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Research Article

Periodic Systems Dependent on Parameters

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This paper is concerned with a periodic system dependent on parameter. We study differentiability with respect to parameters of the periodic solution of the system. Applying a fixed point theorem and the results regarding parameters for C_0-semigroups, we obtained some convenient conditions for determining differentiability with parameters of the periodic solution. The paper is concluded with an application of the obtained results to a periodic boundary value problem.

1. Introduction

One of the fundamental subjects in dynamic systems is the boundary value problem. When studying boundary value problems of differential and integrodifferential equations, we often encounter the problems involving parameters. Take, for example, a periodic boundary value problem

$$\begin{gathered} u_t = u_{xx}, \quad \text{for } t \geq 0, \\ u(x,0) = u_0(x), \quad \text{for } x \in [0,1], \\ k_1 u(0,t) - h_1 u_x(0,t) = f_1(t), \quad k_i, h_i \geq 0 \quad (i = 1,2), \\ k_2 u_t(1,t) + h_2 u_x(1,t) = f_2(t), \quad k_i + h_i > 0 \quad (i = 1,2), \end{gathered} \tag{1.1}$$

on the Banach space $L^2[0,1]$, where $f_1(t)$ and $f_2(t)$ are both ρ-periodic and continuously differentiable. It appears that the boundary conditions contain four scalars k_1, k_2, h_1, and h_2. Because these scalars may vary as the environment of the system changes, they are considered as parameters. Reforming (1.1) (for details, see Section 5), we have the periodic boundary value problem

$$\begin{gathered} w_t = w_{xx} + F(t,\varepsilon), \quad \varepsilon = (k_1,k_2,h_1,h_2) \in R^4, \quad \text{for } t \geq 0, \\ w(x,0) = w_0(x), \quad \text{for } x \in [0,1], \\ k_1 w(0,t) - h_1 w_x(0,t) = 0, \\ k_2 w(1,t) + h_2 w_x(1,t) = 0, \end{gathered} \tag{1.2}$$

where $F(t,\varepsilon)(x) = (1/(k_1(k_2+h_2)+k_2h_1))[(k_1f_2'(t)-k_2f_1'(t))x+(h_2+k_2)f_1'(t)+h_1f_2'(t)]$. Clearly $F(t,\varepsilon)$ is ρ-periodic.

Furthermore, when (1.2) is written as a matrix equation (for details, see Section 5), its associated abstract Cauchy problem has the following form:

$$\begin{gathered} \frac{dz(t)}{dt} = A(\varepsilon)z(t) + f(t,z(t),\varepsilon), \\ z(0) = z_0. \end{gathered} \tag{1.3}$$

This example motivates the discussion on the parameter properties of the general abstract periodic Cauchy Problem (1.3). Since the periodic system (1.3) depends on parameters, it is a natural need for investigating continuity and differentiability with respect to parameters of the solution of the system. Moreover, in applications, the differentiability with respect to parameter is often a typical and necessary condition for studying problems such as bifurcation and inverse problem [1]. It is worth mentioning that (1.1) indicates that the occurrence of parameters in the boundary conditions leads to the dependence of the domain of the operator $A(\varepsilon)$ on the parameters. We have developed some effective methods for dealing with this tricky phenomenon.

In our previous work [2], we have obtained results on continuity in parameters of (1.3). In this paper, we will discuss the differentiability with respect to parameters of solutions of (1.3).

According to the semigroup theory, when $A(\varepsilon)$ generates a C_0-semigroup $T(t,\varepsilon)$, the weak solution of (1.3) can be expressed in terms of the C_0-semigroup $T(t,\varepsilon)$:

$$z(t,\varepsilon) = T(t,\varepsilon)z_0 + \int_0^t T(t-s,\varepsilon)F(s,z(s,\varepsilon),\varepsilon)ds. \tag{1.4}$$

It is clear that the differentiability with respect to parameter ε of semigroup $T(t,\varepsilon)$ will be the key for determining the differentiability with respect to parameter ε of the solution $z(t,\varepsilon)$ of (1.3). Some recent works [3, 4, and reference therein] have obtained fundamental results on the differentiability with respect to parameters of C_0-semigroup. Applying these results together with some fixed point theorem, we are able to prove that (1.3) has a unique periodic solution, which is continuously (Frechét) differentiable with respect to parameter ε.

We now give the outline of the approaches and contents of the paper. The general approach is that we first prove some theorems for the general periodic system (1.3). Then, by applying these results, we derive a theorem concerning (1.2) and thereby we obtain differentiability with respect to the parameter ε of the solution of (1.1). The paper begins with the preliminary section, which presents some differentiability results, a fixed point theorem, and related theorems. These results will be used in proving our theorems in later sections. In order to obtain results for (1.3), we, in Section 3, first study a special case of (1.3)

$$\begin{aligned} z' &= A(\varepsilon)z + f(t,\varepsilon), \\ z(0) &= z_0, \end{aligned} \tag{1.5}$$

where $f(t+\rho,\varepsilon) = f(t,\varepsilon)$ for some $\rho > 0$, and $f(t,\varepsilon)$ is continuous in $(t,\varepsilon) \in R \times P$. After obtaining the differentiability results for (1.5), we, in Section 4, employ a fixed point theorem to attain the differentiability results of (1.3). Lastly, in Section 5, we will apply the obtained abstract results to the periodic boundary value problem (1.1) and use this example to illustrate the obtained results. One will see that the assumptions of the abstract theorems are just natural properties of (1.1).

2. Preliminaries

In this section, we state some existing theorems that will be used in later proofs. We start by giving the results on differentiability with respect to parameters.

Consider the abstract Cauchy problem (1.3), where $A(\varepsilon)$ is a closed linear operator on a Banach space $(X, \|\cdot\|)$ and $\varepsilon \in P$ is a multiparameter (P is an open subset of a finite-dimensional normed linear space $\mathcal{P}$ with norm $|\cdot|$). Let $T(t,\varepsilon)$ be the C_0-semigroup generated by the operator $A(\varepsilon)$. For further information on C_0-semigroup, see [5].

In [3], we obtained a general theorem on differentiability with respect to the parameter ε of C_0-semigroup $T(t,\varepsilon)$ on the entire space X. It is noticed that a major assumption of the theorem is that the resolvent $(\lambda I - A(\varepsilon))^{-1}$ is continuously (Frechét) differentiable with respect to ε. In a recent paper, Grimmer and He [4] have developed several ways to determine differentiability with respect to parameter ε of $(\lambda I - A(\varepsilon))^{-1}$. Here, we include one of such theorems for reference.

Assumption Q. Let $\varepsilon_0 \in P$ be given. Then for each $\varepsilon \in P$ there exist bounded operators $Q_1(\varepsilon), Q_2(\varepsilon) : X \to X$ with bounded inverses $Q_1^{-1}(\varepsilon)$ and $Q_2^{-1}(\varepsilon)$, such that $A(\varepsilon) = Q_1(\varepsilon)A(\varepsilon_0)Q_2(\varepsilon)$.

Note that if $A(\varepsilon_1) = Q_1(\varepsilon_1)A(\varepsilon_0)Q_2(\varepsilon_1)$, then

$$\begin{aligned} A(\varepsilon) &= Q_1(\varepsilon)A(\varepsilon_0)Q_2(\varepsilon) \\ &= Q_1(\varepsilon)Q_1^{-1}(\varepsilon_1)Q_1(\varepsilon_1)A(\varepsilon_0)Q_2(\varepsilon_1)Q_2^{-1}(\varepsilon_1)Q_2(\varepsilon) \\ &= \widetilde{Q}_1(\varepsilon)A(\varepsilon_1)\widetilde{Q}_2(\varepsilon). \end{aligned} \tag{2.1}$$

Thus, having such a relationship for some ε_0 implies a similar relationship at any other $\varepsilon_1 \in P$. Without loss of generality then, we may just consider the differentiating of the semigroup $T(t,\varepsilon)$ at $\varepsilon = \varepsilon_0 \in P$.

Define $R(\varepsilon) = \lambda(\lambda I - A(\varepsilon_0))^{-1}(I - Q_1^{-1}(\varepsilon)Q_2^{-1}(\varepsilon))$, for $\lambda \in \rho(A(\varepsilon_0)) \cap \rho(A(\varepsilon))$, and assume that $I - R(\varepsilon) : X \to D(A(\varepsilon_0))$ is invertible.

Theorem 2.1 (see [3]). *Assume Assumption Q and that*

(1) *there are constants $M \geq 1$ and $\omega \in R$ such that*

$$\left\| (\lambda I - A(\varepsilon))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)}, \quad \text{for } \lambda > \omega, n \in N, \text{ and all } \varepsilon \in P. \tag{2.2}$$

(2) *There is $K_1 > 0$ such that $\|Q_2^{-1}(\varepsilon)x\|_X \leq K_1\|x\|_X$, for all $\varepsilon \in P$.*

(3) *There is $K_2 > 0$ such that $\|(I - R(\varepsilon))^{-1}x\|_X \leq K_2\|x\|_X$, for all $\varepsilon \in P$.*

(4) *For each $x \in X, Q_i^{-1}(\varepsilon)x$ $(i = 1, 2)$ and $(I - R(\varepsilon))^{-1}x$ are (Frechét) differentiable with respect to ε at $\varepsilon = \varepsilon_0$.*

Then for each $x \in X$, $(\lambda I - A(\varepsilon))^{-1}x$ is (Frechét) differentiable with respect to ε at $\varepsilon = \varepsilon_0$.

Theorem 2.2 (see [3]). *Assume the following*

(1) *For some $0 < \delta < \pi/2, \rho(A(\varepsilon)) \supset \sum_\delta = \{\lambda : |\arg \lambda| < \pi/2 + \delta\} \cup \{0\}$, for all $\varepsilon \in P$.*

(2) *For each $\varepsilon \in P$, there exists a constant $M(\varepsilon)$ such that*

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M(\varepsilon)}{|\lambda|} \quad \text{for } \lambda \in \sum_\delta, \ \lambda \neq 0. \tag{2.3}$$

(3) *for each $x \in X$ and each $\lambda \in \sum_\delta \backslash \{0\}, (\lambda I - A(\varepsilon))^{-1}x$ is continuously (Frechét) differentiable with respect to ε on P. Moreover, for any $\varepsilon_0 \in P$, there exists some ball centered at ε_0, say $B(\varepsilon_0, \delta_0), (\delta_0 > 0)$ such that $\varepsilon \in B(\varepsilon_0, \delta_0)$ implies*

$$\left\| D_\varepsilon (\lambda I - A(\varepsilon))^{-1}x \right\| \leq \eta(\lambda, x), \tag{2.4}$$

where $\eta(\lambda, x), \lambda \in \Gamma$, is measurable and for $t > 0$

$$\int_\Gamma \left| e^{\lambda t} \right| \eta(\lambda, x) |d\lambda| < \infty. \tag{2.5}$$

Then for each $x \in X, T(t, \varepsilon)x$ is continuously (Frechét) differentiable with respect to ε on P for $t > 0$. In particular, for $t > 0$

$$D_\varepsilon T(t, \varepsilon)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left[D_\varepsilon (\lambda I - A(\varepsilon))^{-1}x \right] d\lambda, \tag{2.6}$$

where Γ is a smooth curve in $\sum_\delta$ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some θ, $\pi/2 < \theta < \pi/2 + \delta$.

Now we state a fixed point theorem from [6].

Definition 2.3 (see [6, page 6]). Suppose that $\mathcal{F}$ is a subset of a Banach space $(\mathcal{X}, |\cdot|)$, $\mathcal{G}$ is a subset of a Banach space $\mathcal{Y}$, and $\{T_y, y \in \mathcal{G}\}$ is a family of operators taking $\mathcal{F} \to \mathcal{X}$. The operator T_y is said to be a uniform contraction on $\mathcal{F}$ if $T_y : \mathcal{F} \to \mathcal{F}$ and there is a $\lambda, 0 \le \lambda < 1$ such that

$$\left|T_y x - T_y \overline{x}\right| \le \lambda |x - \overline{x}| \quad \forall y \text{ in } \mathcal{G},\ x, \overline{x} \text{ in } \mathcal{F}. \tag{2.7}$$

Theorem 2.4 (see [6, page 7]). *If $\mathcal{F}$ is a closed subset of a Banach space $\mathcal{X}$, $\mathcal{G}$ is a subset of a Banach space $\mathcal{Y}$, $T_y : \mathcal{F} \to \mathcal{F}, y$ in $\mathcal{G}$ is a uniform contraction on $\mathcal{F}$, and $T_y x$ is continuous in y for each fixed x in $\mathcal{F}$, then the unique fixed point $g(y)$ of T_y, y in $\mathcal{G}$, is continuous in y. Furthermore, if $\mathcal{F}, \mathcal{G}$ are the closures of open sets $\mathcal{F}^\circ, \mathcal{G}^\circ$ and $T_y x$ has continuous first derivatives $A(x,y), B(x,y)$ in y, x, respectively, for $x \in \mathcal{F}^\circ, y \in \mathcal{G}^\circ$, then $g(y)$ has a continuous first derivative with respect to y in $\mathcal{G}^\circ$.*

Theorem 2.5 (see [7, page 167]). *Let f be a continuous mapping of an open subset Ω of E into F. f is continuously (Frechét) differentiable in Ω if and only if f is (Frechét) differentiable at each point with respect to the ith $(i = 1, 2, \ldots, n)$ variable, and the mapping $(x_1, \ldots, x_n) \to D_i f(x_1, \ldots, x_n)$ (of Ω into $\mathcal{B}(E_i, F)$) is continuous in Ω. Then at each point $(x_1, \ldots, x_n)$ of Ω, the derivative of f is given by*

$$Df(x_1, \ldots, x_n) \cdot (t_1, \ldots, t_n) = \sum_{i=1}^{n} D_i f(x_1, \ldots, x_n) \cdot t_i, \quad (t_1, \ldots, t_n) \in E. \tag{2.8}$$

Theorem 2.6 (see [4]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and let $\{B(\varepsilon)\}_{\varepsilon \in P} \subset \mathcal{B}(X, Y)$. Assume that*

(A) *for each $x \in X$, $B(\varepsilon)x$ is continuously (Frechét) differentiable in P. In particular, for $\varepsilon_0 \in P$, $[D_\varepsilon B(\varepsilon)x|_{\varepsilon=\varepsilon_0}] \in \mathcal{B}(\mathcal{P}, Y)$ is the (Frechét) derivative of $B(\varepsilon)x$ at $\varepsilon = \varepsilon_0$, and $D_\varepsilon B(\varepsilon)x$ is continuous in P.*

Then, for each $\varepsilon_0 \in P$, there is a constant $H(\varepsilon_0) > 0$ such that

$$\left\| [D_\varepsilon B(\varepsilon)x|_{\varepsilon=\varepsilon_0}] h \right\|_Y \le H(\varepsilon_0) \|x\|_X \cdot |h| \quad \text{for } x \in X,\ h \in \mathcal{P}. \tag{2.9}$$

Lemma 2.7. *Let $B \in \mathcal{B}(X, Y)$. If $\|B\| \le 1/2$, then $(I - B)^{-1}$ exists, and*

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k. \tag{2.10}$$

Moreover, $\|(I - B)^{-1}\| \le 2$.

Proof. The proof is standard and is omitted here. □

3. Differentiability Results of (1.5)

In this section, we study (1.5), which is a special case of (1.3). We will prove that the uniques periodic solution of (1.5) is continuously (Frechét) differentiable with respect to parameter ε.

We first state a theorem from [2]. This result shows that (1.5) has a unique periodic solution which is continuous in parameter ε.

Theorem 3.1 (see [2]). *Assume that*

(1) $T(t,\varepsilon)z$ *is continuous in* ε *for each* $z \in X$, *and*

$$\|T(t,\varepsilon)\| \le M(t_0) \tag{3.1}$$

for some $M(t_0) > 0$ *and all* $\varepsilon \in P, t \in [0,t_0]$, *and*

(2) $\|T(N\rho,\varepsilon)\| \le k < 1$ *for some integer* N *with* $N\rho < t_0$ *and all* $\varepsilon \in P$.

Then there exists a unique ρ*-periodic solution of* (1.5), *say* $z(t,\varepsilon)$, *which is continuous in* ε *for* $\varepsilon \in P$.

Now we will discuss differentiability with respect to parameter ε of the periodic solution of (1.5). The following lemma presents a general result on the differentiability with respect to parameter of the fixed point of a parameter dependent operator.

Lemma 3.2. *Let* $K(\varepsilon) \in \mathcal{B}(X)$ *for each* $\varepsilon \in P$, *and let* $z(\varepsilon)$ *be the fixed point of* $K(\varepsilon)$ *for each* $\varepsilon \in P$, *which is continuous in* ε. *Also, let* $Q(z,\varepsilon) = K(\varepsilon)z$. *If* $Q(z,\varepsilon)$ *has the first partial derivatives* $D_zQ(z,\varepsilon)$ *and* $D_\varepsilon Q(z,\varepsilon)$ *which satisfy*

(1) $D_z(z(\varepsilon),\varepsilon)x$ *is continuous in* ε *for each* $x \in X$ *and* $\|D_zQ(z,\varepsilon)\| \le \alpha < 1$ *for all* $(z,\varepsilon) \in X \times P$, *and*

(2) $D_\varepsilon Q(z,\varepsilon)$ *is continuous in* $(z,\varepsilon) \in X \times P$,

then $z(\varepsilon)$ *is continuously (Frechét) differentiable with respect to* $\varepsilon \in P$.

Proof. We begin by noting that the equation

$$y = D_zQ(z(\varepsilon),\varepsilon)y + D_\varepsilon Q(z(\varepsilon),\varepsilon)h, \quad h \in \mathcal{P}, \tag{3.2}$$

has a unique solution, say $y(\varepsilon,h)$, which is linear in h.

It follows from Lemma 3.2(1) and Theorem 2.4 that

$$y(\varepsilon,h) = (I - D_zQ(z(\varepsilon),\varepsilon))^{-1}D_\varepsilon Q(z(\varepsilon),\varepsilon)h \tag{3.3}$$

is the unique solution of (3.2) for $(\varepsilon,h) \in P \times \mathcal{P}$, which is continuous in (ε,h).

From the uniqueness, one observes that

$$y(\varepsilon,\alpha h_1 + \beta h_2) = \alpha y(\varepsilon,h_1) + \beta y(\varepsilon,h_2), \tag{3.4}$$

for all scalars α,β and $h_1,h_2 \in \mathcal{P}$. That is, $y(\varepsilon,h)$ is linear in h and may be written as $C(\varepsilon)h$, where $C(\varepsilon) : \mathcal{P} \to X$ is a bounded linear operator for each $\varepsilon \in P$.

Now we show that $C(\varepsilon)$ is the derivative of $z(\varepsilon)$.

Let $w = z(\varepsilon + h) - z(\varepsilon) - C(\varepsilon)h$. Since $z(\varepsilon) = Q(z(\varepsilon), \varepsilon)$ by hypothesis, one sees that

$$\begin{aligned} w &= Q(z(\varepsilon+h), \varepsilon+h) - Q(z(\varepsilon), \varepsilon) - C(\varepsilon)h \\ &= Q(z(\varepsilon+h), \varepsilon+h) - Q(z(\varepsilon+h), \varepsilon) + Q(z(\varepsilon+h), \varepsilon) - Q(z(\varepsilon), \varepsilon) - C(\varepsilon)h \\ &= D_\varepsilon Q(z(\varepsilon+h), \varepsilon)h + \circ(h) + D_z Q(z(\varepsilon), \varepsilon)(z(\varepsilon+h) - z(\varepsilon)) \\ &\quad + \circ(z(\varepsilon+h) - z(\varepsilon)) - C(\varepsilon)h. \end{aligned} \tag{3.5}$$

Note that there is a function $k(\varepsilon, h)$ continuous in h and approaching zero as $h \to 0$ such that

$$\circ(z(\varepsilon+h) - z(\varepsilon)) = k(\varepsilon, h)(z(\varepsilon+h) - z(\varepsilon)). \tag{3.6}$$

Now from (3.5) and since $C(\varepsilon)h$ is a solution of (3.2), we have

$$\begin{aligned} w &= D_\varepsilon Q(z(\varepsilon+h), \varepsilon)h + \circ(h) + D_z Q(z(\varepsilon), \varepsilon)(z(\varepsilon+h) - z(\varepsilon)) \\ &\quad + k(\varepsilon, h)(z(\varepsilon+h) - z(\varepsilon)) - C(\varepsilon)h \\ &= [D_\varepsilon Q(z(\varepsilon+h), \varepsilon) - D_\varepsilon Q(z(\varepsilon), \varepsilon)]h + \circ(h) \\ &\quad + [D_z Q(z(\varepsilon), \varepsilon) + k(\varepsilon, h)]w + k(\varepsilon, h)C(\varepsilon)h. \end{aligned} \tag{3.7}$$

Thus

$$[I - D_z Q(z(\varepsilon), \varepsilon) - k(\varepsilon, h)]w = [D_\varepsilon Q(z(\varepsilon+h), \varepsilon) - D_\varepsilon Q(z(\varepsilon), \varepsilon)]h + \circ(h) + k(\varepsilon, h)C(\varepsilon)h. \tag{3.8}$$

Since $D_\varepsilon Q(z(\varepsilon), \varepsilon)$ and $z(\varepsilon)$ are continuous, and $\{p \in \mathcal{P} \mid |p| = 1\}$ is compact, the right-hand side of this expression is $\circ(|h|)$ as $|h| \to 0$. Also, there is a $\gamma_0 > 0$ such that $\|D_z Q(z(\varepsilon), \varepsilon) + k(\varepsilon, h)\| \le \beta < 1$ for $|h| \le \gamma_0$, so $(I - D_z Q(z(\varepsilon), \varepsilon) - k(\varepsilon, h))^{-1}$ is bounded. Thus $|w| = \circ(|h|)$ as $|h| \to 0$. □

Remark 3.3. This proof is based on that of Theorem 3.2 from [6, page 7].

Now we prove the main theorem of the section.

Theorem 3.4. *Assume that*

(1) $\|T(\rho, \varepsilon)\| \le \alpha < 1$ *for all* $\varepsilon \in P$.

(2) $T(t, \varepsilon)z$ *is continuously (Frechét) differentiable with respect to* ε *for each* $z \in X$. *Moreover for any* $\varepsilon_0 \in P$ *there is some* $\delta(\varepsilon_0) > 0$ *such that* $\varepsilon \in B(\varepsilon_0, \delta(\varepsilon_0))$

$$\|D_\varepsilon T(t, \varepsilon)z\| \le \overline{H}(\varepsilon_0)\|z\| \quad \textit{for some } \overline{H}(\varepsilon_0) > 0,\ t \in [0, \rho]. \tag{3.9}$$

(3) $f(t, \varepsilon)$ *is continuously (Frechét) differentiable with respect to* ε.

Then there exists a unique ρ-periodic solution of (1.5)*, say $z(t,\varepsilon)$, which is continuously (Frechét) differentiable with respect to ε for $\varepsilon \in P$.*

Proof. First note that from Theorem 3.1, we have that

$$z(t,\varepsilon) = T(t,\varepsilon)z_0(\varepsilon) + \int_0^t T(t-s,\varepsilon)f(s,\varepsilon)ds \tag{3.10}$$

is the unique ρ-periodic solution of (1.5).

Now we want to show that $z_0(\varepsilon)$ is continuously (Frechét) differentiable with respect to ε by applying Lemma 3.2. To this end, we need to prove the following two claims first.

Claim 1. $T(t-s,\varepsilon)f(s,\varepsilon)$ is continuously (Frechét) differentiable with respect to ε for $t-s, s \in [0,\rho]$. In particular, for any $\varepsilon_0 \in P$,

$$\left[D_\varepsilon T(t-s,\varepsilon)f(s,\varepsilon)\right]\big|_{\varepsilon=\varepsilon_0} = \left[D_\varepsilon T(t-s,\varepsilon)f(s,\varepsilon_0)\right]\big|_{\varepsilon=\varepsilon_0} + T(t-s,\varepsilon_0)\left[D_\varepsilon f(s,\varepsilon)\right]\big|_{\varepsilon=\varepsilon_0}. \tag{3.11}$$

In fact, for any $\varepsilon_0 \in P$ and $h \in \not{P}$ with $\varepsilon_0 + h \in P$,

$$\begin{aligned}
&\frac{1}{|h|}\Big\| T(t-s,\varepsilon_0+h)f(s,\varepsilon_0+h) - T(t-s,\varepsilon_0)f(s,\varepsilon_0) \\
&\qquad - \left\{ \left[D_\varepsilon T(t-s,\varepsilon)f(s,\varepsilon_0)\right]\big|_{\varepsilon=\varepsilon_0} + T(t-s,\varepsilon_0)\left[D_\varepsilon f(s,\varepsilon)\right]\big|_{\varepsilon=\varepsilon_0}\right\} h \Big\| \\
&\quad \le \frac{1}{|h|}\Big\| T(t-s,\varepsilon_0+h)\left\{ f(s,\varepsilon_0+h) - f(s,\varepsilon_0) - \left[D_\varepsilon f(s,\varepsilon)\right]\big|_{\varepsilon=\varepsilon_0}\right\} h \Big\| \\
&\qquad + \frac{1}{|h|}\Big\| T(t-s,\varepsilon_0+h)f(s,\varepsilon_0) - T(t-s,\varepsilon_0)f(s,\varepsilon_0) - \left[D_\varepsilon T(t-s,\varepsilon)f(s,\varepsilon_0)\right]\big|_{\varepsilon=\varepsilon_0} h \Big\| \\
&\qquad + \frac{1}{|h|}\Big\| \{T(t-s,\varepsilon_0+h) - T(s,\varepsilon_0)\}\left[D_\varepsilon f(s,\varepsilon)\right]\big|_{\varepsilon=\varepsilon_0} h \Big\|.
\end{aligned} \tag{3.12}$$

The first two terms on the right go to 0 as $|h| \to 0$ by Theorem 3.4(2) and (3) . The last term on the right goes to 0 because $T(t-s,\varepsilon)z$ is continuous at ε_0 and the set $\{[D_\varepsilon f(s,\varepsilon)]|_{\varepsilon=\varepsilon_0} p \mid |p| = 1\}$ is compact, so (3.11) holds.

Now for each fixed t and s, and any $\varepsilon_0 \in P$, and $\varepsilon \in B(\varepsilon_0, \delta(\varepsilon_0))$, from Theorem 3.4(2)-(3) and (3.11), it is clear that $[D_\varepsilon T(t-s,\varepsilon)f(s,\varepsilon)]$ is continuous at ε_0. This completes the proof of Claim 1.

Based on Claim 1, we have the following claim.

Claim 2. $D_\varepsilon[\int_0^\rho T(\rho-s,\varepsilon)f(s,\varepsilon)ds] = \int_0^\rho [D_\varepsilon T(\rho-s,\varepsilon)f(s,\varepsilon)]ds$.

In fact, from Theorem 2.5 it suffices to show that

$$D_{\varepsilon_i}\left[\int_0^\rho T(\rho-s,\varepsilon)f(s,\varepsilon)ds\right] = \int_0^\rho \left[D_{\varepsilon_i} T(\rho-s,\varepsilon)f(s,\varepsilon)\right]ds \quad (i = 1,\dots,n). \tag{3.13}$$

W.l.o.g. assume that $\mathcal{D} = R^n$. Let $\varepsilon_0 = (\varepsilon_i^0, \dots, \varepsilon_n^0)$ be any point in P. Since $f(s,\varepsilon)$ is continuous on $[0,\rho] \times \overline{B(\varepsilon_0, \delta(\varepsilon_0))}$, there exists $L > 0$ such that

$$\|f(s,\varepsilon)\| \le L \quad \text{for } (s,\varepsilon) \in [0,\rho] \times \overline{B(\varepsilon_0, \delta(\varepsilon_0))}. \tag{3.14}$$

Now from (3.9), we have

$$\begin{aligned} &\int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} \left\| D_{\varepsilon_i} T(\rho - s, \overline{\tau}) f(s, \overline{\tau}) \right\| ds\, d\tau \quad (\overline{\tau} = (\varepsilon_1, \dots, \tau, \dots, \varepsilon_n)) \\ &\quad \le \int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} \left\| D_{\varepsilon} T(\rho - s, \overline{\tau}) \right\| \cdot \left\| f(s, \overline{\tau}) \right\| ds\, d\tau \le \int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} L \cdot \overline{H}(\varepsilon_0) ds\, d\tau < \infty. \end{aligned} \tag{3.15}$$

Thus by a theorem from [8, page 86], we have

$$\int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} D_{\varepsilon_i} T(\rho - s, \overline{\tau}) f(s, \overline{\tau}) ds\, d\tau = \int_0^{\rho} \int_{\varepsilon_i^0}^{\varepsilon_i} D_{\varepsilon_i} T(\rho - s, \overline{\tau}) f(s, \overline{\tau}) d\tau\, ds. \tag{3.16}$$

Furthermore,

$$D_{\varepsilon_i} \int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\overline{\tau}))^{-1} x\, d\lambda\, d\tau = D_{\varepsilon_i} \int_0^{\rho} \int_{\varepsilon_i^0}^{\varepsilon_i} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\overline{\tau}))^{-1} x\, d\tau\, d\lambda. \tag{3.17}$$

Now the left-hand side of (3.17) is

$$D_{\varepsilon_i} \int_{\varepsilon_i^0}^{\varepsilon_i} \int_0^{\rho} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\overline{\tau}))^{-1} x\, d\lambda\, d\tau = \int_0^{\rho} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\varepsilon))^{-1} x\, d\lambda, \tag{3.18}$$

and the right-hand side of (3.17) is

$$\begin{aligned} D_{\varepsilon_i} \int_0^{\rho} \int_{\varepsilon_i^0}^{\varepsilon_i} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\overline{\tau}))^{-1} x\, d\tau\, d\lambda &= D_{\varepsilon_i} \int_0^{\rho} e^{\lambda t} \left[(\lambda I - A(\varepsilon))^{-1} x - \left(\lambda I - A\left(\varepsilon^0\right) \right)^{-1} x \right] d\lambda \\ &= D_{\varepsilon_i} \int_0^{\rho} e^{\lambda t} (\lambda I - A(\varepsilon))^{-1} x\, d\lambda, \end{aligned} \tag{3.19}$$

where $\varepsilon^0 = (\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_i^0, \varepsilon_{i+1}, \dots, \varepsilon_n)$.

That is,

$$D_{\varepsilon_i} \int_0^{\rho} e^{\lambda t} (\lambda I - A(\varepsilon))^{-1} x\, d\lambda = \int_0^{\rho} e^{\lambda t} D_{\varepsilon_i} (\lambda I - A(\varepsilon))^{-1} x\, d\lambda. \tag{3.20}$$

This completes the proof of Claim 2.

Next, consider the operator

$$K(\varepsilon)z = T(\rho,\varepsilon)z + \int_0^\rho T(\rho - s,\varepsilon)F(s,\varepsilon)ds. \tag{3.21}$$

We will apply Lemma 3.2 and Claims 1-2 to show that the operator $K(\varepsilon)$ has $z_0(\varepsilon)$ as the fixed point and $z_0(\varepsilon)$ is continuously (Frechét) differentiable with respect to ε.

Note that the operator $K(\varepsilon)$ has the following properties.

(i) $K(\varepsilon)$ is defined on the Banach space $(X, \|\cdot\|)$.

(ii) $K(\varepsilon)$ is a uniform contraction on X.

In fact, for all $\varepsilon \in P$ and $z_1, z_2 \in X$,

$$\begin{aligned}\|K(\varepsilon)z_1 - K(\varepsilon)z_2\| &= \|T(\rho,\varepsilon)(z_1 - z_2)\| \\ &\le \|T(\rho,\varepsilon)\| \cdot \|z_1 - z_2\| \le \alpha\|z_1 - z_2\| \quad (\text{since } \|T(\rho,\varepsilon)\| \le \alpha < 1).\end{aligned} \tag{3.22}$$

(iii) $K(\varepsilon)z$ is continuous in ε for each fixed $z \in X$. (For the detailed proof, see Theorem 3.2 from [2].)

Applying Theorem 2.4, we have that $z_0(\varepsilon)$ is the fixed point of $K(\varepsilon)$.

Furthermore, it is clear that the first derivative of $K(\varepsilon)z$ with respect to z

$$D_z K(\varepsilon)z = T(\rho,\varepsilon) \tag{3.23}$$

satisfies Lemma 3.2(1) . It is also clear from Claim 2 that the first derivative of $K(\varepsilon)z$ with respect to ε

$$[D_\varepsilon K(\varepsilon)z] = D_\varepsilon[T(\rho,\varepsilon)z] + \int_0^\rho [D_\varepsilon T(\rho - s,\varepsilon)f(s,\varepsilon)]ds \tag{3.24}$$

is continuous in (z,ε) by Theorem 3.4(2) and (3). (Note that using the same argument as in the proof of Claim 1, we can show that $[D_\varepsilon T(\rho - s,\varepsilon)f(s,\varepsilon)]$ is continuous at $\varepsilon = \varepsilon_0$ and thereby $\int_0^\rho [D_\varepsilon T(\rho - s,\varepsilon)f(s,\varepsilon)]ds$ is continuous at ε_0.)

Finally applying Lemma 3.2, we have that $z_0(\varepsilon)$ is continuously (Frechét) differentiable with respect to ε. Using a similar argument as that in Claim 1 and Claim 2 we can show that $T(\rho,\varepsilon)z_0(\varepsilon)$ and $\int_0^t T(\rho - s,\varepsilon)f(s,\varepsilon)ds$ are continuously (Frechét) differentiable with respect to ε. Thus,

$$z(t,\varepsilon) = T(t,\varepsilon)z_0(\varepsilon) + \int_0^t T(t - s,\varepsilon)f(s,\varepsilon)ds \tag{3.25}$$

is continuously (Frechét) differentiable with respect to ε for $\varepsilon \in P$. □

Now we present a theorem with an assumption on the resolvent of the operator $A(\varepsilon)$ instead of the C_0-semigroup $T(t,\varepsilon)$.

Theorem 3.5. *Assume Theorem 3.4(1) and (3) and that*

(1) *for some* $0 < \delta < \pi/2, \rho(A(\varepsilon)) \supset \sum_\delta = \{\lambda : |\arg \lambda| < \pi/2 + \delta\} \cup \{0\}$ *for all* $\varepsilon \in P$,

(2) *there exists a constant* M *such that* $\|(\lambda I - A(\varepsilon))^{-1}\| \leq M/|\lambda|$ *for* $\lambda \in \sum_\delta$, $\lambda \neq 0$ *and all* $\varepsilon \in P$,

(3) *for each* $z \in X$ *and each* $\lambda \in \sum_\delta \backslash \{0\}, (\lambda I - A(\varepsilon))^{-1} z$ *is continuously (Frechét) differentiable with respect to* ε *on* P. *Moreover, for any* $\varepsilon_0 \in P$ *there exists* $\delta(\varepsilon_0) > 0$ *such that* $\varepsilon \in B(\varepsilon_0, \delta(\varepsilon_0))$ *implies*

$$\left\| D_\varepsilon (\lambda I - A(\varepsilon))^{-1} z \right\| \leq \eta(\lambda, z), \tag{3.26}$$

where $\eta(\lambda, z), \lambda \in \Gamma$, *is measurable and for* $t > 0$

$$\int_\Gamma \eta(\lambda, z) \left| e^{\lambda t} \right| |d\lambda| < \infty. \tag{3.27}$$

Then there exists a unique ρ*-periodic solution of* (1.5), *say* $z(t, \varepsilon)$, *which is continuously (Frechét) differentiable with respect to* ε *for* $\varepsilon \in P$.

Proof. First note that from Theorem 2.2 we have, for each $z \in X$,

$$D_\varepsilon T(t, \varepsilon) z = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left[D_\varepsilon (\lambda I - A(\varepsilon))^{-1} z \right] d\lambda, \tag{3.28}$$

where Γ is a smooth curve in $\sum_\delta$ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some $\theta, \pi/2 < \theta < \pi/2 + \delta$.

Moreover, since $D_\varepsilon (\lambda I - A(\varepsilon))^{-1} z$ is continuous in ε, it is clear from (3.28) that $D_\varepsilon T(t, \varepsilon) z$ is continuous in ε, so Theorem 3.4(2) is satisfied. Also it is clear from (3.28) that $D_\varepsilon T(t, \varepsilon) z$ is continuous in (t, ε). By Theorem 2.6, we have

$$\|D_\varepsilon T(t, \varepsilon) z\| \leq H(t, \varepsilon) \|z\|, \quad \text{for some } H(\mathrm{t}, \varepsilon) > 0. \tag{3.29}$$

Now by the Principle of Uniform Boundedness, there is a $H(\varepsilon_0) > 0$ such that

$$\|D_\varepsilon T(t, \varepsilon) z\| \leq H(\varepsilon_0) \|z\| \quad \forall (\mathrm{t}, \varepsilon) \in [0, \rho] \times \mathrm{B}(\varepsilon_0, \delta(\varepsilon_0)), \tag{3.30}$$

thus (3.9) is satisfied. Now the desired result follows from Theorem 3.4. □

4. Differentiability Results of (1.3)

In this section, we discuss the general (1.3). Let $P = B(0, 1) \in \wp$.

Lemma 4.1. *Assume that Theorem 3.4(1) and (2) are satisfied. Then* $(I - T(\rho, \varepsilon))^{-1} z$ *is continuously (Frechét) differentiable with respect to* ε *for each* $z \in X$.

Proof. First note that from Theorem 3.4(1), we see that $(I - T(\rho,\varepsilon))^{-1}$ exists by Lemma 2.7. Also,

$$\left\|(I - T(\rho,\varepsilon))^{-1}\right\| \leq \frac{1}{1-\alpha} \doteq H. \tag{4.1}$$

Next consider the operator defined on X:

$$K(\varepsilon)z = T(\rho,\varepsilon)z + y, \quad \text{where } y \text{ is a given point in } X. \tag{4.2}$$

Then we have

$$\|K(\varepsilon)z_1 - K(\varepsilon)z_2\| \leq \|T(\rho,\varepsilon)\| \cdot \|z_1 - z_2\| \leq \alpha\|z_1 - z_2\|, \tag{4.3}$$

so $K(\varepsilon)$ is a uniform contraction. Also it is obvious that $K(\varepsilon)z$ is continuous in ε by Theorem 3.4(2). Therefore from Theorem 2.4 it follows that there is a unique fixed point of $K(\varepsilon)$, say $z(\varepsilon)$.

Furthermore, since

$$\begin{gathered} D_\varepsilon K(\varepsilon)z = D_\varepsilon T(\rho,\varepsilon)z, \\ D_z K(\varepsilon)z = T(\rho,\varepsilon), \end{gathered} \tag{4.4}$$

which clearly satisfy Lemma 3.2(1) and (2), so by Lemma 3.2, we have that

$$z(\varepsilon) = (I - T(\rho,\varepsilon))^{-1} y \tag{4.5}$$

is continuously (Frechét) differentiable w.r.t. ε. □

Let $PC[R,\rho] = \{g \in C(R) \mid g(t+\rho) = g(t)\}$.

Consider the equation

$$\begin{gathered} z(t)' = A(\varepsilon)z(t) + f(t, g(t), \varepsilon) \\ z(0) = z_0 \end{gathered} \tag{4.6}$$

on a Banach space $(X, \|\cdot\|)$, where $f(t+\rho, g, \varepsilon) = f(t, g, \varepsilon)$ for some $\rho > 0$ and $g \in PC[R,\rho]$, and $f(t, g, \varepsilon)$ is continuous in $(t, g, \varepsilon) \in R \times PC[R,\rho] \times P$.

Lemma 4.2. *Assume that Lemma 3.2(1) and Theorem 3.4(2) and* (3.9) *are satisfied and*

(K) $f(t, z, \varepsilon)$ *is continuously (Frechét) differentiable with respect to* ε.

Then there exists a unique ρ-periodic solution of (4.6)*, say $z(t,\varepsilon,g)$, which is continuously (Frechét) differentiable with respect to ε for $\varepsilon \in P$. Also*

$$z(0,\varepsilon,g) = (I - T(\rho,\varepsilon))^{-1} \int_0^{\rho} T(\rho - s,\varepsilon) f(s, g(s),\varepsilon) ds. \tag{4.7}$$

which is continuously (Frechét) differentiable with respect to ε.

Proof. Let $F(t,\varepsilon) = f(t,g(t),\varepsilon)$. Then $F(t+\rho,\varepsilon) = F(t,\varepsilon)$. Also it is obvious that $F(t,\varepsilon)$ satisfies Theorem 3.4(3) . Therefore by Theorem 3.4, there is a unique ρ-solution $z(t,\varepsilon,g)$ of (4.6) which is continuously (Frechét) differentiable with respect to ε. In particular, $z(0,\varepsilon,g)$ is continuously (Frechét) differentiable with respect to ε. Moreover, using the same argument as that in the proof of Theorem 3.4 we see that

$$z(0,\varepsilon,g) = T(\rho,\varepsilon) z(0,\varepsilon,g) + \int_0^{\rho} T(\rho - s,\varepsilon) f(s, g(s),\varepsilon) ds. \tag{4.8}$$

Thus

$$z(0,\varepsilon,g) = (I - T(\rho,\varepsilon))^{-1} \int_0^{\rho} T(\rho - s,\varepsilon) f(s, g(s),\varepsilon) ds, \tag{4.9}$$

which is continuously (Frechét) differentiable w.r.t. ε by Lemma 4.1. □

Define $K(\varepsilon) : PC[R,\rho] \rightarrow PC[R,\rho]$ by

$$K(\varepsilon)g(t) = T(t,\varepsilon) z(0,\varepsilon,g) + \int_0^{t} T(t - s,\varepsilon) f(s, g(s),\varepsilon) ds. \tag{4.10}$$

Lemma 4.3. *Assume that Theorem 3.4(1)-(2) and* (3.9) *and Lemma 4.2(K) are satisfied. In addition, assume that*

(1) *$T(t,\varepsilon)z$ is continuous in ε for each $z \in X$, and*

$$\|T(t,\varepsilon)\| \leq M(t_0) \tag{4.11}$$

for some $M(t_0 > 0)$ and all $\varepsilon \in P, t \in [0,t_0]$.

(2) *$\|f(t,z_1,\varepsilon) - f(t,z_2,\varepsilon)\| \leq L(\varepsilon)\|z_1 - z_2\|$, where $L(\varepsilon)$ is continuous in $\varepsilon \in P$ and $L(0) = 0$.*

(3) *$f_2(t,g,\varepsilon) = (\partial/\partial g) f(t,g,\varepsilon)$ is continuous in (t,g,ε).*

Then the operator $K(\varepsilon)$ has a unique fixed point $g(\cdot,\varepsilon) \in PC[R,\rho]$ which is continuously (Frechét) differentiable with respect to ε.

Proof. It is clear that $(PC[R,\rho], \|\cdot\|_\infty)$ is a Banach space. Since $L(0) = 0$, then, by the continuity of $L(\varepsilon)$, there is δ_0 such that $\varepsilon \in B(0,\delta_0)$ implies

$$L(\varepsilon) \leq \frac{1}{4M(t_0)H[M(t_0)H + 1]}. \tag{4.12}$$

Now for $\varepsilon \in P$,

$$\begin{aligned}
&\|T(t,\varepsilon)\| \cdot \|z(0,g_1,\varepsilon) - z(0,g_2,\varepsilon)\| \\
&\quad = \|T(t,\varepsilon)\| \cdot \left\| (I - T(\rho,\varepsilon))^{-1} \int_0^{\rho} T(\rho - s,\varepsilon)[f(s,g_1,\varepsilon) - f(s,g_2,\varepsilon)]ds \right\| \quad \text{(by (4.7))} \\
&\quad \le M(t_0) \left\| (I - T(\rho,\varepsilon))^{-1} \right\| \\
&\qquad \times \int_0^{\rho} \|T(\rho - s,\varepsilon)\| \cdot \|f(s,g_1,\varepsilon) - f(s,g_2,\varepsilon)\| ds \quad \text{(by Lemma 4.3(1))} \\
&\quad \le M(t_0) \cdot H \cdot M(t_0) \int_0^{\rho} \|f(s,g_1,\varepsilon) - f(s,g_2,\varepsilon)\| ds \quad \text{(by Lemma 4.1)} \\
&\quad \le H \cdot M^2(t_0) \cdot \rho L(\varepsilon) \|g_1 - g_2\| \le \frac{1}{4} \|g_1 - g_2\| \quad \text{(by (4.12))},
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\int_0^t \|T(t-s,\varepsilon)\| \cdot \|f(s,g_1(s),\varepsilon) - f(s,g_2(s),\varepsilon)\| ds \\
&\quad \le L(\varepsilon) \int_0^t \|g_1(s) - g_2(s)\| ds \quad \text{(by Lemma 4.3(1)-(2))} \\
&\quad \le M \cdot \rho L(\varepsilon) \|g_1 - g_2\| \le \frac{1}{4} \|g_1 - g_2\| \quad \text{(by (4.12))}.
\end{aligned} \tag{4.14}$$

Hence,

$$\begin{aligned}
\|K(\varepsilon)g_1 - K(\varepsilon)g_2\| &\le \|T(t,\varepsilon)\| \|z(0,g_1,\varepsilon) - z(0,g_2,\varepsilon)\| \\
&\quad + \int_0^t \|T(t-s,\varepsilon)\| \cdot \|f(s,g_1(s),\varepsilon) - f(s,g_2(s),\varepsilon)\| ds \\
&\le \frac{1}{2} \|g_1 - g_2\| \quad \text{(by(4.13) and (4.14))}.
\end{aligned} \tag{4.15}$$

Therefore $K(\varepsilon)$ is a uniform contraction.

Furthermore, $K(\varepsilon)g$ is continuous in ε for fixed g, and also

$$\begin{aligned}
D_g K(\varepsilon)g &= T(t,\varepsilon) D_g z(0,\varepsilon,g) + \int_0^t T(t-s,\varepsilon) f_2(s,g(s),\varepsilon) ds, \\
&= T(t,\varepsilon)(I - T(\rho,\varepsilon))^{-1} \int_0^{\rho} f_2(s,g(s),\varepsilon) ds + \int_0^t T(t-s,\varepsilon) f_2(s,g(s),\varepsilon) ds,
\end{aligned} \tag{4.16}$$

$$D_{\varepsilon} K(\varepsilon)g = [D_{\varepsilon} T(t,\varepsilon) z(0,\varepsilon,g)] + \int_0^t [D_{\varepsilon} T(t-s,\varepsilon) f(s,g(s),\varepsilon)] ds$$

are continuous in (g, ε). Therefore from Theorem 2.4 it follows that $K(\varepsilon)$ has a unique fixed point, say $g(\cdot, \varepsilon) \in PC[R, \rho]$, which is continuously (Frechét) differentiable with respect to ε. □

Now we present the main theorem for (1.3).

Theorem 4.4. *Assume that Theorem 3.4(1)-(2) and* (3.5)*, Lemmas 4.2(K), and 4.3(1)–(3) are satisfied.*

Then there exists a unique ρ-periodic solution of (1.3)*, say $z(t, \varepsilon)$, which is continuously (Frechét) differentiable with respect to ε for $\varepsilon \in P$.*

Proof. This is an immediate result from Lemmas 4.2 and 4.3. □

5. Application to a Periodic Boundary Value Problem

Consider the periodic boundary value problem (1.1) on the Banach space $L^2[0,1]$, where $f_1(t+\rho) = f_1(t)$, $f_2(t+\rho) = f_2(t)$, for some $\rho > 0$ and $f_1, f_2 \in C^1(R)$.

Let

$$w = u - mx - b, \tag{5.1}$$

where

$$\begin{aligned} m &= \frac{k_1 f_2 - k_2 f_1}{k_1(k_2+h_2) + k_2 h_1}, \\ b &= \frac{(k_2+h_2) f_1 + h_1 f_2}{k_1(k_2+h_2) + k_2 h_1}. \end{aligned} \tag{5.2}$$

Then (1.1) becomes

$$\begin{aligned} w_t &= w_{xx} + F(t, \varepsilon), \quad \varepsilon = (k_1, k_2, h_1, h_2) \in R_+^4, \quad \text{for } t \geq 0, \\ w(x, 0) &= w_0(x) \quad \text{for } x \in [0,1], \\ k_1 w(0,t) - h_1 w_x(0,t) &= 0, \\ k_2 w(1,t) + h_2 w_x(1,t) &= 0, \end{aligned} \tag{5.3}$$

where $F(t,\varepsilon)(x) = (1/(k_1(k_2+h_2) + k_2 h_1))[(k_1 f_2'(t) - k_2 f_1'(t))x + (h_2 + k_2) f_1'(t) + h_1 f_2'(t)]$.

Assume $k_1, k_2 > 0$ and let $\alpha = h_1/k_1$ and $\beta = h_2/k_2$. Then the associated abstract Cauchy problem is

$$\begin{aligned} \frac{dw(t)}{dt} &= A(\varepsilon) w(t) + F(t, \varepsilon), \\ w(0) &= f, \end{aligned} \tag{5.4}$$

on $X = (L^2[0,1], \|\cdot\|_{L^2}), t \in R$, where

$$A(\varepsilon) = \frac{d^2}{dx^2}, \qquad \varepsilon = (\alpha,\beta) \in R_+^2 = \left\{(\alpha,\beta) \in R^2 \mid \ \alpha,\beta \geq 0\right\},$$
$$D(A(\varepsilon)) = \left\{w \in H^2[0,1] \mid w(0) - \alpha w'(0) = 0,\ w(1) + \beta w'(1) = 0\right\}. \tag{5.5}$$

We now show that (5.4) satisfies all assumptions of Theorem 3.5.

It is well known that the operator $A(\varepsilon)$ generates an analytic semigroup. The resolvent $(\lambda I - A(\varepsilon))^{-1}$ of $A(\varepsilon)$ satisfies, for all $\varepsilon \in R_+^2$ and $\lambda \in \sum_{\pi/4} = \{\lambda \mid \ |\arg\lambda| < (3\pi)/4\}$,

$$\left\|(\lambda I - A(\varepsilon))^{-1}\right\| \leq \frac{M}{|\lambda|}, \quad \text{where } M = \sqrt{2}. \tag{5.6}$$

Thus Assumptions Theorem 3.5(1) and (2) are satisfied. Also, refer to [3, Section 4], we have shown that $(\lambda I - A(\varepsilon))^{-1}x$ is continuously (Frechét) differentiable with respect to ε. So Assumption Theorem 3.5(3) is satisfied. Furthermore, from the expression of F, it is obvious that $F(t,\varepsilon)$ is continuous in (t,ε) and is continuously (Frechét) differentiable with respect to ε, so Theorem 3.4(3) is satisfied. Now to apply Theorem 3.5, we only need to show that Theorem 3.4(1) is satisfied.

In fact, for $w_0 = \sum_{n=1}^{\infty} a_n\phi_n(x)$,

$$T(t,\varepsilon)w_0 = \sum_{n=1}^{\infty} a_n e^{\lambda_n t}\phi_n(x), \tag{5.7}$$

where $\lambda_n < 0$, and a_n, λ_n, and ϕ_n depend on ε. Moreover,

$$\begin{aligned}
\|T(\rho,\varepsilon)w_0\|^2 &= \left\|\sum_{n=1}^{\infty} a_n e^{\lambda_n\rho}\phi_n(x)\right\|^2 \\
&\leq \left[e^{\lambda_1\rho}\left\|\sum_{n=1}^{\infty} a_n e^{(\lambda_n-\lambda_1)\rho}\phi_n(x)\right\|\right]^2 = e^{2\lambda_1\rho}\sum_{n=1}^{\infty}|a_n|^2 e^{2(\lambda_n-\lambda_1)\rho} \\
&\leq e^{2\lambda_1\rho}\sum_{n=1}^{\infty}|a_n|^2 \quad \left(\text{since } e^{2(\lambda_n-\lambda_1)\rho} \leq 1\right) \\
&= e^{2\lambda_1\rho}\|w_0\|^2 = \alpha^2\|w_0\|^2 \quad \left(\text{since } \alpha \doteq e^{\lambda_1\rho} < 1\right).
\end{aligned} \tag{5.8}$$

Thus Theorem 3.4(1) is satisfied. Now all the assumptions of Theorem 3.5 are satisfied, therefore (5.4) has a unique ρ-periodic solution, say $w(t,\varepsilon)$, which is continuously (Frechét) differentiable with respect to ε. Moreover, $u(t,\varepsilon) = w(t,\varepsilon) + mx + b$ is the unique ρ-periodic solution of (1.1) and it is continuously (Frechét) differentiable with respect to ε.

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