} C_s^{[j]}(0) &= \binom{j}{s-2} a^{(j-s+2)}(0) + \binom{j}{s-1} b^{(j-s+1)}(0) + \binom{j}{s} c^{(j-s)}(0) \\ &\quad + \frac{1}{s} \left(\binom{j}{s-1} d^{(j-s+1)}(0) + \binom{j}{s-2} 2\rho A_n^{(j-s+2)}(0) \right). \end{aligned} \quad (2.19)$$

Proof. Let n be odd. Then we will consider (2.6). Since $q_{n-1}^{(j)}(0) = p_n^{(j+1)}(0)/(j+1)$, we have

$$\begin{aligned} & \left(d(x)q_{n-1}(x) + 2\rho A_n(x)q_{n-1}'(x) \right)^{(j)} \Big|_{x=0} \\ &= 2\rho A_n(0) \frac{p_n^{(j+2)}(0)}{j+2} + (d(0) + 2j\rho A_n'(0)) \frac{p_n^{(j+1)}(0)}{j+1} \\ &\quad + \sum_{s=2}^j \left(\binom{j}{s-1} d^{(j-s+1)}(0) + \binom{j}{s-2} 2\rho A_n^{(j-s+2)}(0) \right) \frac{p_n^{(s)}(0)}{s} + d^{(j)}(0)p_n'(0), \end{aligned} \quad (2.20)$$

and we have

$$\begin{aligned} & \left(a(x)p_n''(x) + b(x)p_n'(x) + c(x)p_n(x) \right)^{(j)} \Big|_{x=0} \\ &= a(0)p_n^{(j+2)}(0) + (ja'(0) + b(0))p_n^{(j+1)}(0) \\ &\quad + \sum_{s=0}^j \left(\binom{j}{s-2} a^{(j-s+2)}(0) + \binom{j}{s-1} b^{(j-s+1)}(0) + \binom{j}{s} c^{(j-s)}(0) \right) p_n^{(s)}(0). \end{aligned} \quad (2.21)$$

Therefore, we have the result from (2.6). \square

In the rest of this paper, we let $\rho > -1/2$ and $w(x) = \exp(-Q(x)) \in \tilde{\mathcal{F}}_\nu(C^2_+)$ for positive integer $\nu \geq 1$ and assume that $1 + 2\rho - \delta \geq 0$ for $\rho < 0$ and

$$a_n \lesssim n^{1/(1+\nu-\delta)}, \quad (2.22)$$

where $0 \leq \delta < 1$ is defined in (1.13).

In Section 3, we will estimate the higher-order derivatives of orthonormal polynomials at the zeros of orthonormal polynomials with respect to exponential-type weights.

3. Estimation of Higher-Order Derivatives of Orthonormal Polynomials

From [3, Theorem 4.2] we know that there exist C and $n_0 > 0$ such that for $n \geq n_0$ and $|x| \leq a_n(1 + \eta_n)$,

$$\frac{A_n(x)}{2b_n} \sim \varphi_n(x)^{-1} \left(a_n^2(1 + 2\eta_n)^2 - x^2 \right)^{-1/2}, \quad |B_n(x)| \lesssim A_n(x). \tag{3.1}$$

If $T(x)$ is unbounded, then (2.22) is trivially satisfied. Additionally we have, from [17, Theorem 1.3], that if we assume that $Q''(x)$ is nondecreasing, then for $|x| \leq \varepsilon a_n$ with $0 < \varepsilon < 1/2$

$$|B_n(x)| < \lambda(\varepsilon, n) A_n(x), \tag{3.2}$$

where there exists a constant $C > 0$ such that

$$\lambda(\varepsilon, n) = C \cdot \max \left\{ \left(\frac{1}{n\theta} + 1 \right) \theta^{\Lambda-1}, \varepsilon^{(1-1/\Lambda)(\Lambda-1)}, \varepsilon^{1/\Lambda}, \lambda(n) \right\}, \tag{3.3}$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \lambda(\varepsilon, n) = 0. \tag{3.4}$$

Here, $\theta = \varepsilon^{(\Lambda-1)/2\Lambda}$ and $\lambda(n) = O(e^{-n^c})$ for some $C > 0$.

For the higher derivatives of $A_n(x)$ and $B_n(x)$, we have the following results in [17, Theorem 1.8].

Theorem 3.1 (see[17, Theorem 1.4]). *For $|x| \leq a_n(1 + \eta_n)$ and $j = 0, \dots, \nu - 1$*

$$\left| A_n^{(j)}(x) \right| \lesssim A_n(x) \left(\frac{T(a_n)}{a_n} \right)^j, \quad \left| B_n^{(j)}(x) \right| \lesssim A_n(x) \left(\frac{T(a_n)}{a_n} \right)^j. \tag{3.5}$$

Moreover, there exists $\varepsilon(n) > 0$ such that for $|x| \leq a_n/2$ and $j = 1, \dots, \nu - 1$,

$$\left| A_n^{(j)}(x) \right| \leq \varepsilon(n) A_n(x) \left(\frac{n}{a_n} \right)^j, \quad \left| B_n^{(j)}(x) \right| \leq \varepsilon(n) A_n(x) \left(\frac{n}{a_n} \right)^j, \tag{3.6}$$

with $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$.

Corollary 3.2. *Let $0 < \beta_1 < 1/2$. Then there exists a positive constant $C \neq C(n)$ such that one has for $|x| \leq \beta_1 a_n$ and $j = 1, \dots, \nu - 1$,*

$$\left| A_n^{(j)}(x) \right| \leq C A_n(x) \left(\frac{n}{a_n} \right)^j, \quad \left| B_n^{(j)}(x) \right| \leq C A_n(x) \left(\frac{n}{a_n} \right)^j. \tag{3.7}$$

In the following, we have the estimation of the higher-order derivatives of orthonormal polynomials.

Theorem 3.3. *Let $1 \leq 2s + 1 \leq \nu$ and $0 < \alpha < 1/2$. Then for $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$ the following equality holds for n large enough:*

$$p_n^{(2s+1)}(x_{kn}) = (-1)^s \beta^s(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2s} (1 + \tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)) p_n'(x_{kn}), \quad (3.8)$$

where

$$\beta(x, n) := \frac{b_n}{b_{n-1}} \left(\frac{a_n}{n}\right)^2 A_n(x) A_{n-1}(x), \quad (3.9)$$

and $|\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. Moreover, for $1 \leq 2s \leq \nu$

$$\left| p_n^{(2s)}(x_{kn}) \right| \lesssim C \mu_1(\alpha, n) \left(\frac{n}{a_n}\right)^{2s-1} |p_n'(x_{kn})|. \quad (3.10)$$

Here,

$$\begin{aligned} \mu_1(\alpha, n) &:= (\varepsilon(n) + \alpha^{\Lambda-1} + \alpha), & \mu_2(\alpha, n) &:= \frac{\log n}{n} + \varepsilon(n) + \alpha \lambda(\alpha, n) + \alpha^2, \\ \mu_3(\alpha, n) &:= \lambda(\alpha, n) \lambda(\alpha, n-1) + \alpha \lambda(\alpha, n) + \varepsilon(n) + \varepsilon(n) \lambda(\alpha, n) + \frac{1}{n}. \end{aligned} \quad (3.11)$$

Corollary 3.4. *Suppose the same assumptions as Theorem 3.3. Given any $\delta > 0$, there exists a small fixed positive constant $0 < \alpha_0(\delta) < 1/2$ such that (3.8) holds satisfying $|\tilde{\rho}_{2s+1}(\alpha_0, x_{kn}, n)| \leq \delta$ and*

$$\left| p_n^{(2s)}(x_{kn}) \right| \leq \delta \left(\frac{n}{a_n}\right)^{2s-1} |p_n'(x_{kn})| \quad (3.12)$$

for $a_n/\alpha_0 n \leq |x_{kn}| \leq \alpha_0 a_n$.

Corollary 3.5. *For $|x_{kn}| \leq a_n/2$ and $1 \leq j \leq \nu$*

$$\left| p_n^{(j)}(x_{kn}) \right| \lesssim \left(\frac{n}{a_n}\right)^{j-1} |p_n'(x_{kn})|. \quad (3.13)$$

Theorem 3.6. *Let $0 < |x_{kn}| \leq a_n(1 + \eta_n)$ and let $\nu = 2, 3, \dots, j = 1, 2, \dots, \nu - 2$. Then*

$$\left| p_n^{(j+2)}(x_{kn}) \right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{j+1} |p_n'(x_{kn})|, \quad (3.14)$$

and especially if j is even, then

$$|p_n^{(j+2)}(x_{kn})| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^j |p'_n(x_{kn})|. \tag{3.15}$$

We note that for n large enough,

$$|x_{kn}| < a_n(1 + \eta_n), \quad k = 1, 2, \dots, n, \tag{3.16}$$

because we know that $x_{1n} < a_{n+\rho/2}$ from [3, Theorem 2.2] and

$$\begin{aligned} a_{n+\rho/2} - a_n &= a_{n+\rho/2} \left(1 - \frac{a_n}{a_{n+\rho/2}} \right) \leq C_1 \frac{a_{n+\rho/2}}{T(a_n)} \log \left(1 + \frac{\rho/2}{n} \right) \\ &\leq C_2 \frac{a_n}{nT(a_n)} \leq a_n o(\eta_n). \end{aligned} \tag{3.17}$$

To prove these results we need some lemmas.

Lemma 3.7. (a) For $s \geq r > 0$

$$T(a_r) \left(1 - \frac{a_r}{a_s} \right) \leq C \log \frac{s}{r}. \tag{3.18}$$

(b) For $|x| \leq (1/2)a_n$

$$|Q'(x)| \leq C \left(\frac{x}{a_n} \right)^{\Lambda-1} \frac{n}{a_n}. \tag{3.19}$$

(c) For $|x| \leq a_n(1 + \eta_n)$

$$|A_n(x)| \sim \frac{n}{a_{2n} - |x|}. \tag{3.20}$$

(d) Let $0 \leq j \leq \nu - 1$. Then for $|x| \leq a_n/2$

$$|Q^{(j+1)}(x)| \lesssim |Q'(a_n/2)| \left(\frac{T(a_n/2)}{a_n} \right)^j, \tag{3.21}$$

and for $a_n/2 \leq |x| \leq a_n(1 + \eta_n)$

$$|Q^{(j+1)}(x)| \lesssim |Q'(x)| \left(\frac{T(a_n)}{a_n} \right)^j. \tag{3.22}$$

Proof. (a) It is [1, Lemma 3.11(c)]. (b) It is [1, Lemma 3.8(c)]. (c) It comes from (3.1). (d) Since $j + 1 \leq \nu$, $Q^{(j+1)}(x)$ is increasing. So, we obtain (d) by (1.12). \square

Lemma 3.8. Let $a(x), b(x), c(x), d(x)$, and $e_i(x)$, $i = 1, 2$, be defined in Theorem 2.1.

(a) For $|x| \leq a_n/2$ and $1 \leq k \leq \nu - 1$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that

$$\left| a^{(k)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+1}. \quad (3.23)$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$ and $1 \leq k \leq \nu - 1$,

$$\left| a^{(k)}(x) \right| \lesssim \left(\frac{T(a_n)}{a_n} \right)^k A_n(x). \quad (3.24)$$

(b) For $|x| \leq a_n/2$ and $1 \leq k \leq \nu - 2$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that

$$\left| b^{(k)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+2}. \quad (3.25)$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$ and $1 \leq k \leq \nu - 1$,

$$\left| b^{(k)}(x) \right| \lesssim \left(Q'(x) + \frac{n}{a_n} \right) \left(\frac{T(a_n)}{a_n} \right)^k A_n(x). \quad (3.26)$$

(c) For $|x| \leq a_n/2$ and $1 \leq k \leq \nu - 3$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that

$$\left| c_i^{(k)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+3}, \quad i = 1, 2, 3, 4, 5, 6. \quad (3.27)$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$ and $1 \leq k \leq \nu - 3$,

$$\left| c_i^{(k)}(x) \right| \lesssim \left(\frac{T(a_n)}{a_n} \right)^k A_n^3(x), \quad i = 1, 2, 3, 4, 5, 6. \quad (3.28)$$

(d) For $|x| \leq a_n/2$ and $1 \leq k \leq \nu - 3$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that

$$\left| d^{(k)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+2}, \quad \left| e_i^{(k)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+1}, \quad i = 1, 2. \quad (3.29)$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$ and $0 \leq k \leq \nu - 3$,

$$\begin{aligned}
 |d^{(k)}(x)| &\lesssim \left(A_n(x) + \frac{T(a_n)}{a_n} \right) \left(\frac{T(a_n)}{a_n} \right)^k A_n(x), \\
 |e_i^{(k)}(x)| &\lesssim \left(\frac{T(a_n)}{a_n} \right)^k A_n(x), \quad i = 1, 2.
 \end{aligned}
 \tag{3.30}$$

Proof. (a) Since $a(x) = A_n(x)$, we prove it by Theorem 3.1.

(b) For $1 \leq k \leq \nu - 2$, we see

$$b^{(k)}(x) = - \left(A_n^{(k+1)}(x) + 2 \sum_{p=0}^k \binom{k}{p} Q^{(p+1)}(x) A_n^{(k-p)}(x) \right).
 \tag{3.31}$$

From (3.18), we know that $T(a_n/2) \lesssim \log n$. Therefore by (3.19), (3.21), and (3.6) we have for $0 \leq x \leq a_n/2$

$$\left| Q^{(p+1)}(x) A_n^{(k-p)}(x) \right| \lesssim \left| Q' \left(\frac{a_n}{2} \right) \right| \left(\frac{T(a_n/2)}{a_n} \right)^p \left| A_n^{(k-p)}(x) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+2},
 \tag{3.32}$$

and for $|x| \leq a_n(1 + \eta_n)$ we have by (3.21) and (3.22)

$$\left| Q^{(p+1)}(x) A_n^{(k-p)}(x) \right| \lesssim \left(Q'(x) + \frac{n}{a_n} \right) \left(\frac{T(a_n)}{a_n} \right)^k A_n(x).
 \tag{3.33}$$

Consequently we have (b).

(c) Next we estimate $c^{(k)}(x)$. Suppose $|x| \leq a_n/2$. Let us set $c(x) = \sum_{i=1}^6 c_i(x)$. By (3.6) and (3.20) we have

$$\begin{aligned}
 |c_1^{(k)}(x)| &\lesssim \sum_{t,u,v,t+u+v=k} A_n^{(t)}(x) A_n^{(u)}(x) A_{n-1}^{(v)}(x) \\
 &\lesssim \varepsilon(n) \sum_{t,u,v,t+u+v=k} \left(\frac{n}{a_n} \right)^k A_n^3(x) \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+3}.
 \end{aligned}
 \tag{3.34}$$

For $c_i^{(k)}(x)$ ($i = 2, 3, 4, 5, 6$), we obtain the same estimate as $c_1^{(k)}$:

$$|c_i^{(k)}(x)| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{k+3}, \quad i = 2, 3, 4, 5, 6.
 \tag{3.35}$$

For $|x| \leq a_n(1 + \eta_n)$, we have similarly to the case of $|x| \leq a_n/2$

$$|c_i^{(k)}(x)| \lesssim \left(\frac{T(a_n)}{a_n} \right)^k A_n^3(x), \quad i = 1, 2, 3, 4, 5, 6.
 \tag{3.36}$$

(d) It is similar to (c). Consequently we have the following lemma. □

Lemma 3.9. Let $0 < \alpha < 1/2$, $0 \leq j \leq \nu - 2$, and $L_1 > 0$. Let $a_n/\alpha n \leq |x| \leq \alpha a_n$. Then

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C\mu_1(\alpha, n) \frac{n}{a_n}, \quad (3.37)$$

where $\mu_1(\alpha, n)$ is defined in Theorem 3.3 and for $L_1(a_n/n) \leq |x| \leq a_n/2$

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C \frac{n}{a_n}. \quad (3.38)$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$,

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \lesssim \frac{T(a_n)}{a_n} + |Q'(x)| + \frac{1}{|x|}. \quad (3.39)$$

Proof. Since

$$\begin{aligned} |B_{j+1}^{[j]}(x)| &= \left| ja'(x) + b(x) + \frac{e_1(x)}{x} \right| \\ &\lesssim |(j-1)A'_n(x) - 2Q'(x)A_n(x)| + \left| \frac{A_n(x)}{x} \right|, \end{aligned} \quad (3.40)$$

we have (3.39) for $|x| \leq a_n(1 + \eta_n)$ by (3.5). For $a_n/\alpha n \leq |x| \leq \alpha a_n$ we have from (3.6) and (3.19) that

$$|B_{j+1}^{[j]}(x)| \leq (\varepsilon(n) + C_1\alpha^{\Lambda-1} + C_2\alpha) \frac{n}{a_n} A_n(x) \leq C\mu_1(\alpha, n) \frac{n}{a_n} A_n(x). \quad (3.41)$$

Moreover, we can obtain (3.38) for $L_1(a_n/n) \leq |x| \leq a_n/2$ from the above easily. \square

Lemma 3.10. Let $0 < \alpha < 1/2$ and $0 \leq j \leq \nu - 2$. Let $a_n/\alpha n \leq |x| \leq \alpha a_n$. Then for $a_n/\alpha n \leq |x| \leq \alpha a_n$

$$-\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} = (-1)\beta(x, n)(1 + f_j(\alpha, x_{kn}, n)) \left(\frac{n}{a_n} \right)^2 \quad (3.42)$$

with $|f_j(\alpha, x_{kn}, n)| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n))$, where $\mu_2(\alpha, n)$, $\mu_3(\alpha, n)$, and $\beta(x, n)$ are defined in Theorem 3.3. For $L_1(a_n/n) \leq |x| \leq (1/2)a_n$ one has

$$\left| \frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C \left(\frac{n}{a_n} \right)^2. \quad (3.43)$$

On the other hand, one has for $L_1(a_n/n) < |x| \leq a_n(1 + \eta_n)$,

$$\left| \frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n} \right)^2. \tag{3.44}$$

Proof. First, we know that

$$\begin{aligned} B_j^{[j]}(x) &= \frac{j(j-1)}{2} a''(x) + j b'(x) + c(x) \\ &\quad + d(x)x^{-1} + j e_1'(x)x^{-1} - j e_1(x)x^{-2} + e_2(x)x^{-2}. \end{aligned} \tag{3.45}$$

Suppose $a_n/\alpha n \leq |x| \leq \alpha a_n$. Since from (3.18) and (3.19)

$$\left| Q''\left(\frac{a_n}{2}\right) \right| \lesssim \frac{\log n}{n} \left(\frac{n}{a_n}\right)^2, \quad \left| Q'\left(\frac{a_n}{2}\right) \right| \lesssim \frac{n}{a_n}, \tag{3.46}$$

we have from (3.6)

$$\left| \frac{j(j-1)}{2} a''(x) + j b'(x) \right| \leq C_1 \left(\frac{\log n}{n} + \varepsilon(n) \right) \left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.47}$$

Since

$$|d(x)| \leq C_1 (\lambda(\alpha, n) + \varepsilon(n)) \frac{n}{a_n} A_n(x), \tag{3.48}$$

we know from (3.6) that

$$\left| d(x)x^{-1} + j e_1'(x)x^{-1} - j e_1(x)x^{-2} + e_2(x)x^{-2} \right| \leq C\alpha (\lambda(\alpha, n) + \varepsilon(n) + \alpha) \left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.49}$$

Therefore we have for $a_n/\alpha n \leq |x| \leq \alpha a_n$

$$\left| B_j^{[j]}(x) - c(x) \right| \leq C\mu_2(\alpha, n) \left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.50}$$

Since from (3.3)

$$\begin{aligned} |c_2(x) + c_3(x)| &= \left| A_n(x) B_n(x) B_{n-1}(x) + \frac{x}{b_{n-1}} A_n(x) A_{n-1}(x) B_n(x) \right| \\ &\leq C (\lambda(\alpha, n) \lambda(\alpha, n-1) + \alpha \lambda(\alpha, n)) \left(\frac{n}{a_n}\right)^2 A_n(x) \end{aligned} \tag{3.51}$$

and similarly

$$|c_4(x) + c_5(x) + c_6(x)| \leq C \left(\varepsilon(n) + \varepsilon(n)\lambda(\alpha, n) + \frac{1}{n} \right) \left(\frac{n}{a_n} \right)^2 A_n(x), \quad (3.52)$$

we have

$$|c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x)| \leq C\mu_3(\alpha, n) \left(\frac{n}{a_n} \right)^2 A_n(x). \quad (3.53)$$

Then we have

$$\left| \frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} - \frac{c_1(x)}{B_{j+2}^{[j]}(x)} \right| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n)) \left(\frac{n}{a_n} \right)^2. \quad (3.54)$$

Therefore, since

$$\frac{c_1(x)}{B_{j+2}^{[j]}(x)} = \beta(x, n) \left(\frac{n}{a_n} \right)^2, \quad (3.55)$$

there exist constants $f_j(\alpha, x_{kn}, n)$ with $|f_j(\alpha, x_{kn}, n)| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n))$ such that we have for $a_n/an \leq |x| \leq \alpha a_n$

$$-\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} = (-1)\beta(x, n)(1 + f_j(\alpha, x_{kn}, n)) \left(\frac{n}{a_n} \right)^2. \quad (3.56)$$

Especially, from the above estimates we can see (3.43) for $L_1(a_n/n) \leq |x| \leq a_n/2$. On the other hand, suppose $L_1(a_n/n) \leq |x| \leq a_n(1 + \eta_n)$. Then since from Theorem 2.1 and (3.5)

$$|c(x)| \lesssim A_n^3(x) + \frac{T(a_n)}{a_n} A_n^2(x) \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n} \right)^2 A_n(x) \quad (3.57)$$

and $|Q'(x)| + n/a_n \lesssim A_n(x)$, we have from Lemma 3.8

$$\left| B_j^{[j]}(x) \right| \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n} \right)^2 A_n(x). \quad (3.58)$$

Therefore, we have (3.44) for $L_1(a_n/n) < |x| \leq a_n(1 + \eta_n)$. \square

Lemma 3.11. *Let $0 < \alpha < 1/2$ and $1 \leq j \leq \nu - 2$. Let $L_1(a_n/n) \leq |x| \leq a_n/2$. Then for $\ell = 1, 2, \dots, j - 1$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that*

$$\left| \frac{B_\ell^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq \varepsilon(n) \left(\frac{n}{a_n} \right)^{j-\ell+2}. \tag{3.59}$$

Moreover, one has for $L_1(a_n/n) \leq |x| \leq a_n(1 + \eta_n)$,

$$\left| \frac{B_\ell^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \lesssim \frac{T(a_n)}{a_n} \left(A_n(x) + \frac{T(a_n)}{a_n} \right)^{j-\ell+1}. \tag{3.60}$$

Proof. For $\ell = 1, 2, \dots, j - 1$ we have from Lemma 3.8 that there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \rightarrow 0$ as $n \rightarrow 0$ such that

$$\begin{aligned} |B_\ell^{[j]}(x)| &= |a^{(j-\ell+2)}(x)| + |b^{(j-\ell+1)}(x)| + |c^{(j-\ell)}(x)| \\ &\quad + |x|^{-1} \sum_{i=\ell}^j |d^{(j-i)}(x)| + |x|^{-1} \sum_{i=\ell-1}^j |e_1^{(j-i)}(x)| + |x|^{-2} \sum_{i=\ell}^j |e_2^{(j-i)}(x)| \\ &\leq \varepsilon(n) \left(\frac{n}{a_n} \right)^{j-\ell+3} + \varepsilon(n) \frac{\alpha n}{a_n} \left(\frac{n}{a_n} \right)^{j-\ell+2} + \varepsilon(n) \left(\frac{\alpha n}{a_n} \right)^2 \left(\frac{n}{a_n} \right)^{j-\ell+1} \\ &\leq \varepsilon(n) \left(\frac{n}{a_n} \right)^{j-\ell+3}. \end{aligned} \tag{3.61}$$

Similarly, for $\ell = 1, 2, \dots, j - 1$ and $L_1(a_n/n) < |x| \leq a_n(1 + \eta_n)$,

$$|B_\ell^{[j]}(x)| \lesssim \frac{T(a_n)}{a_n} \left(A_n(x) + \frac{T(a_n)}{a_n} \right)^{j-\ell+1} A_n(x). \tag{3.62}$$

Therefore, we have the results. □

Proof of Theorem 3.3. First we know that the following differential equation is satisfied:

$$\begin{aligned} p_n^{(j+2)}(x_{kn}) &= -\frac{B_{j+1}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})} p_n^{(j+1)}(x_{kn}) - \frac{B_j^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})} p_n^{(j)}(x_{kn}) \\ &\quad - \frac{B_{j-1}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})} p_n^{(j-1)}(x_{kn}) - \dots - \frac{B_1^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})} p_n'(x_{kn}). \end{aligned} \tag{3.63}$$

Suppose $L_1(a_n/n) \leq |x_{kn}| \leq (1/2)a_n$. Then since we see from (3.63) and (3.38) that

$$|p_n''(x_{kn})| \leq C \frac{n}{a_n} |p_n'(x_{kn})|, \tag{3.64}$$

we have by (3.63) and mathematical induction

$$\left| p_n^{(j+1)}(x_{kn}) \right| \lesssim \left(\frac{n}{a_n} \right)^j |p_n'(x_{kn})|. \quad (3.65)$$

Next, suppose $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$. More precisely, from Lemma 3.9 we have

$$|p_n''(x_{kn})| \leq C\mu_1(\alpha, n) \frac{n}{a_n} |p_n'(x_{kn})|. \quad (3.66)$$

Then by (3.63), (3.42), and (3.66) there exists a constant $\tilde{\rho}_1(\alpha, x_{kn}, n)$ with

$$|\tilde{\rho}_1(\alpha, x_{kn}, n)| \leq |f_1(\alpha, x_{kn}, n) + C\mu_1(\alpha, x_{kn})| \leq C \sum_{i=1}^3 \mu_i(\alpha, n), \quad (3.67)$$

such that we have that

$$p_n^{(3)}(x_{kn}) = (-1)\beta(x_{kn}, n) \left(\frac{n}{a_n} \right)^2 (1 + \tilde{\rho}_1(\alpha, x_{kn}, n)) p_n'(x_{kn}). \quad (3.68)$$

Suppose that there exist constants $\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)$ with $|\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$ such that

$$p_n^{(2s-1)}(x_{kn}) = (-1)^{s-1} \beta^{s-1}(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s-2} (1 + \tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)) p_n'(x_{kn}), \quad (3.69)$$

$$\left| p_n^{(2s)}(x_{kn}) \right| \leq C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s-1} |p_n'(x_{kn})|. \quad (3.70)$$

Then we have by (3.38) and (3.70)

$$\left| \frac{B_{2s}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} p_n^{(2s)}(x_{kn}) \right| \lesssim C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s+1} |p_n'(x_{kn})|, \quad (3.71)$$

and we have by (3.42) and (3.69)

$$-\frac{B_{2s-1}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} p_n^{(2s-1)}(x_{kn}) = (-1)^s \beta^s(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s} (1 + \tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)) p_n'(x_{kn}), \quad (3.72)$$

where $\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n) = f_{2s-1}(\alpha, x_{kn}, n)\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n) + f_{2s-1}(\alpha, x_{kn}, n) + \tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)$ and $|\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. Also, we have by (3.59) that for $1 \leq \ell \leq 2s-2$

$$\left| \frac{B_\ell^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} p_n^{(\ell)}(x_{kn}) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{2s} |p'_n(x_{kn})|. \tag{3.73}$$

Therefore, there exists $\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)$ satisfying $|\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$ such that

$$p_n^{(2s+1)}(x_{kn}) = (-1)^s \beta^s(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s} (1 + \tilde{\rho}_{2s+1}(x_{kn}, n)) p'_n(x_{kn}). \tag{3.74}$$

Moreover, we have by (3.37) and (3.65)

$$\left| \frac{B_{2s+1}^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(2s+1)}(x_{kn}) \right| \lesssim C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s+1} |p'_n(x_{kn})|, \tag{3.75}$$

and by (3.43) and (3.70)

$$\left| \frac{B_{2s}^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(2s)}(x_{kn}) \right| \leq C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s} |p'_n(x_{kn})|. \tag{3.76}$$

Also we obtain by (3.59) and (3.65) that for $1 \leq \ell \leq 2s-1$

$$\left| \frac{B_\ell^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(\ell)}(x_{kn}) \right| \leq \varepsilon(n) \left(\frac{n}{a_n} \right)^{2s+1} |p'_n(x_{kn})|. \tag{3.77}$$

Therefore, since we have by (3.63) that

$$\left| p_n^{(2s+2)}(x_{kn}) \right| \leq C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s+1} |p'_n(x_{kn})|, \tag{3.78}$$

we proved the results. □

Proof of Theorem 3.4. From (3.3), Theorem 3.1, and the definitions of $\mu_i(\alpha, n)$ ($i = 1, 2, 3$) in Theorem 3.3, if for any $\delta > 0$ we choose a fixed constant $\alpha_0(\delta) > 0$ small enough, then there exists an integer $N = N(\alpha_0)$ such that we can make $\mu_1(\alpha_0, n)$, $\mu_2(\alpha_0, n)$, and $\mu_3(\alpha_0, n)$ small enough for $a_n/\alpha_0 n \leq |x| \leq \alpha_0 a_n$ with $n > N$. □

Proof of Corollary 3.5. Since we have from Lemma 3.8 that $|C_{j+2}^{[j]}(0)| \sim n/a_n$, $|C_{j+1}^{[j]}(0)| \lesssim (n/a_n)^2$ for $j \geq 0$ and $|C_s^{[j]}(0)| \lesssim (n/a_n)^{j+3-s}$ for $1 \leq s \leq j$, we obtain using the mathematical induction that

$$|p_n^{(j+1)}(0)| \lesssim \left(\frac{n}{a_n}\right)^j |p_n'(0)|. \quad (3.79)$$

Therefore, from (3.65) we prove the result easily. \square

Proof of Theorem 3.6. We know that from (3.39)

$$|p_n''(x_{kn})| \leq \left| \frac{B_1^{[0]}(x_{kn})}{B_2^{[0]}(x_{kn})} \right| |p_n'(x_{kn})| \leq \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) |p_n'(x_{kn})| \quad (3.80)$$

and from (3.44)

$$|p_n^{(3)}(x_{kn})| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^2 |p_n'(x_{kn})|. \quad (3.81)$$

Suppose

$$\begin{aligned} |p_n^{(2s-1)}(x_{kn})| &\lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s-2} |p_n'(x_{kn})|, \\ |p_n^{(2s)}(x_{kn})| &\lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s-2} |p_n'(x_{kn})|. \end{aligned} \quad (3.82)$$

Then since

$$\begin{aligned} \left| \frac{B_{2s}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} \right| |p_n^{(2s)}(x_{kn})| &\lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right)^2 \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s-2} |p_n'(x_{kn})|, \\ \left| \frac{B_{2s-1}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} \right| |p_n^{(2s-1)}(x_{kn})| &\lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s} |p_n'(x_{kn})|, \\ \left| \frac{B_s^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} \right| |p_n^{(s)}(x_{kn})| &\lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s-1} |p_n'(x_{kn})|, \end{aligned} \quad (3.83)$$

we have

$$|p_n^{(2s+1)}(x_{kn})| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s} |p_n'(x_{kn})|. \quad (3.84)$$

Here, we used that $T(a_n)/a_n + |Q'(x_{kn})| + 1/|x_{kn}| \lesssim A_n(x_{kn}) + T(a_n)/a_n$. Similarly, since

$$\left| \frac{B_s^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} \right| \left| p_n^{(s)}(x_{kn}) \right| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s} |p_n'(x_{kn})|, \quad (3.85)$$

we have

$$\left| p_n^{(2s+2)}(x_{kn}) \right| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{2s} |p_n'(x_{kn})|. \quad (3.86)$$

□

4. Estimation of the Coefficients of Higher-Order Hermite-Fejér Interpolation

Let l, m be nonnegative integers with $0 \leq l < m \leq \nu$. For $f \in C^{(l)}(\mathbb{R})$ we define the (l, m) -order Hermite-Fejér interpolation polynomials $L_n(l, m, f; x) \in \mathcal{P}_{mn-1}$ as follows: for each $k = 1, 2, \dots, n$,

$$\begin{aligned} L_n^{(j)}(l, m, f; x_{k,n,\rho}) &= f^{(j)}(x_{k,n,\rho}), \quad j = 0, 1, 2, \dots, l, \\ L_n^{(j)}(l, m, f; x_{k,n,\rho}) &= 0, \quad j = l+1, l+2, \dots, m-1. \end{aligned} \quad (4.1)$$

Especially for each $P \in \mathcal{P}_{mn-1}$ we see $L_n(m-1, m, P; x) = P(x)$. The fundamental polynomials $h_{s,k,n,\rho}(m; x) \in \mathcal{P}_{mn-1}$, $k = 1, 2, \dots, n$ of $L_n(l, m, f; x)$ are defined by

$$h_{s,k,n,\rho}(l, m; x) = l_{k,n,\rho}^m(x) \sum_{i=s}^{m-1} e_{s,i}(l, m, k, n) (x - x_{k,n,\rho})^i. \quad (4.2)$$

Here, $l_{k,n,\rho}(x)$ is fundamental Lagrange interpolation polynomial of degree $n-1$ (cf. [18, page 23]) given by

$$l_{k,n,\rho}(x) = \frac{p_n(w_\rho^2; x)}{(x - x_{k,n,\rho}) p_n'(w_\rho^2; x_{k,n,\rho})}, \quad (4.3)$$

and $h_{s,k,n,\rho}(l, m; x)$ satisfies

$$h_{s,k,n,\rho}^{(j)}(l, m; x_{p,n,\rho}) = \delta_{s,j} \delta_{k,p} \quad j, s = 0, 1, \dots, m-1, p = 1, 2, \dots, n. \quad (4.4)$$

Then

$$L_n(l, m, f; x) = \sum_{k=1}^n \sum_{s=0}^l f^{(s)}(x_{k,n,\rho}) h_{s,k,n,\rho}(l, m; x). \quad (4.5)$$

In this section, we often denote $l_{kn}(x) := l_{k,n,\rho}(x)$ and $h_{skn}(x) := h_{s,k,n,\rho}(x)$ if it does not confuse us. Then we will first estimate $(l_{kn}^m)^{(j)}(x_{kn})$ for $0 \leq j \leq \nu - 1$. Since we have

$$l_{kn}^{(j)}(x) = \frac{p_n^{(j+1)}(x_{kn})}{(j+1)p_n'(x_{kn})} \quad (4.6)$$

by induction on m , we can estimate $(l_{kn}^m)^{(j)}(x_{kn})$.

Theorem 4.1. *Let $0 \leq j \leq \nu - 1$. Then one has for $|x_{kn}| \leq a_n/2$*

$$\left| (l_{kn}^m)^{(j)}(x_{kn}) \right| \leq C \left(\frac{n}{a_n} \right)^j. \quad (4.7)$$

In addition, one has that for $|x_{kn}| \leq a_n(1 + \eta_n)$

$$\left| (l_{kn}^m)^{(j)}(x_{kn}) \right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^j \quad (4.8)$$

and if j is odd, then one has that for $0 < |x_{kn}| \leq a_n(1 + \eta_n)$

$$\left| (l_{kn}^m)^{(j)}(x_{kn}) \right| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{j-1}. \quad (4.9)$$

For $j = 0, 1, \dots$ define $\phi_j(1) := (2j+1)^{-1}$ and for $k \geq 2$

$$\phi_j(k) := \sum_{r=0}^j \frac{1}{2j-2r+1} \binom{2j}{2r} \phi_r(k-1). \quad (4.10)$$

Theorem 4.2 (cf. [10, Lemma 10]). *Let $0 < \alpha < 1/2$ and let $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$. Then for $0 \leq 2s \leq \nu - 2$ there exists uniquely a sequence $\{\phi_j(m)\}_{j=0}^\infty$ of positive numbers*

$$(l_{kn}^m)^{(2s)}(x_{kn}) = (-1)^s \phi_s(m) \beta^s(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s} (1 + \xi_s(m, \alpha, x_{kn}, n)) \quad (4.11)$$

and $|\xi_s(m, \alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. Moreover, one has for $1 \leq 2s - 1 \leq \nu - 1$

$$\left| (l_{kn}^m)^{(2s-1)}(x_{kn}) \right| \leq C \mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s-1}. \quad (4.12)$$

Theorem 4.3. *Suppose the same assumptions as Theorem 4.2. Given any $\delta > 0$, there exists a small fixed positive constant $0 < \alpha_0(\delta) < 1/2$ such that (4.11) holds satisfying $|\xi_j(m, \alpha, x_{kn}, n)| \leq \delta$ and*

$$\left| (l_{kn}^m)^{(2j+1)}(x_{kn}) \right| \leq \delta \left(\frac{n}{a_n} \right)^{2j+1} \quad (4.13)$$

for $a_n/\alpha_0 n \leq |x_{kn}| \leq \alpha_0 a_n$.

Theorem 4.4. Let $0 \leq s \leq i \leq m - 1$. Then one has for $|x_{kn}| \leq a_n/2$

$$|e_{s,i}(l, m, k, n)| \leq C \left(\frac{n}{a_n} \right)^{i-s}. \quad (4.14)$$

On the other hand, one has for $|x_{kn}| \leq a_n(1 + \eta_n)$

$$|e_{s,i}(l, m, k, n)| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{i-s}. \quad (4.15)$$

Especially, if $i - s$ is odd, then one has

$$|e_{s,i}(l, m, k, n)| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{i-s-1}. \quad (4.16)$$

Especially, for $f \in C(\mathbb{R})$ we define the m -order Hermite-Fejér interpolation polynomials $L_n(m, f; x) \in \mathcal{P}_{mn-1}$ as the $(0, m)$ -order Hermite-Fejér interpolation polynomials $L_n(0, m, f; x)$. Then we know that

$$L_n(m, f; x) = \sum_{k=1}^n f(x_{k,n,\rho}) h_{k,n,\rho}(m; x), \quad (4.17)$$

where $e_i(m, k, n) := e_{0,i}(0, m, k, n)$ and

$$h_{k,n,\rho}(m; x) = l_{k,n,\rho}^m(x) \sum_{i=0}^{m-1} e_i(m, k, n) (x - x_{k,n,\rho})^i. \quad (4.18)$$

Then for the convergence theorem with respect to $L_n(m, f; x)$ we have the following corollary.

Corollary 4.5. Let $0 \leq i \leq m - 1$. Then one has for $|x_{kn}| \leq a_n/2$

$$|e_i(m, k, n)| \leq C \left(\frac{n}{a_n} \right)^i. \quad (4.19)$$

On the other hand, one has for $|x_{kn}| \leq a_n(1 + \eta_n)$

$$|e_i(m, k, n)| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^i. \quad (4.20)$$

Especially, if i is odd, then one has

$$|e_i(m, k, n)| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|} \right) \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n} \right)^{i-1}. \quad (4.21)$$

Proof of Theorem 4.1. Theorem 4.1 is shown by induction with respect to m . The case of $m = 1$ follows from (4.6), Corollary 3.5, and Theorem 3.6. Suppose that for the case of $m - 1$ the results hold. Then from the following relation:

$$(l_{kn}^m)^{(j)}(x_{kn}) = \sum_{r=0}^j \binom{j}{r} (l_{kn}^{m-1})^{(r)}(x_{kn}) l_{kn}^{(j-r)}(x_{kn}), \quad (4.22)$$

we have (4.7) and (4.8). Moreover, we obtain (4.9) from the following: for $1 \leq 2s - 1 \leq \nu - 1$

$$\begin{aligned} (l_{kn}^m)^{(2s-1)}(x_{kn}) &= \sum_{r=0}^s \binom{2s-1}{2r} (l_{kn}^{m-1})^{(2r)}(x_{kn}) l_{kn}^{(2s-2r-1)}(x_{kn}) \\ &\quad + \sum_{r=0}^s \binom{2s-1}{2r+1} (l_{kn}^{m-1})^{(2r+1)}(x_{kn}) l_{kn}^{(2s-2r-2)}(x_{kn}). \end{aligned} \quad (4.23)$$

□

Proof of Theorem 4.2. Similarly to Theorem 4.1, we use mathematical induction with respect to m . From Theorem 3.3 we know that for $0 \leq 2s \leq \nu - 1$

$$l_{kn}^{(2s)}(x_{kn}) = (-1)^s \phi_s(1) \beta^s(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s} (1 + \xi_s(1, \alpha, x_{kn}, n)) \quad (4.24)$$

and for $1 \leq 2s - 1 \leq \nu - 1$

$$\left| l_{kn}^{(2s-1)}(x_{kn}) \right| \leq C \mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s-1}, \quad (4.25)$$

where $\xi_s(1, \alpha, x_{kn}, n) = \tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)$ and

$$|\xi_s(1, \alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n)). \quad (4.26)$$

Then from the following relations:

$$\begin{aligned} (l_{kn}^m)^{(j)}(x_{kn}) &= \sum_{0 \leq 2r \leq j} \binom{j}{2r} (l_{kn}^{m-1})^{(2r)}(x_{kn}) l_{kn}^{(j-2r)}(x_{kn}) \\ &\quad + \sum_{1 \leq 2r-1 \leq j} \binom{j}{2r-1} (l_{kn}^{m-1})^{(2r-1)}(x_{kn}) l_{kn}^{(j-2r+1)}(x_{kn}). \end{aligned} \quad (4.27)$$

we have the results by induction with respect to m . □

Proof of Theorem 4.3. It is proved by the same reason as the proof of Corollary 3.4. □

Proof of Theorem 4.4. To prove the result, we proceed by induction on i . From (4.2) and (4.4) we know that $e_{s,s}(l, m, k, n) = 1/s!$ and the following recurrence relation; for $s + 1 \leq i \leq m - 1$

$$e_{s,i}(l, m, k, n) = - \sum_{p=s}^{i-1} \frac{1}{(i-p)!} e_{s,p}(l, m, k, n) \left(l_{k,n,\rho}^m \right)^{(i-p)} (x_{k,n,\rho}). \tag{4.28}$$

When $i = s$, $e_{s,s}(l, v, k, n) = 1/s!$ so that (4.14) and (4.15) are satisfied for $i = s$. From (4.7), (4.8), (4.28), and assumption of induction on i , for $s + 1 \leq i \leq m - 1$, we have the results easily. When $i - s$ is odd, we know that

$$\begin{aligned} i - p : \text{odd,} & \quad \text{if } p - s : \text{even,} \\ i - p : \text{even,} & \quad \text{if } p - s : \text{odd.} \end{aligned} \tag{4.29}$$

Therefore, similarly we have (4.16) from (4.8), (4.9), (4.28), and assumption of induction on i . □

Proof of Corollary 4.5. Since $e_i(m, k, n) = e_{0,i}(0, m, k, n)$, it is trivial from Theorem 4.4. □

We rewrite the relation (4.10) in the form for $v = 1, 2, 3, \dots$,

$$\phi_0(v) := 1 \tag{4.30}$$

and for $j = 1, 2, 3, \dots, v = 2, 3, 4, \dots$,

$$\phi_j(v) - \phi_j(v - 1) = \frac{1}{2j + 1} \sum_{r=0}^{j-1} \binom{2j + 1}{2r} \phi_r(v - 1). \tag{4.31}$$

Now, for every j we will introduce an auxiliary polynomial determined by $\{\Psi_j(y)\}_{j=1}^\infty$ as the following lemma.

Lemma 4.6 (see[10, Lemma 11]). (i) For $j = 0, 1, 2, \dots$, there exists a unique polynomial $\Psi_j(y)$ of degree j such that

$$\Psi_j(v) = \phi_j(v), \quad v = 1, 2, 3, \dots \tag{4.32}$$

(ii) $\Psi_0(y) = 1$ and $\Psi_j(0) = 0, j = 1, 2, \dots$

Since $\Psi_j(y)$ is a polynomial of degree j , we can replace $\phi_j(v)$ in (4.10) with $\Psi_j(y)$, that is,

$$\Psi_j(y) = \sum_{r=0}^j \frac{1}{2j - 2r + 1} \binom{2j}{2r} \Psi_r(y - 1), \quad j = 0, 1, 2, \dots, \tag{4.33}$$

for an arbitrary y and $j = 0, 1, 2, \dots$. We use the notation $F_{kn}(x, y) = (l_{kn}(x))^y$ which coincides with $l_{kn}^y(x)$ if y is an integer. Since $l_{kn}(x_{kn}) = 1$, we have $F_{kn}(x, t) > 0$ for x in a neighborhood of x_{kn} and an arbitrary real number y .

We can show that $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is a polynomial of degree at most j with respect to y for $j = 0, 1, 2, \dots$, where $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is the j th partial derivative of $F_{kn}(x, y)$ with respect to x at (x_{kn}, y) (see [6, page 199]). We prove these facts by induction on j . For $j = 0$ it is trivial. Suppose that it holds for $j \geq 0$. To simplify the notation, let $F(x) = F_{kn}(x, y)$ and $l(x) = l_{kn}(x)$ for a fixed y . Then $F'(x)l(x) = y l'(x)F(x)$. By Leibniz's rule, we easily see that

$$F^{(j+1)}(x_{kn}) = -\sum_{s=0}^{j-1} F^{(s+1)}(x_{kn})l^{(j-s)}(x_{kn}) + y \sum_{s=0}^j l^{(s+1)}(x_{kn})F^{(j-s)}(x_{kn}), \tag{4.34}$$

which shows that $F^{(j+1)}(x_{kn})$ is a polynomial of degree at most $j + 1$ with respect to y . Let $P_{kn}^{[j]}(y)$, $j = 0, 1, 2, \dots$ be defined by

$$\left(\frac{\partial}{\partial x}\right)^{2j} F_{kn}(x_{kn}, y) = (-1)^j \beta^j(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2j} \Psi_j(y) + P_{kn}^{[j]}(y). \tag{4.35}$$

Then $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$.
By Theorem 4.2 we have the following.

Lemma 4.7 (see[10, Lemma 12]). *Let $j = 0, 1, 2, \dots$, and let M be a positive constant. If $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$ and $|y| \leq M$, then*

$$\left|\left(\frac{\partial}{\partial y}\right)^s P_{kn}^{[j]}(y)\right| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n)) \left(\frac{n}{a_n}\right)^{2j}, \quad s = 0, 1, \tag{4.36}$$

$$\left|\left(\frac{\partial}{\partial y}\right)^{2j+1} F_{kn}(x_{kn}, y)\right| \leq C\mu_1(\alpha, n) \left(\frac{n}{a_n}\right)^{2j+1}. \tag{4.37}$$

Lemma 4.8 (see[10, Lemma 13]). *If $y < 0$, then for $j = 0, 1, 2, \dots$,*

$$(-1)^j \Psi_j(y) > 0. \tag{4.38}$$

Lemma 4.9. *For positive integers s and m with $1 \leq m \leq \nu$*

$$\sum_{r=0}^s \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) = 0. \tag{4.39}$$

Proof. If we let $C_s(y) = \sum_{r=0}^s \binom{2s}{2r} \Psi_r(-y) \Psi_{s-r}(y)$, then it suffices to show that $C_s(m) = 0$. For every s

$$\begin{aligned} 0 &= (I_{kn}^{-m+m})^{2s}(x_{kn}) = \sum_{i=0}^{2s} \binom{2s}{i} (I_{kn}^{-m})^{(i)}(x_{kn}) (I_{kn}^m)^{(2s-i)}(x_{kn}) \\ &= \sum_{r=0}^s \binom{2s}{2r} \left(\frac{\partial}{\partial x}\right)^{2r} F_{kn}(x_{kn}, -m) (I_{kn}^m)^{(2s-2r)}(x_{kn}) \\ &\quad + \sum_{r=0}^{s-1} \binom{2s}{2r+1} \left(\frac{\partial}{\partial x}\right)^{2r+1} F_{kn}(x_{kn}, -m) (I_{kn}^m)^{(2s-2r-1)}(x_{kn}). \end{aligned} \tag{4.40}$$

By (4.24), (4.35), and (4.36), we see that the first sum $\sum_{r=0}^s$ has the form of

$$\sum_{r=0}^s = (-1)^s \beta^s(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2s} \left(\sum_{r=0}^s \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) + \tilde{\eta}_s(-m, \alpha, x_{kn}, n)\right). \tag{4.41}$$

Then since

$$\begin{aligned} \tilde{\eta}_s(-m, \alpha, x_{kn}, n) &= \sum_{r=0}^s \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) \xi_{s-r}(m, \alpha, x_{kn}, n) \\ &\quad + \sum_{r=0}^s \binom{2s}{2r} (-1)^{-r} \beta^{-r}(x_{kn}, n) \left(\frac{n}{a_n}\right)^{-2r} \phi_{s-r}(m) P_{kn}^{[j]}(m) (1 + \xi_{s-r}(m, \alpha, x_{kn}, n)), \end{aligned} \tag{4.42}$$

we know that $|\tilde{\eta}_s(-m, \alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. By (4.37) and (4.7), the second sum $\sum_{r=0}^{s-1}$ is bounded by $C\mu_1(\alpha, n)(n/a_n)^{2s}$. Here, we can make $C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n)) < \delta$ for arbitrary positive δ . Therefore, we obtain the following result: for every s

$$0 = \sum_{r=0}^s \binom{2s}{2r} \Psi_r(-m) \Psi_{s-r}(m). \tag{4.43}$$

□

Then the following theorem is important to show a divergence theorem with respect to $L_n(m, f; x)$ where m is an odd integer.

Theorem 4.10 (cf. [10, (4.16)] and [15]). *For $j = 0, 1, 2, \dots$, there is a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j \Psi_j(-m) > 0$ for $m = 1, 3, 5, \dots$ and the following relation holds. Let $0 < \alpha < 1/2$.*

Then one has an expression for $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$, and $0 \leq 2s \leq m-1$:

$$e_{2s}(m, k, n) = (-1)^s \frac{1}{(2s)!} \Psi_s(-m) \beta^s(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2s} (1 + \eta_s(m, \alpha, x_{kn}, n)), \quad (4.44)$$

where $\eta_s(m, \alpha, x_{kn}, n)$ satisfies that for $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$ and for $s = 0, 1, 2, \dots$

$$|\eta_s(m, \alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n)). \quad (4.45)$$

Proof. We prove (4.44) by induction on s . Since $e_0(m, k, n) = 1$ and $\Psi_0(y) = 1$, (4.44) holds for $s = 0$. From (4.28) we write $e_{2s}(m, k, n)$ in the form of

$$\begin{aligned} e_{2s}(m, k, n) &= -\sum_{r=0}^{s-1} \frac{1}{(2s-2r)!} e_{2r}(m, k, n) (I_{kn}^m)^{(2s-2r)}(x_{kn}) \\ &\quad - \sum_{r=1}^s \frac{1}{(2s-2r+1)!} e_{2r-1}(m, k, n) (I_{kn}^m)^{(2s-2r+1)}(x_{kn}) \\ &=: I + II. \end{aligned} \quad (4.46)$$

Then by (4.12) and (4.14), $|II|$ is bounded by $C\mu_1(\alpha, n)(n/a_n)^{2s}$. For $0 \leq i < s$ we suppose (4.44) and (4.45). Then we have for I

$$\sum_{r=0}^{s-1} = \frac{(-1)^{s+1}}{(2s)!} \beta^s(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2s} \sum_{r=0}^{s-1} \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) (1 + \eta_r)(1 + \xi_{s-r}), \quad (4.47)$$

where $\xi_{s-r} := \xi_{s-r}(m, \alpha, x_{kn}, n)$ and $\eta_r := \eta_r(m, \alpha, x_{kn}, n)$ which are defined in (4.11) and (4.44). Then using Lemma 4.9 and $\phi_0(m) = 1$ we have the following form:

$$e_{2s}(m, k, n) = \frac{(-1)^s}{(2s)!} \Psi_s(-m) \beta^s(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2s} (1 + \eta_s(m, \alpha, x_{kn}, n)). \quad (4.48)$$

Here, since

$$\begin{aligned} \eta_s(m, \alpha, x_{kn}, n) &= \sum_{r=0}^{s-1} \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) (\eta_r + \xi_{s-r} + \eta_r \xi_{s-r}) \\ &\quad + (-1)^s \beta^{-s}(x_{kn}, n) \left(\frac{n}{a_n}\right)^{-2s} II, \end{aligned} \quad (4.49)$$

we see that $|\eta_s(v, \alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. Therefore, we proved the result. \square

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