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Research Article

Positive Semidefinite Matrices, Exponential Convexity for Majorization, and Related Cauchy Means

M. Anwar,[1] N. Latif,[2] and J. Pečarić[2,3]

[1] *Center for Advanced Mathematics and Physics, NUST Campus at College of Electrical & Mechanical Engineering, Peshawar Road, Rawalpindi 48000, Pakistan*
[2] *Abdus Salam School of Mathematical Sciences, GC University, Lahore 5400, Pakistan*
[3] *Faculty of Textile Technology, University of Zagreb, 1002 Zagreb, Croatia*

Correspondence should be addressed to M. Anwar, matloob_t@yahoo.com

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We prove positive semidefiniteness of matrices generated by differences deduced from majorization-type results which implies exponential convexity and log-convexity of these differences and also obtain Lyapunov's and Dresher's inequalities for these differences. We introduce new Cauchy means and show that these means are monotone.

1. Introduction and Preliminaries

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$ denote two sequences of positive real numbers with $\sum_{i=1}^{n} p_i = 1$. The well-known Jensen inequality for convex function [1, page 43] gives that, for $t < 0$ or $t > 1$,

$$\sum_{i=1}^{n} p_i x_i^t \geq \left(\sum_{i=1}^{n} p_i x_i \right)^t, \tag{1.1}$$

and vice versa for $0 < t < 1$.

In [2], the following generalization of this theorem is given.

Theorem 1.1. *For $-\infty < r < s < t < +\infty$,*

$$\lambda_s^{t-r} \leq \lambda_r^{t-s} \lambda_t^{s-r}, \tag{1.2}$$

where

$$\lambda_t := \begin{cases} \dfrac{\sum_{i=1}^{n} p_i x_i^t - \left(\sum_{i=1}^{n} p_i \ x_i\right)^t}{t(t-1)}, & t \neq 0, 1, \\ \log\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \log x_i, & t = 0, \\ \sum_{i=1}^{n} p_i x_i \log x_i - \left(\sum_{i=1}^{n} p_i x_i\right) \log\left(\sum_{i=1}^{n} p_i x_i\right), & t = 1. \end{cases} \tag{1.3}$$

For fixed $n \geq 2$ let

$$\mathbf{x} = (x_1, \ldots, x_n), \qquad \mathbf{y} = (y_1, \ldots, y_n) \tag{1.4}$$

denote two n-tuples. Let

$$\begin{gathered} x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}, \qquad y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}, \\ x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}, \qquad y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)} \end{gathered} \tag{1.5}$$

be their ordered components.

Definition 1.2 (see [1, page 319]). $\mathbf{y}$ is said to majorize $\mathbf{x}$ (or $\mathbf{x}$ is said to be majorized by $\mathbf{y}$), in symbol, $\mathbf{y} \succ \mathbf{x}$, if

$$\sum_{i=1}^{m} x_{[i]} \leq \sum_{i=1}^{m} y_{[i]} \tag{1.6}$$

holds for $m = 1, 2, \ldots, n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1.7}$$

Note that (1.6) is equivalent to

$$\sum_{i=n-m+1}^{n} x_{(i)} \leq \sum_{i=n-m+1}^{n} y_{(i)} \tag{1.8}$$

for $m = 1, 2, \ldots, n-1$.

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin [3, page11] (see also [1, page 320]).

Theorem 1.3. *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}$, $\mathbf{y}$ be two n-tuples such that x_i, $y_i \in I$ $(i = 1, \ldots, n)$. Then*

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i) \tag{1.9}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ if and only if $\mathbf{y} \succ \mathbf{x}$ holds.

Remark 1.4 (see [4]). If $\phi(x)$ is a strictly convex function, then equality in (1.9) is valid iff $x_{[i]} = y_{[i]}$, $i = 1, \ldots, n$.

The following theorem can be regarded as a generalization of Theorem 1.3 and is proved by Fuchs in [5] (see also [1, page 323]).

Theorem 1.5. *Let $\mathbf{x}$, $\mathbf{y}$ be two decreasing real n-tuples, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a real n-tuple such that*

$$\begin{gathered} \sum_{i=1}^{k} p_i x_i \leq \sum_{i=1}^{k} p_i y_i \quad \text{for } k = 1, \ldots, n-1, \\ \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i. \end{gathered} \tag{1.10}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, one has

$$\sum_{i=1}^{n} p_i \phi(x_i) \leq \sum_{i=1}^{n} p_i \phi(y_i). \tag{1.11}$$

Definition 1.6. A function $h : (a, b) \to \mathbb{R}$ is exponentially convex function if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geq 0 \tag{1.12}$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i \in (a, b)$, $i = 1, \ldots, n$ such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

The following proposition is given in [6].

Proposition 1.7. *Let $h : (a, b) \to \mathbb{R}$. The following propositions are equivalent.*

(i) *h is exponentially convex.*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.13}$$

for every $n \in \mathbb{N}$, every $\xi_i \in \mathbb{R}$, and every $x_i, x_j \in (a, b)$, $1 \leq i, j \leq n$.

Corollary 1.8. *If h is exponentially convex, then*

$$\det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0, \tag{1.14}$$

for every $n \in \mathbb{N}$ and every $x_i \in (a,b)$, $i = 1,\ldots,n$.

Corollary 1.9. *If $h : (a,b) \to \mathbb{R}^+$ is exponentially convex function, then h is a* log*-convex function in Jensens sense:*

$$h\left(\frac{x+y}{2}\right) \leq \sqrt{h(x)h(y)}, \quad \forall x,y \in (a,b). \tag{1.15}$$

In this paper, we prove positive semidefiniteness of matrices generated by differences deduced from majorization-type results (1.9), (1.11), (4.2), and (4.5) which implies exponential convexity and log-convexity of these differences and also obtain Lyapunov's and Dresher's inequalities for these differences. In [7], new Cauchy means are introduced. By using these means, a generalization of (1.2) was given (see [7]). In the present paper, we give related results in discrete and indiscrete cases and some new means of the Cauchy type.

2. Main Results

Lemma 2.1. *Define the function*

$$\varphi_s(x) := \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x\log x, & s = 1. \end{cases} \tag{2.1}$$

Then $\varphi_s'' = x^{s-2}$, that is, φ_s is convex for $x > 0$.

Definition 2.2. It is said that a positive function f is log-convex in the Jensen sense on some interval $I \subseteq \mathbb{R}$ if

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right) \tag{2.2}$$

holds for every $s,t \in I$.

The following lemma gives an equivalent condition for convexity of function f [1, page 2].

Lemma 2.3. *If ϕ is convex on an interval $I \subseteq \mathbb{R}$, then*

$$\phi(s_1)(s_3-s_2) + \phi(s_2)(s_1-s_3) + \phi(s_3)(s_2-s_1) \geq 0 \tag{2.3}$$

holds for every $s_1 < s_2 < s_3$, $s_1,s_2,s_3 \in I$.

Theorem 2.4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive* n*-tuples,* $\mathbf{y} \succ \mathbf{x}$*,*

$$\Lambda_t = \Lambda_t(\mathbf{x};\mathbf{y}) := \sum_{i=1}^{n}\varphi_t(y_i) - \sum_{i=1}^{n}\varphi_t(x_i), \tag{2.4}$$

and all $x_{[i]}$*'s and* $y_{[i]}$*'s are not equal.*

Then the following statements are valid.

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, \dots, s_n \in \mathbb{R}$*, the matrix* $[\Lambda_{(s_i+s_j)/2}]_{i,j=1}^{n}$ *is a positive semidefinite matrix. Particularly*

$$\det\left[\Lambda_{(s_i+s_j)/2}\right]_{i,j=1}^{k} \geq 0 \tag{2.5}$$

for $k = 1, \dots, n$*.*

(b) *The function* $s \mapsto \Lambda_s$ *is exponentially convex.*

(c) *The function* $s \mapsto \Lambda_s$ *is* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$*:*

$$\Lambda_s^{t-r} \leq \Lambda_r^{t-s}\ \Lambda_t^{s-r}. \tag{2.6}$$

Proof. (a) Consider the function

$$\mu(x) = \sum_{i,j}^{k} u_i u_j \varphi_{s_{ij}}(x) \tag{2.7}$$

for $k = 1, \dots, n$, $x > 0$, $u_i \in \mathbb{R}$, $s_{ij} \in \mathbb{R}$, where $s_{ij} = (s_i + s_j)/2$ and $\varphi_{s_{ij}}$ is defined in (2.1).

We have

$$\mu''(x) = \sum_{i,j}^{k} u_i u_j x^{s_{ij}-2} = \left(\sum_{i}^{k} u_i x^{s_i/2-1}\right)^2 \geq 0, \quad x \geq 0. \tag{2.8}$$

This shows that μ is a convex function for $x \geq 0$.

Using Theorem 1.3,

$$\sum_{m=1}^{n}\mu(y_m) - \sum_{m=1}^{n}\mu(x_m) \geq 0. \tag{2.9}$$

This implies that

$$\sum_{m=1}^{n}\left(\sum_{i,j}^{k} u_i u_j \varphi_{s_{ij}}(y_m)\right) - \sum_{m=1}^{n}\left(\sum_{i,j}^{k} u_i u_j \varphi_{s_{ij}}(x_m)\right) \geq 0, \tag{2.10}$$

or equivalently

$$\sum_{i,j}^{k} u_i u_j \Lambda_{s_{ij}} \geq 0. \tag{2.11}$$

From last inequality, it follows that the matrix $[\Lambda_{(s_i+s_j)/2}]_{i,j=1}^{n}$ is a positive semidefinite matrix, that is, (2.5) is valid.

(b) Note that Λ_s is continuous for $s \in \mathbb{R}$. Then by using Proposition 1.7, we get exponentially convexity of the function $s \to \Lambda_s$.

(c) Since $\varphi_t(x)$ is continuous and strictly convex function for $x > 0$ and all $x_{[i]}$'s and $y_{[i]}$'s are not equal, therefore by Theorem 1.3 with $\phi = \varphi_t$ we have

$$\sum_{i=1}^{n} \varphi_t(y_i) > \sum_{i=1}^{n} \varphi_t(x_i). \tag{2.12}$$

This implies

$$\Lambda_t = \Lambda_t(\mathbf{x}; \mathbf{y}) = \sum_{i=1}^{n} \varphi_t(y_i) - \sum_{i=1}^{n} \varphi_t(x_i) > 0, \tag{2.13}$$

that is, Λ_t is a positive-valued function.

A simple consequence of Corollary 1.9 is that Λ_s is log-convex; then by definition

$$\log \Lambda_s^{t-r} \leq \log \Lambda_r^{t-s} + \log \Lambda_t^{s-r}, \tag{2.14}$$

which is equivalent to (2.6). □

Theorem 2.5. *Let Λ_t be defined as in Theorem 2.4 and $t, s, u, v \in \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then*

$$\left(\frac{\Lambda_t}{\Lambda_s}\right)^{1/(t-s)} \leq \left(\frac{\Lambda_v}{\Lambda_u}\right)^{1/(v-u)}. \tag{2.15}$$

Proof. For a convex function φ, a simple consequence of (2.3) is the following inequality [1, page 2]:

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{2.16}$$

with $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$. Since by Theorem 2.4(c) and Λ_t is log-convex, we can set in (2.16) $\varphi(x) = \log \Lambda_t,\ x_1 = s,\ x_2 = t,\ y_1 = u$, and $y_2 = v$. We get

$$\frac{\log \Lambda_t - \log \Lambda_s}{t - s} \leq \frac{\log \Lambda_v - \log \Lambda_u}{v - u}, \tag{2.17}$$

from which (2.15) follows. □

Theorem 2.6. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive decreasing* n*-tuples, let* $\mathbf{p} = (p_1, \ldots, p_n)$ *be a real* n*-tuple and let*

$$\lambda_t = \lambda_t(\mathbf{x}, \mathbf{y}; \mathbf{p}) \ := \sum_{i=1}^{n} p_i \varphi_t(y_i) - \sum_{i=1}^{n} p_i \varphi_t(x_i) \tag{2.18}$$

such that conditions (1.10) *are satisfied and* λ_t *is positive.*

Then the following statements are valid.

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in \mathbb{R}$, *the matrix* $[\lambda_{(s_i+s_j)/2}]_{i,j=1}^{n}$ *is a positive semidefinite matrix. Particularly*

$$\det\left[\lambda_{(s_i+s_j)/2}\right]_{i,j=1}^{k} \geq 0 \tag{2.19}$$

for $k = 1, \ldots, n$.

(b) *The function* $s \mapsto \lambda_s$ *is exponentially convex.*

(c) *The function* $s \mapsto \lambda_s$ *is* log-convex *on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$\lambda_s^{t-r} \leq \lambda_r^{t-s}\ \lambda_t^{s-r}. \tag{2.20}$$

Proof. As in the proof of Theorem 2.4, we use Theorem 1.5 instead of Theorem 1.3. □

Theorem 2.7. *Let* λ_t *be defined as in Theorem 2.6 and* $t, s, u, v \in \mathbb{R}$ *such that* $s \leq u$, $t \leq v$, $s \neq t$, *and* $u \neq v$. *Then*

$$\left(\frac{\lambda_t}{\lambda_s}\right)^{1/(t-s)} \leq \left(\frac{\lambda_v}{\lambda_u}\right)^{1/(v-u)}. \tag{2.21}$$

Proof. Similar to the proof of Theorem 2.5. □

3. Cauchy Means

Let us note that (2.15) and (2.21) have the form of some known inequalities between means (e.g., Stolarsky means, Gini means, etc). Here we will prove that expressions on both sides of (2.15) and (2.21) are also means.

Lemma 3.1. *Let $f \in C^2(I)$, I interval in $\mathbb{R}$, be such that*

$$m \leq f''(x) \leq M. \tag{3.1}$$

Consider the functions ϕ_1, ϕ_2 defined as

$$\begin{aligned}\phi_1(x) &= \frac{Mx^2}{2} - f(x),\\ \phi_2(x) &= f(x) - \frac{mx^2}{2},\end{aligned} \tag{3.2}$$

then $\phi_i(x)$ for $i = 1, 2$ are convex.

Proof. Since

$$\begin{aligned}\phi_1''(x) &= M - f''(x) \geq 0,\\ \phi_2''(x) &= f''(x) - m \geq 0,\end{aligned} \tag{3.3}$$

that is, ϕ_i for $i = 1, 2$ are convex. □

Denote

$$I_1 = [m_1, M_1], \quad \text{where } m_1 = \min\{m_{\mathbf{x}}, m_{\mathbf{y}}\},\ M_1 = \max\{M_{\mathbf{x}}, M_{\mathbf{y}}\}. \tag{3.4}$$

In the above expression, $m_{\mathbf{x}}$ and $m_{\mathbf{y}}$ are the minimums of $\mathbf{x}$ and $\mathbf{y}$, respectively. Similarly, $M_{\mathbf{x}}$ and $M_{\mathbf{y}}$ are the maximums of $\mathbf{x}$ and $\mathbf{y}$ respectively.

Theorem 3.2. *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive n-tuples, $\mathbf{y} \succ \mathbf{x}$, all $x_{[i]}$'s and $y_{[i]}$'s are not equal, and $f \in C^2(I_1)$, with I_1 being defined as in (3.4), then there exists $\xi \in I_1$ such that*

$$\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i) = \frac{f''(\xi)}{2}\left\{\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} x_i^2\right\}. \tag{3.5}$$

Proof. Since $f \in C^2(I_1)$ and I_1 is compact, then $m_1 \leq f''(x) \leq M_1$ for $x \in I_1$. Then by applying ϕ_1 and ϕ_2 defined in Lemma 3.1 for ϕ in Theorem 1.3, we have

$$\begin{aligned}\sum_{i=1}^{n} \phi_1(x_i) &\leq \sum_{i=1}^{n} \phi_1(y_i),\\ \sum_{i=1}^{n} \phi_2(x_i) &\leq \sum_{i=1}^{n} \phi_2(y_i),\end{aligned} \tag{3.6}$$

that is,

$$\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i) \le \frac{M_1}{2}\left\{\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} x_i^2\right\}, \tag{3.7}$$

$$\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i) \ge \frac{m_1}{2}\left\{\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} x_i^2\right\}. \tag{3.8}$$

By combining (3.7) and (3.8)

$$m_1 \le 2\left(\frac{\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} {x_i}^2}\right) \le M_1. \tag{3.9}$$

$\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} {x_i}^2 \neq 0$ because $x_{[i]} \neq y_{[i]}$, for $i = 1, \ldots, n$ by using Remark 1.4. Using the fact that for $m_1 \le \rho \le M_1$, there exists $\xi \in I_1$ such that $f''(\xi) = \rho$, we get (3.5). □

Theorem 3.3. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive* n*-tuples,* $\mathbf{y} \succ \mathbf{x}$*, all* $x_{[i]}$*'s and* $y_{[i]}$*'s are not equa,and* $f, g \in C^2(I_1)$*, with* I_1 *being defined as in* (3.4)*, then there exists* $\xi \in I_1$ *such that*

$$\frac{\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} g(y_i) - \sum_{i=1}^{n} g(x_i)} = \frac{f''(\xi)}{g''(\xi)}, \tag{3.10}$$

provided that $g''(x) \neq 0$ *for every* $x \in I_1$.

Proof. Let a function $k \in C^2(I_1)$ be defined as

$$k = c_1 f - c_2 g, \tag{3.11}$$

where c_1 and c_2 are defined as

$$\begin{aligned} c_1 &= \sum_{i=1}^{n} g(y_i) - \sum_{i=1}^{n} g(x_i), \\ c_2 &= \sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i). \end{aligned} \tag{3.12}$$

Then, using Theorem 3.2 with $f = k$, we have

$$0 = \left(c_1 \frac{f''(\xi)}{2} - c_2 \frac{g''(\xi)}{2}\right)\left\{\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} {x_i}^2\right\}. \tag{3.13}$$

Since

$$\sum_{i=1}^{n} {y_i}^2 - \sum_{i=1}^{n} {x_i}^2 \neq 0, \tag{3.14}$$

therefore, (3.13) gives

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}. \tag{3.15}$$

After putting values, we get (3.10). The denominator of left-hand side is nonzero by using $f = g$ in Theorem 3.2. □

Corollary 3.4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive* n*-tuples such that* $\mathbf{y} \succ \mathbf{x}$ *and all* $x_{[i]}$*'s and* $y_{[i]}$*'s are not equal, then for* $-\infty < s \neq t \neq 0,\ 1 \neq s < +\infty$ *there exists* $\xi \in I_1$*, with* I_1 *being defined as in* (3.4)*, such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} y_i{}^t - \sum_{i=1}^{n} x_i{}^t}{\sum_{i=1}^{n} y_i{}^s - \sum_{i=1}^{n} x_i{}^s}. \tag{3.16}$$

Proof. Setting $f(x) = x^t$ and $g(x) = x^s$, $t \neq s \neq 0, 1$ in (3.10), we get (3.16). □

Remark 3.5. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, then from (3.16) we have

$$m_1 \leq \left\{ \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} y_i{}^t - \sum_{i=1}^{n} x_i{}^t}{\sum_{i=1}^{n} y_i{}^s - \sum_{i=1}^{n} x_i{}^s} \right\}^{1/(t-s)} \leq M_1. \tag{3.17}$$

In fact, similar result can also be given for (3.10). Namely, suppose that f''/g'' has inverse function. Then from (3.10), we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1} \left(\frac{\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} g(y_i) - \sum_{i=1}^{n} g(x_i)} \right). \tag{3.18}$$

So, we have that the expression on the right-hand side of (3.18) is also a mean. By the inequality (3.17), we can consider for positive n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{x}$,

$$M_{t,s} = \left(\frac{\Lambda_t}{\Lambda_s}\right)^{1/(t-s)} \tag{3.19}$$

for $-\infty < s \neq t < +\infty$, as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{gathered} M_{s,s} = \exp\left(\frac{\sum_{i=1}^{n} y_i{}^s \log y_i - \sum_{i=1}^{n} x_i{}^s \log x_i}{\sum_{i=1}^{n} y_i{}^s - \sum_{i=1}^{n} x_i{}^s} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ M_{0,0} = \exp\left(\frac{\sum_{i=1}^{n} \log^2 y_i - \sum_{i=1}^{n} \log^2 x_i}{2\ \left[\sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} \log x_i\right]} + 1 \right), \\ M_{1,1} = \exp\left(\frac{\sum_{i=1}^{n} y_i \log^2 y_i - \sum_{i=1}^{n} x_i \log^2 x_i}{2\ \left[\sum_{i=1}^{n} y_i \log y_i - \sum_{i=1}^{n} x_i \log x_i\right]} - 1 \right). \end{gathered} \tag{3.20}$$

Theorem 3.6. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following inequality is valid:*

$$M_{t,s} \leq M_{u,v}. \tag{3.21}$$

Proof. Since Λ_s is log-convex, therefore by (2.15) we get (3.21). □

Theorem 3.7. *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive decreasing n-tuples, let $\mathbf{p}$ be a real n-tuple such that conditions* (1.10) *are satisfied, λ_t is positive defined as in Theorem 2.6, and $f \in C^2(I_1)$, with I_1 being defined as in* (3.4), *then there exists $\xi \in I_1$ such that*

$$\sum_{i=1}^{n} p_i f(y_i) - \sum_{i=1}^{n} p_i f(x_i) = \frac{f''(\xi)}{2}\left\{\sum_{i=1}^{n} p_i {y_i}^2 - \sum_{i=1}^{n} p_i x_i^2\right\}, \tag{3.22}$$

Proof. As in the proof of Theorem 3.2, we use Theorem 1.5 instead of Theorem 1.3. □

Theorem 3.8. *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive decreasing n-tuples, $\mathbf{p}$ be a real n-tuple such that conditions* (1.10) *are satisfied, λ_t is positive defined as in Theorem 2.6 and $f, g \in C^2(I_1)$, I_1 is defined as in* (3.4). *Then there exists $\xi \in I_1$ such that*

$$\frac{\sum_{i=1}^{n} p_i f(y_i) - \sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i g(y_i) - \sum_{i=1}^{n} p_i g(x_i)} = \frac{f''(\xi)}{g''(\xi)}. \tag{3.23}$$

provided that $g''(x) \neq 0$ for every $x \in I_1$.

Proof. Similar to the proof of Theorem 3.3. □

Corollary 3.9. *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive decreasing n-tuples, let $\mathbf{p}$ be a real n-tuple such that conditions* (1.10) *are satisfied and λ_t is positive defined as in Theorem 2.6, then for $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$ there exists $\xi \in I_1$, with I_1 being defined as in* (3.4), *such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} p_i {y_i}^t - \sum_{i=1}^{n} p_i {x_i}^t}{\sum_{i=1}^{n} p_i {y_i}^s - \sum_{i=1}^{n} p_i {x_i}^s}. \tag{3.24}$$

Proof. Setting $f(x) = x^t$ and $g(x) = x^s$, $t \neq s \neq 0, 1$ in (3.23), we get (3.24). □

Remark 3.10. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, then from (3.24) we have

$$m_1 \leq \left\{\frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} p_i {y_i}^t - \sum_{i=1}^{n} p_i {x_i}^t}{\sum_{i=1}^{n} p_i {y_i}^s - \sum_{i=1}^{n} p_i {x_i}^s}\right\}^{1/(t-s)} \leq M_1. \tag{3.25}$$

In fact, similar result can also be given for (3.23). Namely, suppose that f''/g'' has inverse function. Then from (3.23), we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\sum_{i=1}^{n} p_i f(y_i) - \sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i g(y_i) - \sum_{i=1}^{n} p_i g(x_i)}\right). \tag{3.26}$$

So, we have that the expression on the right-hand side of (3.26) is also a mean. By the inequality (3.25), we can consider for positive n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that conditions (1.10) are satisfied, and

$$\widetilde{M}_{t,s} = \left(\frac{\lambda_t}{\lambda_s}\right)^{1/(t-s)} \tag{3.27}$$

for $-\infty < s \neq t < +\infty$, as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned}
\widetilde{M}_{s,s} &= \exp\left(\frac{\sum_{i=1}^{n} p_i \; y_i{}^s \log y_i - \sum_{i=1}^{n} p_i x_i{}^s \log x_i}{\sum_{i=1}^{n} p_i y_i{}^s - \sum_{i=1}^{n} p_i x_i{}^s} - \frac{2s-1}{s(s-1)}\right), \quad s \neq 0, 1. \\
\widetilde{M}_{0,0} &= \exp\left(\frac{\sum_{i=1}^{n} p_i \log^2 y_i - \sum_{i=1}^{n} p_i \log^2 x_i}{2\left(\sum_{i=1}^{n} p_i \log y_i - \sum_{i=1}^{n} p_i \log x_i\right)} + 1\right). \\
\widetilde{M}_{1,1} &= \exp\left(\frac{\sum_{i=1}^{n} p_i y_i \log^2 y_i - \sum_{i=1}^{n} p_i x_i \log^2 x_i}{2\left(\sum_{i=1}^{n} p_i y_i \log y_i - \sum_{i=1}^{n} p_i x_i \log x_i\right)} - 1\right).
\end{aligned} \tag{3.28}$$

Theorem 3.11. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following inequality is valid:*

$$\widetilde{M}_{t,s} \leq \widetilde{M}_{u,v}. \tag{3.29}$$

Proof. Since λ_s is log-convex, therefore by (2.21) we get (3.29). □

4. Some Related Results

Let $x(\tau)$, $y(\tau)$ be real valued functions defined on an interval $[a, b]$ such that $\int_a^s x(\tau)d\tau$, $\int_a^s y(\tau)d\tau$ both exist for all $s \in [a, b]$.

Definition 4.1 (see [1, page 324]). $y(\tau)$ is said to majorize $x(\tau)$, in symbol, $y(\tau) \succ x(\tau)$, for $\tau \in [a, b]$ if they are decreasing in $\tau \in [a, b]$ and

$$\int_a^s x(\tau)d\tau \leq \int_a^s y(\tau)d\tau \quad \text{for } s \in [a, b], \tag{4.1}$$

and equality in (4.1) holds for $s = b$.

The following theorem can be regarded as a majorization theorem in integral case [1, page 325].

Theorem 4.2. *$y(\tau) \succ x(\tau)$ for $\tau \in [a,b]$ iff they are decreasing in $[a,b]$ and*

$$\int_a^b \phi(x(\tau))d\tau \leq \int_a^b \phi(y(\tau))d\tau \tag{4.2}$$

holds for every ϕ that is continuous, and convex in $[a,b]$ such that the integrals exist.

The following theorem is a simple consequence of Theorem 12.14 in [8] (see also [1, page 328]):

Theorem 4.3. *Let $x(\tau), y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are continuous and increasing and let $G : [a,b] \to \mathbb{R}$ be a function of bounded variation.*

(a) *If*

$$\int_\nu^b x(\tau)dG(\tau) \leq \int_\nu^b y(\tau)dG(\tau) \quad \text{for every } \nu \in [a,b], \tag{4.3}$$

$$\int_a^b x(\tau)dG(\tau) = \int_a^b y(\tau)dG(\tau) \tag{4.4}$$

hold, then for every continuous convex function f, one has

$$\int_a^b f(x(\tau))dG(\tau) \leq \int_a^b f(y(\tau))dG(\tau). \tag{4.5}$$

(b) *If* (4.3) *holds, then* (4.5) *holds for every continuous increasing convex function f.*

Theorem 4.4. *Let $x(\tau)$ and $y(\tau)$ be two positive functions defined on an interval $[a,b]$, decreasing in $[a,b]$, $y(\tau) \succ x(\tau)$,*

$$\beta_t(x(\tau); y(\tau)) := \int_a^b \varphi_t(y(\tau))d\tau - \int_a^b \varphi_t(x(\tau))d\tau, \tag{4.6}$$

and β_t is positive.

Then the following statements are valid.

(a) *For every $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{R}$, the matrix $[\beta_{(s_i+s_j)/2}]_{i,j=1}^n$ is a positive semidefinite matrix. Particularly*

$$\det\left[\beta_{(s_i+s_j)/2}\right]_{i,j=1}^k \geq 0 \tag{4.7}$$

for $k = 1, \ldots, n$.

(b) *The function $s \mapsto \beta_s$ is exponentially convex.*

(c) *The function $s \mapsto \beta_s$ is log-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\beta_s^{t-r} \le \beta_r^{t-s} \beta_t^{s-r}. \tag{4.8}$$

Proof. As in the proof of Theorem 2.4, we use Theorem 4.2 instead of Theorem 1.3. □

Theorem 4.5. *Let β_t be defined as in Theorem 4.4 and $t, s, u, v \in \mathbb{R}$ such that $s \le u$, $t \le v$, $s \ne t$, and $u \ne v$. Then*

$$\left(\frac{\beta_t}{\beta_s}\right)^{1/(t-s)} \le \left(\frac{\beta_v}{\beta_u}\right)^{1/(v-u)}. \tag{4.9}$$

Proof. Similar to the proof of Theorem 2.5. □

Denote

$$I_2 = [m_2, M_2], \quad \text{where } m_2 = \min\{m_{x(\tau)}, m_{y(\tau)}\},\ M_2 = \max\{M_{x(\tau)}, M_{y(\tau)}\}. \tag{4.10}$$

In the above expression, $m_{x(\tau)}$ and $m_{y(\tau)}$ are the minimums of $x(\tau)$ and $y(\tau)$, respectively. Similarly, $M_{x(\tau)}$ and $M_{y(\tau)}$ are the maximums of $x(\tau)$ and $y(\tau)$, respectively.

Theorem 4.6. *Let $x(\tau)$ and $y(\tau)$ be two positive decreasing functions in $[a, b]$ such that $y(\tau) \succ x(\tau)$, β_t is positive defined as in Theorem 4.4, and $f \in C^2(I_2)$, with I_2 being defined as in* (4.10)*, then there exists $\xi \in I_2$ such that*

$$\int_a^b f(y(\tau))d\tau - \int_a^b f(x(\tau))d\tau = \frac{f''(\xi)}{2}\left\{\int_a^b y^2(\tau)d\tau - \int_a^b x^2(\tau)d\tau\right\}. \tag{4.11}$$

Proof. As in the proof of Theorem 3.2, we use Theorem 4.2 instead of Theorem 1.3. □

Theorem 4.7. *Let $x(\tau)$ and $y(\tau)$ be two positive decreasing functions in $[a, b]$ such that $y(\tau) \succ x(\tau)$, β_t is positive defined as in Theorem 4.4, and $f, g \in C^2(I_2)$, with I_2 being defined as in* (4.10)*. Then there exists $\xi \in I_2$ such that*

$$\frac{\int_a^b f(y(\tau))d\tau - \int_a^b f(x(\tau))d\tau}{\int_a^b g(y(\tau))d\tau - \int_a^b g(x(\tau))d\tau} = \frac{f''(\xi)}{g''(\xi)}, \tag{4.12}$$

provided that $g''(z) \ne 0$ for every $z \in I_2$.

Proof. Similar to the proof of Theorem 3.3. □

Corollary 4.8. *Let $x(\tau)$ and $y(\tau)$ be two positive decreasing functions in $[a,b]$ such that $y(\tau) > x(\tau)$ and β_t is positive defined as in Theorem 4.4, then for $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$ there exists $\xi \in I_2$, with I_2 being defined as in* (4.10)*, such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\int_a^b y^t(\tau)d\tau - \int_a^b x^t(\tau)d\tau}{\int_a^b y^s(\tau)d\tau - \int_a^b x^s(\tau)d\tau}. \tag{4.13}$$

Proof. Set $f(x) = x^t$ and $g(x) = x^s$, $t \neq s \neq 0, 1$ in (4.12), we get (4.13). □

Remark 4.9. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, then from (4.13) we have

$$m_2 \leq \left\{ \frac{s(s-1)}{t(t-1)} \frac{\int_a^b y^t(\tau)d\tau - \int_a^b x^t(\tau)d\tau}{\int_a^b y^s(\tau)d\tau - \int_a^b x^s(\tau)d\tau} \right\}^{1/(t-s)} \leq M_2. \tag{4.14}$$

In fact, similar result can also be given for (4.12). Namely, suppose that f''/g'' has inverse function. Then from (4.12), we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1} \left(\frac{\int_a^b f(y(\tau))d\tau - \int_a^b f(x(\tau))d\tau}{\int_a^b g(y(\tau))d\tau - \int_a^b g(x(\tau))d\tau} \right). \tag{4.15}$$

So, we have that the expression on the right-hand side of (4.15) is also a mean. By the inequality (4.14), we can consider for positive functions $x(\tau)$ and $y(\tau)$ such that $y(\tau) > x(\tau)$, and

$$\widehat{M}_{t,s} = \left(\frac{\beta_t}{\beta_s}\right)^{1/(t-s)} \tag{4.16}$$

for $-\infty < s \neq t < +\infty$, as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned} \widehat{M}_{s,s} &= \exp\left(\frac{\int_a^b y^s(\tau) \log y(\tau)d\tau - \int_a^b x^s(\tau) \log x(\tau)d\tau}{\int_a^b y^s(\tau)d\tau - \int_a^b x^s(\tau)d\tau} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ \widehat{M}_{0,0} &= \exp\left(\frac{\int_a^b \log^2 y(\tau)d\tau - \int_a^b \log^2 x(\tau)d\tau}{2\left[\int_a^b \log y(\tau)d\tau - \int_a^b \log x(\tau)d\tau\right]} + 1 \right), \\ \widehat{M}_{1,1} &= \exp\left(\frac{\int_a^b y(\tau)\log^2 y(\tau)d\tau - \int_a^b x(\tau)\log^2 x(\tau)d\tau}{2\left[\int_a^b y(\tau) \log y(\tau)d\tau - \int_a^b x(\tau) \log x(\tau)d\tau\right]} - 1 \right). \end{aligned} \tag{4.17}$$

Theorem 4.10. *Let $t, s, u, v \in \mathbb{R}$ such that $t \le u$, $s \le v$, then the following inequality is valid:*

$$\widehat{M}_{t,s} \le \widehat{M}_{u,v}. \tag{4.18}$$

Proof. Since β_s is log-convex, therefore by (4.9) we get (4.18). □

Theorem 4.11. *Let $x(\tau), y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are positive, continuous, and increasing and let $G : [a,b] \to \mathbb{R}$ be a function of bounded variation. Also let*

$$\Gamma_t(x(\tau), y(\tau);\ G(\tau)) := \int_a^b \varphi_t(y(\tau))dG(\tau) - \int_a^b \varphi_t(x(\tau))dG(\tau), \tag{4.19}$$

such that conditions (4.3) *and* (4.4) *are satisfied and Γ_t is positive.*

Then the following statements are valid.

(a) *For every $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{R}$, the matrix $[\Gamma_{(s_i+s_j)/2}]_{i,j=1}^n$ is a positive semidefinite matrix. Particularly*

$$\det\left[\Gamma_{(s_i+s_j)/2}\right]_{i,j=1}^k \ge 0 \tag{4.20}$$

for $k = 1, \ldots, n$.

(b) *The function $s \mapsto \Gamma_s$ is exponentially convex.*

(c) *The function $s \mapsto \Gamma_s$ is* log-convex *on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\Gamma_s^{t-r} \le \Gamma_r^{t-s}\Gamma_t^{s-r}. \tag{4.21}$$

Proof. As in the proof of Theorem 2.4, we use Theorem 4.3 instead of Theorem 1.3. □

Theorem 4.12. *Let Γ_t be defined as in Theorem 4.11 and $t, s, u, v \in \mathbb{R}$ such that $s \le u$, $t \le v$, $s \ne t$, and $u \ne v$. Then*

$$\left(\frac{\Gamma_t}{\Gamma_s}\right)^{1/(t-s)} \le \left(\frac{\Gamma_v}{\Gamma_u}\right)^{1/(v-u)}. \tag{4.22}$$

Proof. Similar to the proof of Theorem 2.5. □

Theorem 4.13. *Let $x(\tau)$ and $y(\tau)$ be positive, continuous, and increasing functions in $[a,b]$ such that conditions* (4.3) *and* (4.4) *are satisfied, Γ_t is positive defined as in Theorem 4.11, $f \in C^2(I_2)$, with*

I_2 being defined as in (4.10)*, and $G : [a,b] \to \mathbb{R}$ be a function of bounded variation, then there exists $\xi \in I_2$ such that*

$$\int_a^b f(y(\tau))dG(\tau) - \int_a^b f(x(\tau))dG(\tau) = \frac{f''(\xi)}{2}\left\{\int_a^b y^2(\tau)dG(\tau) - \int_a^b x^2(\tau)dG(\tau)\right\}. \tag{4.23}$$

Proof. As in the proof of Theorem 3.2, we use Theorem 4.3 instead of Theorem 1.3. □

Theorem 4.14. *Let $x(\tau)$ and $y(\tau)$ be positive, continuous and increasing functions in $[a,b]$ such that conditions* (4.3) *and* (4.4) *are satisfied, $G : [a,b] \to \mathbb{R}$ be a function of bounded variation, Γ_t is positive defined as in Theorem 4.11, and $f, g \in C^2(I_2)$, with I_2 being defined as in* (4.10)*. Then there exists $\xi \in I_2$ such that*

$$\frac{\int_a^b f(y(\tau))dG(\tau) - \int_a^b f(x(\tau))dG(\tau)}{\int_a^b g(y(\tau))dG(\tau) - \int_a^b g(x(\tau))dG(\tau)} = \frac{f''(\xi)}{g''(\xi)}, \tag{4.24}$$

provided that $g''(z) \neq 0$ for every $z \in I_2$.

Proof. Similar to the proof of Theorem 3.3. □

Corollary 4.15. *Let $x(\tau)$ and $y(\tau)$ be positive, continuous, and increasing functions in $[a,b]$ such that conditions* (4.3) *and* (4.4) *be satisfied, Γ_t is positive defined as in Theorem 4.11, and $G : [a,b] \to \mathbb{R}$ be a function of bounded variation, then for $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$ there exists $\xi \in I_2$, with I_2 being defined as in* (4.10)*, such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)}\frac{\int_a^b y^t(\tau)dG(\tau) - \int_a^b x^t(\tau)dG(\tau)}{\int_a^b y^s(\tau)dG(\tau) - \int_a^b x^s(\tau)dG(\tau)}. \tag{4.25}$$

Proof. Setting $f(x) = x^t$ and $g(x) = x^s$, $t \neq s \neq 0, 1$ in (4.24), we get (4.25). □

Remark 4.16. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, then from (4.25) we have

$$m_2 \le \left\{\frac{s(s-1)}{t(t-1)}\frac{\int_a^b y^t(\tau)dG(\tau) - \int_a^b x^t(\tau)dG(\tau)}{\int_a^b y^s(\tau)dG(\tau) - \int_a^b x^s(\tau)dG(\tau)}\right\}^{1/(t-s)} \le M_2. \tag{4.26}$$

In fact, similar result can also be given for (4.24). Namely, suppose that f''/g'' has inverse function. Then from (4.24), we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\int_a^b f(y(\tau))\ dG(\tau) - \int_a^b f(x(\tau))\ dG(\tau)}{\int_a^b g(y(\tau))\ dG(\tau) - \int_a^b g(x(\tau))\ dG(\tau)}\right). \tag{4.27}$$

So, we have that the expression on the right-hand side of (4.27) is also a mean. By the inequality (4.26), we can consider for positive functions $x(\tau)$ and $y(\tau)$ such that conditions (4.3) and (4.4) are satisfied, and

$$\mathcal{M}_{t,s} = \left(\frac{\Gamma_t}{\Gamma_s}\right)^{1/(t-s)} \tag{4.28}$$

for $-\infty < s \neq t < +\infty$, as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned}
\mathcal{M}_{s,s} &= \exp\left(\frac{\int_a^b y^s(\tau)\log y(\tau)dG(\tau) - \int_a^b x^s(\tau)\log x(\tau)dG(\tau)}{\int_a^b y^s(\tau)dG(\tau) - \int_a^b x^s(\tau)dG(\tau)} - \frac{2s-1}{s(s-1)}\right), \quad s \neq 0,1, \\
\mathcal{M}_{0,0} &= \exp\left(\frac{\int_a^b \log^2 y(\tau)dG(\tau) - \int_a^b \log^2 x(\tau)dG(\tau)}{2\left[\int_a^b \log y(\tau)dG(\tau) - \int_a^b \log x(\tau)dG(\tau)\right]} + 1\right), \\
\mathcal{M}_{1,1} &= \exp\left(\frac{\int_a^b y(\tau)\log^2 y(\tau)dG(\tau) - \int_a^b x(\tau)\log^2 x(\tau)dG(\tau)}{2\left[\int_a^b y(\tau)\log y(\tau)dG(\tau) - \int_a^b x(\tau)\log x(\tau)dG(\tau)\right]} - 1\right).
\end{aligned} \tag{4.29}$$

Theorem 4.17. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following inequality is valid:*

$$\mathcal{M}_{t,s} \leq \mathcal{M}_{u,v}. \tag{4.30}$$

Proof. Since Γ_s is log-convex, therefore by (4.22) we get (4.30). □

Remark 4.18. Let $x = \sum_{i=1}^n p_i y_i / \sum_{i=1}^n p_i$ such that $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. If we substitute in Theorem 2.6 $(x_1; x_2; \ldots; x_n) = (x; x; \ldots; x)$, we get (1.2). In fact in such results we have that $\mathbf{y}$ is monotonic n-tuple. But since the weights are positive, our results are also valid for arbitrary $\mathbf{y}$.

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