Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 697954, 18 pages
doi:10.1155/2010/697954

Research Article

On Isoperimetric Inequalities in Minkowski Spaces

Horst Martini[1] and Zokhrab Mustafaev[2]

[1] *Faculty of Mathematics, University of Technology Chemnitz, 09107 Chemnitz, Germany*
[2] *Department of Mathematics, University of Houston-Clear Lake, Houston, TX 77058, USA*

Correspondence should be addressed to Horst Martini, horst.martini@mathematik.tu-chemnitz.de

Received 11 July 2009; Revised 2 December 2009; Accepted 4 March 2010

Academic Editor: Ulrich Abel

The purpose of this expository paper is to collect some (mainly recent) inequalities, conjectures, and open questions closely related to isoperimetric problems in real, finite-dimensional Banach spaces (= Minkowski spaces). We will also show that, in a way, Steiner symmetrization could be used as a useful tool to prove Petty's conjectured projection inequality.

1. Introductory General Survey

In Geometric Convexity, but also beyond its limits, isoperimetric inequalities have always played a central role. Applications of such inequalities can be found in Stochastic Geometry, Functional Analysis, Fourier Analysis, Mathematical Physics, Discrete Geometry, Integral Geometry, and various further mathematical disciplines.

We will present a survey on isoperimetric inequalities in real, finite-dimensional Banach spaces, also called *Minkowski spaces*. In the introductory part a very general survey on this topic is given, where we refer to historically important papers and also to results from Euclidean geometry that are potential to be extended to *Minkowski geometry*, that is, to the geometry of Minkowski spaces of dimension $d \geq 2$. The second part of the introductory survey then refers already to Minkowski spaces.

1.1. Historical Aspects and Results Mainly from Euclidean Geometry

Some of the isoperimetric inequalities have a long history, but many of them were also established in the second half of the 20th century. The most famous isoperimetric inequality is of course the classical one, establishing that among all simple closed curves of given length in the Euclidean plane the circle of the same circumference encloses maximum area; the respective inequality is given by

$$L^2 \geq 4\pi A, \tag{1.1}$$

with A being the area enclosed by a curve of length L and, thus, with equality if and only if the curve is a circle. In 3-space the analogous inequality states that if S is the surface area of a compact (convex) body of volume V, then

$$S^3 \geq 36\pi V^2 \tag{1.2}$$

holds, with equality if and only if the body is a ball. Note already here that the extremal bodies with respect to isoperimetric problems are usually called *isoperimetrices*.

Osserman [1] gives an excellent survey of many theoretical aspects of the classical isoperimetric inequality, explaining it first in the plane, extending it then to domains in $\mathbb{R}^n$, and describing also various applications (the reader is also referred to [2–4]). In the survey [5] the historical development of the classical isoperimetric problem in the plane is presented, and also different solution techniques are discussed. The author of [6] goes back to the early history of the isoperimetric problem. The paper [7] of Ritoré and Ros is a survey on the classical isoperimetric problem in $\mathbb{R}^3$, and the authors give also a modified version of this problem in terms of "free boundary". A further historical discussion of the isoperimetric problem is presented in [8]. In Chapters 8 and 9 of the book [9] many aspects and applications of isoperimetric problems are discussed, including also related inequalities, the Wulff shape (see the references given there and, in particular, of [10, Chapter 10]), and equilibrium capillary surfaces.

Isoperimetric inequalities appear in a large variety of contexts and have been proved in different ways; the occurring methods are often purely technical, but very elegant approaches exist, too. And also new isoperimetric inequalities are permanently obtained, even nowadays. In [11] (see also [12]), the authors prove L_p versions of Petty's projection inequality and the Busemann-Petty centroid inequality (see [13] and below for a discussion of these known inequalities) by using the method of Steiner symmetrization with respect to smooth L_p-projection bodies. In [14] equivalences of some affine isoperimetric inequalities, such as "duals" of L_p versions of Petty's projection inequality and "duals" of L_p versions of the Busemann-Petty inequality, are established; see also [15]. Here we also mention the paper [16], where the method of Steiner symmetrization is discussed and many references are given.

If K is a convex body in $\mathbb{R}^d$ with surface area S and volume V, then for $d = 2$ the Bonnesen inequality states that $S^2 - 4\pi V \geq \pi^2(R-r)^2$, where S is length, V is area, and r and R stand for in- and circumradius of K relative to the Euclidean unit ball (see also the definitions below), with equality if and only if K is a ball. In [17], Diskant extends Bonnesen's inequality (estimating the *isoperimetric deficit*, $(S(K)/d\epsilon_d r^{d-1})^d - (V(K)/\epsilon_d r^d)^{d-1}$, from below) for higher-dimensional spaces. Osserman establishes in [2] the following versions of the isoperimetric deficit in $\mathbb{R}^d$:

$$\begin{gathered}\left(\frac{S}{d\epsilon_d r^{d-1}}\right)^{d/(d-1)} - \frac{V}{\epsilon_d r^d} \geq \left[\left(\frac{S}{d\epsilon_d r^{d-1}}\right)^{1/(d-1)} - 1\right]^d, \\ \left(\frac{S}{d\epsilon_d r^{d-1}}\right)^d - \left(\frac{V}{\epsilon_d r^d}\right)^{d-1} \geq \left[\left(\frac{S}{d\epsilon_d r^{d-1}}\right)^{1/(d-1)} - 1\right]^{d(d-1)}.\end{gathered} \tag{1.3}$$

It is well known that a convex n-gon P_n with perimeter $L(P_n)$ and area $A(P_n)$ satisfies the isoperimetric inequality $L^2(P_n)/A(P_n) \geq 4n \tan(\pi/n)$. In [18] it is shown that this inequality can be embedded into a larger class of inequalities by applying a class of certain differential equations. Another interesting recent paper on isoperimetric properties of polygons is [19].

In [20] it is proved that if P is a simplicial polytope (i.e., a convex polytope all of whose proper faces are simplices) in $\mathbb{R}^d$ and $\zeta_k(P)$ is the total k-dimensional volume of the k-faces of P with $k \in \{1, \ldots, d\}$, then

$$\frac{\zeta_s^{1/s}(P)}{\zeta_r^{1/r}(P)} \leq \left(\frac{\binom{d-r}{d-s}}{\binom{s+1}{r+1}}\right)^{1/r} \frac{\left((1/s!)\sqrt{(s+1)/2^s}\right)^{1/s}}{\left((1/r!)\sqrt{(r+1)/2^r}\right)^{1/r}}, \tag{1.4}$$

where r and s are integers with $1 \leq r \leq s \leq d$, with equality if and only if P is a regular s-simplex.

The authors of [21] study the problem of maximizing A/L^2 for smooth closed curves C in $\mathbb{R}^d$, where L is again the length of C and A is an expression of signed areas which is determined by the orthogonal projections of C onto the coordinate-planes. They prove that $L^2 - (4\pi/\lambda)|A| \geq 0$, where λ is the largest positive number such that $i\lambda$ is an eigenvalue of the skew symmetric matrix with entries 0, 1, and -1.

An interesting and natural *reverse isoperimetric problem* was solved by Ball (see [22, 23, Lecture 6]). Namely, given a convex body $K \subset \mathbb{R}^d$, how small can the surface area of K be made by applying affine, volume-preserving transformations? In the general case the extremal body (with largest surface area) is the simplex, and for centrally symmetric K it is the cube. In [23, Lecture 5] a consequence of the Brunn-Minkowski inequality (see below) involving parallel bodies is discussed, and it is shown how it yields the isoperimetric inequality. Further important results in the direction of reverse isoperimetric inequalities are given in [11, 24]. The latter paper deals with L_p analogues of centroid and projection inequalities; a direct approach to the reverse inequalities for the unit balls of subspaces of L_p is given, with complete clarification of the extremal cases.

In [25] the authors prove that if K and M are compact, convex sets in the Euclidean plane, then $V(K, M) \leq L(K)L(M)/8$ with equality if and only if K and M are orthogonal segments or one of the sets is a point (here $V(K, M)$ denotes the *mixed volume* of K and M, defined below). They also show that $V(K, -K) \leq (\sqrt{3}/18)L^2(K)$; the equality case is known only when K is a polygon.

1.2. The Isoperimetric Problem in Normed Spaces

For (normed or) *Minkowski planes* the isoperimetric problem can be stated in the following way: among all simple closed curves of given *Minkowski length* (= length measured in the norm) find those enclosing largest area. Here the Minkowski length of a closed curve C can also be interpreted as the mixed area of C and the polar reciprocal of the Minkowskian unit circle with respect to the Euclidean unit circle rotated through 90°. In [26] (as well as in [27]) the *solution of the isoperimetric problem for Minkowski planes* is established. Namely, these extremal curves, called *isoperimetrices* I_B, are translates of the rotated polar reciprocals as described above. Conversely, the same applies to curves of minimal Minkowski length enclosing a given fixed area.

In [28] it is proved that for the Minkowski metric $ds = (dx^n + dy^n)^{1/n}$, where $n \geq 2$ is an integer, the solutions of the isoperimetric problem have the form $(x-A)^{n/(n-1)} + (B-y)^{n/(n-1)} = c$, and in [29] the particular case of taxicab geometry is studied.

In [30] the following isoperimetric inequality for a convex n-gon P in a Minkowski plane with unit disc B and isoperimetrix I_B is obtained: if P^* is the n-gon whose sides are parallel to those of P and which is circumscribed about I_B, then $L^2(P) - 4A(P)A(P^*) \geq 0$, with equality if and only if P is circumscribed about an anticircle of radius r, where L stands for the Minkowskian perimeter and A for area. (An *anticircle* of radius r is any translate of a homothetical copy of I_B with homothety ratio r.)

In [31] the isoperimetric problem in Minkowski planes is discussed for the case that the isoperimetrix is the polar reciprocal of unit discs related to duals of L_p-spaces.

In [32] some families of smooth curves in Minkowski planes are studied. It is shown that if C is a closed convex curve with length $L(C)$ enclosing area $A(C)$, and C' is an anticircle with radius $r > 0$ enclosing area $A(C')$, then $r^2L^2(C) \geq 4A(C)A(C')$. This inequality is also extended to closed nonconvex curves.

In [33] star-shaped domains in $\mathbb{R}^d$, presented in polar coordinates by equations of the form $R = 1+u(e)$, are investigated, with e being vector from the unit sphere. The isoperimetric deficit $\Delta := (S/d\epsilon_d)(V/\epsilon_d)^{-(d-1)/d} - 1$ of these domains is estimated for various norms of u, where again S and V denote surface area and volume of the domain and ϵ_d stands for the volume of the standard Euclidean ball.

Since a Minkowski space is a normed space, the given norm defines a usual metric m in such a space. In [34] it is proved that if J is a rectifiable Jordan curve of Minkowski length $L_m(J)$, that is, with respect to the Minkowski metric m, then there is, up to translation, a centrally symmetric curve C_J such that $L_m(C_J) = L_m(J)$ for all m. Also, the isoperimetric problem for rectifiable Jordan curves is solved here. Here C_J encloses the largest area in the class of rectifiable Jordan curves $\{K \in \mathbb{R}^2 : L_m(K) = L_m(J), \text{ for any } m\}$.

In [35] the notion of Minkowski space is extended by considering unit spheres as closed, but in general nonsymmetric hypersurfaces, also called *gauges*. The author gives a suitable definition of volume and applies this definition for solving this generalized form of the isoperimetric problem.

Strongly related to isoperimetric problems, in [36] the lower bound for the geometric dilation of a rectifiable simple closed curve C in Minkowski planes is obtained; note that the *geometric dilation* is the supremum of the quotient between the Minkowski length of the shorter part of C between two different points p and q of it, and the normed distance between these points. In [36] it is proved that for rectifiable simple closed curves in a Minkowski plane $\mathbb{M}^2$ this lower bound is a quarter of the circumference of the unit circle of $\mathbb{M}^2$, and that (in contrast to the Euclidean subcase) this lower bound can also be attained by curves that are not Minkowskian circles. Furthermore, it is shown that precisely in the subcase of strictly convex normed planes only Minkowskian circles can reach that bound. If p, q split C into two parts of equal Minkowskian lengths, then the normed distance of these points is called *halving distance of* C *in direction* $p - q$. In [37] several inequalities are established which show the relation between halving distances of a simple rectifiable closed curve C in Minkowski planes and other Minkowskian quantities, such as minimum width, inradius, and circumradius of C.

Conversely considered, generalized classes of isoperimetric problems in higher-dimensional Minkowski spaces refer to all convex bodies of given mixed volume having minimum surface area. In d-dimensional Minkowski spaces, $d \geq 3$, there are several notions of surface area and volume, for each combination of which there is, up to translation, a unique solution of the corresponding isoperimetric problem. Again, this convex body is called the respective *isoperimetrix* and also denoted by I_B; see [38, Chapter 5], for a broad representation of the isoperimetric problem in $\mathbb{M}^d$, $d \geq 3$, and types of isoperimetrices for correspondingly different definitions of surface area and volume. In [39] the stability of the solution of the isoperimetric problem in d-dimensional Minkowski spaces $\mathbb{M}^d$ is verified (see also [40]). Namely, some upper estimate for the term $\mu_B^d(\partial K) - d^d \mu_B(I_B)\mu_B^{d-1}(K)$ is obtained when $\mu_B(K) = \mu_B(I_B)$ holds. Here $\mu_B(\partial K)$ and $\mu_B(K)$ stand for surface area and volume of a convex body K in a Minkowski space $\mathbb{M}^d$, respectively. In [41] sharpenings of the isoperimetric problem in $\mathbb{M}^d$ are established. For instance, one of them is given by

$$\begin{aligned} &\mu_B^{d/(d-1)}(\partial K) - \left(d^d \mu_B(I_B)\right)^{1/(d-1)} \mu_B(K) \\ &\quad \geq \left(\mu_B^{1/(d-1)}(\partial K) - \rho\left(d\mu_B(I_B)\right)^{1/(d-1)}\right)^d - \left(d^d \mu_B(I_B)\right)^{1/(d-1)} \mu_B\left(K_\rho(I_B)\right), \end{aligned} \tag{1.5}$$

where $K_\rho(I_B)$ is the inner parallel body of K relative to I_B at distance ρ (see [42, page 134], for more about inner/outer parallel bodies).

In the recent book [43] one can find a discussion on how to involve the following version of the isoperimetric inequality into the theory of partial differential equations: let Ω be a bounded domain in $\mathbb{R}^d$, and let $S(\partial\Omega)$ be a suitable $(d-1)$-dimensional area measure of the boundary $\partial\Omega$ of Ω. Then

$$S(\partial\Omega) \geq d\epsilon_d^{1/d} V(\Omega)^{1-1/d}, \tag{1.6}$$

with equality only for the ball. The relation to Sobolev's inequality is also discussed. Another side of isoperimetric inequalities is presented in [44]: namely, the isoperimetric problem for product probability measures is investigated there.

Finally we mention once more that the monograph [38] contains a wide and deep discussion of the isoperimetric problem for different definitions of surface area and volume in higher dimensions, showing (also with many nice figures) that the isoperimetrices for the Holmes-Thompson definition and the Busemann definition given below belong to important classes of convex bodies known as projection bodies (= centered zonoids) and intersection bodies, respectively; see Section 2 for definitions of these notions. Corresponding isoperimetric inequalities are discussed there, too.

We will continue by discussing recently established isoperimetric inequalities for Minkowski spaces more detailed, also in view of their applications, and we will also pose related conjectures and open questions. Our attention will be restricted to affine isoperimetric inequalities in Minkowski spaces; we will almost ignore (with minor exceptions) asymptotic affine inequalities.

2. Definitions and Preliminaries

Recall that a *convex body* K is a compact, convex set with nonempty interior in $\mathbb{R}^d$, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$.

Let $(\mathbb{R}^d, \|\cdot\|) =: \mathbb{M}^d$, $d \geq 2$, be a d-dimensional real Banach space, that is, a (normed linear or) *Minkowski space* with *unit ball* B, where B is a convex body centered at the origin. The *unit sphere* of $\mathbb{M}^d$ is the boundary of B and denoted by ∂B. The standard Euclidean unit ball of $\mathbb{R}^d$ will be denoted by E_d, its volume by ϵ_d, and as usual we denote by S^{d-1} the standard Euclidean unit sphere in $\mathbb{R}^d$.

Let λ be the *Lebesgue measure* induced by the standard Euclidean structure in $\mathbb{R}^d$. We will refer to this measure as d-dimensional volume (area in $\mathbb{R}^2$) and denote it by $\lambda(\cdot)$. The measure λ gives rise to consider a dual measure λ^* on the family of convex subsets of the *dual space* $\mathbb{R}^{d*}$ (i.e., the vector space of linear functionals on $\mathbb{R}^d$, i.e., all linear mappings from $\mathbb{R}^d$ into $\mathbb{R}$ with the usual pointwise operations; see [38, Chapter 0]). However, using the standard basis we will identify $\mathbb{R}^d$ and $\mathbb{R}^{d*}$, and in that case λ and λ^* coincide in $\mathbb{R}^d$. We write λ_i for the *i-dimensional Lebesgue measure* in $\mathbb{R}^d$, with $1 \leq i \leq d$, and therefore we simply write λ instead of λ_d; again the identification of $\mathbb{R}^d$ and $\mathbb{R}^{d*}$ via the standard basis implies that λ_i and λ_i^* coincide in $\mathbb{R}^d$ as well. If $u \in S^{d-1}$, we denote by $u^\perp$ the $(d-1)$-dimensional subspace orthogonal to u, and by l_u the line through the origin parallel to u. By $\lambda_1(K, u)$ we denote the usual one-dimensional inner cross-section measure or *maximal chord length* of K in direction u.

One of the well-known inequalities regarding volumes of convex bodies under (vector or) *Minkowski addition*, defined by $K_1 + K_2 := \{x + y : x \in K_1, y \in K_2\}$ for convex bodies K_1, K_2 in $\mathbb{R}^d$, is the *Brunn-Minkowski inequality* which states that, for $0 \leq t \leq 1$,

$$\lambda^{1/d}((1-t)K_1 + tK_2) \geq (1-t)\lambda^{1/d}(K_1) + t\lambda^{1/d}(K_2) \tag{2.1}$$

holds. Here equality is obtained if and only if K_1 and K_2 are homothetic to each other. In [45], Gardner gives an excellent survey on this inequality, its applications, and extensions.

A Minkowski space $\mathbb{M}^d$ possesses a *Haar measure* μ, and this measure is unique up to multiplication of the Lebesgue measure with a positive constant, that is,

$$\mu = \sigma_B \lambda. \tag{2.2}$$

Choosing the "correct" multiple, which can depend on orientation, is not as easy as it seems at first glance, but the two measures μ and λ have, of course, to coincide in the standard Euclidean space.

For a convex body K in $\mathbb{R}^d$, we define the *polar body* K° of K by

$$K^\circ = \left\{ y \in \mathbb{R}^d : \langle x, y \rangle \leq 1,\ x \in K \right\}. \tag{2.3}$$

If K is a convex body in $\mathbb{R}^d$, then the *support function* h_K of K is defined by

$$h_K(u) = \sup\{\langle u, y\rangle : y \in K\}, \quad u \in S^{d-1}, \tag{2.4}$$

giving the distance from o to the supporting hyperplane of K with outward normal u. Note that $K_1 \subset K_2$ if and only if $h_{K_1} \leq h_{K_2}$ for any $u \in S^{d-1}$.

If $o \in K$, then its *radial function* $\rho_K(u)$ is defined by

$$\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\}, \quad u \in S^{d-1}, \tag{2.5}$$

giving the distance from o to $l_u \cap \partial K$ in direction u. Note again that $K_1 \subset K_2$ if and only if $\rho_{K_1} \leq \rho_{K_2}$ for any $u \in S^{d-1}$. For $\alpha_1, \alpha_2 \geq 0$ and any direction u these functions satisfy

$$\begin{aligned} h_{\alpha_1 K_1 + \alpha_2 K_2}(u) &= \alpha_1 h_{K_1}(u) + \alpha_2 h_{K_2}(u), \\ \rho_{\alpha_1 K_1 + \alpha_2 K_2}(u) &\geq \alpha_1 \rho_{K_1}(u) + \alpha_2 \rho_{K_2}(u) \ . \end{aligned} \tag{2.6}$$

In view of the latter inequality, we always have $\rho_{\alpha K} = \alpha \rho_K$.

We mention the relation

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \quad u \in S^{d-1}, \tag{2.7}$$

between the support function of a convex body K and the inverse of the radial function of K° (see [38, 42, 46, 47] for properties of and results on support and radial functions).

For convex bodies $K_1, \ldots, K_{n-1}, K_n$ in $\mathbb{R}^d$ we denote by $V(K_1, \ldots, K_n)$ their *mixed volume*, defined by

$$V(K_1, \ldots, K_n) = \frac{1}{d} \int_{S^{d-1}} h_{K_n} dS(K_1, \ldots, K_{n-1}, u) \tag{2.8}$$

with $dS(K_1, \ldots, K_{n-1}, \cdot)$ being *mixed surface area element* of $K_1, \ldots, K_{n-1}$; see [38, 42, 46–48] for many interesting properties of mixed volumes.

Note that we have $V(K_1, K_2, \ldots, K_n) \leq V(L_1, K_2, \ldots, K_n)$ if $K_1 \subset L_1$, that $V(\alpha K_1, \ldots, K_n) = \alpha V(K_1, \ldots, K_n)$ if $\alpha \geq 0$, and that $V(K, K, \ldots, K) = \lambda(K)$. Furthermore, we will write $V(K[d-i], L[i])$ instead of $V(\underbrace{K, K, \ldots, K}_{d-i}, \underbrace{L, L, \ldots, L}_{i})$.

We would also like to mention *Steiner's formula for mixed volumes* (see, e.g., [42, Section 4]), given by

$$\lambda(K + \alpha E_d) = \sum_{i=0}^{n} \binom{n}{i} V(K[d-i], E_d[i]) \alpha^i. \tag{2.9}$$

Minkowski's inequality for mixed volumes states that if K_1 and K_2 are convex bodies in $\mathbb{R}^d$, then

$$V^d(K_1[d-1], K_2) \geq \lambda^{d-1}(K_1)\lambda(K_2), \tag{2.10}$$

with equality if and only if K_1 and K_2 are homothetic (see [38, 42, 46–48]). If K_2 is the standard unit ball in $\mathbb{R}^d$, then this inequality becomes the standard isoperimetric inequality.

Another inequality referring to mixed volumes is the *Aleksandrov-Fenchel inequality*, stating that for convex bodies $K_1, K_2, \dots, K_d$ in $\mathbb{R}^d$

$$\lambda(K_1, K_2, \dots, K_d)^2 \geq \lambda(K_1[2], K_3, \dots, K_d)\lambda(K_2[2], K_3, \dots, K_d) \tag{2.11}$$

holds. Here one has equality if K_1 and K_2 are homothetic. In general, the equality case is still an open question (see [42, Section 6]).

If K is a convex body in $\mathbb{R}^d$, then the *projection body* ΠK of K is defined via its support function by

$$h_{\Pi K}(u) = \lambda_{d-1}\left(K \mid u^{\perp}\right) \tag{2.12}$$

for each $u \in S^{d-1}$, where $K \mid u^{\perp}$ is the orthogonal projection of K onto $u^{\perp}$, and $\lambda_{d-1}(K \mid u^{\perp})$ is called the $(d-1)$-*dimensional outer cross-section measure* or *brightness* of K at u. We note that any projection body is a centered zonoid, and that for centered convex bodies K_1, K_2 the equality $\Pi K_1 = \Pi K_2$ implies $K_1 = K_2$; see [42, 47] for more information about projection bodies. (*Zonoids* are the limits, in the Hausdorff sense, of *zonotopes*, i.e., of vector sums of finitely many line segments.)

The *intersection body* IK of a convex body K in $\mathbb{R}^d$ is defined via its radial function by

$$\rho_{IK}(u) = \lambda_{d-1}\left(K \cap u^{\perp}\right) \tag{2.13}$$

for each $u \in S^{d-1}$. Note that if K_1 and K_2 are centered convex bodies in $\mathbb{R}^d$, then from $IK_1 = IK_2$ it follows that $K_1 = K_2$ (see [47, 49]).

We should also say that any projection body is dual to some intersection body, and that the converse is not true. The reader can also consult the book [50] of Koldobsky about a Fourier analytic characterization of intersection bodies.

Let K and L be convex bodies in $\mathbb{R}^d$. Then the *relative inradius* $r(K, L)$ and the *relative circumradius* $R(K, L)$ of K with respect to L are defined by

$$\begin{aligned} r(K, L) &:= \sup\left\{\alpha : \exists x \in \mathbb{R}^d, \alpha L + x \subseteq K\right\}, \\ R(K, L) &:= \inf\left\{\alpha : \exists x \in \mathbb{R}^d, \alpha L + x \supseteq K\right\}, \end{aligned} \tag{2.14}$$

respectively.

3. Surface Areas, Volumes, and Isoperimetrices in Minkowski Spaces

As already announced, there are different definitions of measures in higher-dimensional Minkowski spaces (see [38, 51, 52], but also [53] for a variant). We define now the most important ones.

Definition 3.1. If K is a convex body in $\mathbb{M}^d$, then the *d-dimensional Holmes-Thompson volume* of K is defined by

$$\mu_B^{\mathrm{HT}}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\epsilon_d}, \quad \text{that is, } \sigma_B = \frac{\lambda(B^\circ)}{\epsilon_d}. \tag{3.1}$$

Definition 3.2. If K is a convex body in $\mathbb{R}^d$, then the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{\mathrm{Bus}}(K) = \frac{\epsilon_d}{\lambda(B)}\lambda(K), \quad \text{that is, } \sigma_B = \frac{\epsilon_d}{\lambda(B)}. \tag{3.2}$$

Note that these definitions coincide with the standard notion of volume if the space is Euclidean, and that $\mu_B^{\mathrm{Bus}}(B) = \epsilon_d$.

Let M be a surface in $\mathbb{R}^d$ with the property that at each point x of M there is a unique tangent hyperplane, and that u_x is the unit normal vector to this hyperplane at x. Then the *Minkowski surface area* of M is defined by

$$\mu_B(M) := \int_M \sigma_B(u_x)dS(x). \tag{3.3}$$

For the *Holmes-Thompson surface area*, the quantity $\sigma_B(u)$ is defined by

$$\sigma_B(u) = \frac{\lambda\Big(\big(B\cap u^\perp\big)^\circ\Big)}{\epsilon_{d-1}}. \tag{3.4}$$

For the *Busemann surface area*, $\sigma_B(u)$ is defined by

$$\sigma_B(u) = \frac{\epsilon_{d-1}}{\lambda\big(B\cap u^\perp\big)}. \tag{3.5}$$

If K is a convex body in $\mathbb{M}^d$, then the *Minkowski surface area* of K can also be defined by

$$\mu_B(\partial K) = dV(K[d-1], I_B), \tag{3.6}$$

where I_B is that convex body whose support function is σ_B. The convex body I_B plays the central role regarding the solution of the *isoperimetric problem* in Minkowski spaces; see again [38] and the definitions below. Recall once more that in two-dimensional Minkowski spaces

I_B is the polar reciprocal of B with respect to the Euclidean unit circle, rotated through 90° (see [38, 54–56]).

For the Holmes-Thompson measure, I_B is defined by

$$I_B^{\mathrm{HT}} = \frac{\Pi(B^\circ)}{\epsilon_{d-1}} \tag{3.7}$$

and therefore a centered zonoid. For the Busemann measure we have

$$I_B^{\mathrm{Bus}} = \epsilon_{d-1}(IB)^\circ. \tag{3.8}$$

Among the homothetic images of I_B we want to specify a unique one, called the *isoperimetrix* $\widehat{I}_B$ and determined by $\mu_B(\partial\widehat{I}_B) = d\mu_B(\widehat{I}_B)$ (see [38]).

Definition 3.3. The *isoperimetrix for the Holmes-Thompson measure* is defined by

$$\widehat{I}_B^{\mathrm{HT}} = \frac{\epsilon_d}{\lambda(B^\circ)} I_B^{\mathrm{HT}}. \tag{3.9}$$

Definition 3.4. The *isoperimetrix for the Busemann measure* is defined by

$$\widehat{I}_B^{\mathrm{Bus}} = \frac{\lambda(B)}{\epsilon_d} I_B^{\mathrm{Bus}}. \tag{3.10}$$

4. Inequalities in Minkowski Spaces

One of the fundamental theorems in geometric convexity refers to the *Blaschke-Santaló inequality* and states that if K is a centrally symmetric convex body in $\mathbb{R}^d$, then

$$\lambda(K)\lambda(K^\circ) \le \epsilon_d^2 \tag{4.1}$$

with equality if and only if K is an ellipsoid. See also [57, 58] for some new results in this direction.

The sharp lower bound on the product $\lambda(K)\lambda(K^\circ)$ is known only for certain classes of convex bodies, for example, yielding the *Mahler-Reisner Theorem*. This theorem states that if K is a zonoid in $\mathbb{R}^d$, then

$$\frac{4^d}{d!} \le \lambda(K)\lambda(K^\circ), \tag{4.2}$$

with equality if and only if K is a parallelotope. Mahler proved this inequality for $d = 2$, and Reisner established it for the class of zonoids (see [59]). In [60], Saint-Raymond established this inequality for convex bodies with d hyperplanes of symmetry whose normals are linearly independent.

In [61] it is proved that there is a constant c, independent of d, such that $\lambda(K)\lambda(K^\circ) \geq c^d \epsilon_d^2$. In recent years, there have been attempts to extend the Mahler-Reisner Theorem to all convex bodies (see, e.g., [62, 63]).

If K is a convex body in $\mathbb{R}^d$, then

$$\binom{2d}{d} d^{-d} \leq \lambda^{d-1}(K)\lambda\left((\Pi K)^\circ\right) \leq \left(\frac{\epsilon_d}{\epsilon_{d-1}}\right)^d \tag{4.3}$$

holds, with equality on the right side if and only if K is an ellipsoid, and with equality on the left side if and only if K is a simplex.

The right inequality is called *Petty's projection inequality*, and the left one was established by Zhang (see [47, 64]).

The following question has been raised several times and is still open (see [65]).

Problem 1. What is the sharp lower bound on $\lambda^{d-1}(K)\lambda((\Pi K)^\circ)$, when K is a centrally symmetric convex body in $\mathbb{R}^d$?

In [66], it was conjectured that this sharp lower bound is attained when K is a parallelotope.

In [67] (see also [68]), Schmuckenschläger defines the convolution square F_K of K as the convolution of the indicator function I_K of K and I_{-K}, and the distribution function $V_K(\delta)$ of this convolution is defined by

$$\begin{aligned} V_K(\delta) : &= \lambda(F_K > \delta) := \lambda\left(\left\{x \in \mathbb{R}^d : F_K(x) > \delta\right\}\right) \\ &= \lambda\left(\left\{x \in \mathbb{R}^d : \lambda(K \cap (K + x)) > \delta\right\}\right). \end{aligned} \tag{4.4}$$

Based on this, Schmuckenschläger proves that if K is a convex body in $\mathbb{R}^d$, then

$$\lim_{\delta \to \lambda(K)} \frac{V_K(\delta)}{(\lambda(K) - \delta)^d} = \lambda\left((\Pi K)^\circ\right). \tag{4.5}$$

Furthermore, he proves the following version of Petty's projection inequality: if K is a convex body in $\mathbb{R}^d$ such that $\lambda(K) = \lambda(E_d)$, then

$$\lambda\left((\Pi K)^\circ\right) \leq \lambda\left((\Pi E_d)^\circ\right). \tag{4.6}$$

Another proof of this inequality is given in [69]. In this proof one has to take n random segments in K and to consider then their Minkowski average D (recall that the *Minkowski average* of the segments $[x_i, y_i] \subset K$ with $1 \leq i \leq n$ is the zonotope defined by $D(x_1, y_1, \ldots, x_n, y_n) := (1/n)([x_1, y_1] + \cdots + [x_n, y_n])$). Then it is shown that, for $\lambda(K)$ fixed, the supremum of $\lambda(D)$ is minimal for K an ellipsoid. This result implies Petty's projection inequality referring to $\max \lambda((\Pi K)^\circ)$.

Setting $K = B^\circ$ in Petty's projection inequality, one obtains

$$\lambda\left(\left(\widehat{I}_B^{\mathrm{HT}}\right)^\circ\right) \leq \lambda(B^\circ), \tag{4.7}$$

with equality if and only if B is an ellipsoid (see also [38]).

Petty's conjectured projection inequality states that if K is a convex body in $\mathbb{R}^d$ with $d \geq 3$, then

$$\epsilon_d^{-2}\lambda(\Pi K)\lambda^{1-d}(K) \geq \left(\frac{\epsilon_{d-1}}{\epsilon_d}\right)^d \tag{4.8}$$

with equality if and only if K is an ellipsoid; see [70]. In [71] (see also [13]) Lutwak says that this conjectured inequality is one of the major open problems in the field of affine isoperimetric inequalities. In [72], Schneider discusses applications of this conjecture in Stochastic Geometry. In [73] (see also [74]) Brannen proves that this inequality holds for 3-dimensional convex cylindrical bodies.

In [75] it is proved that Petty's conjectured projection inequality is equivalent to another open problem (namely the isoperimetric problem for the Holmes-Thompson measure) over the class of origin-symmetric convex bodies, since the following statement is proved there.

Theorem 4.1. *Let B be the unit ball of $\mathbb{M}^d$. Then Petty's conjectured projection inequality is true for B if and only if*

$$\frac{\mu_B^d\left(\partial I_B^{HT}\right)}{\mu_B^{d-1}\left(I_B^{HT}\right)} \geq d^d \epsilon_d, \tag{4.9}$$

and equality holds if and only if B is an ellipsoid.

Similar to (4.7), we state the following conjecture that would also follow from Petty's conjectured projection inequality (see [75]).

Conjecture 4.2. *If B is the unit ball of $\mathbb{M}^d$, then we have*

$$\lambda\left(\widehat{I}_B^{HT}\right) \geq \lambda(B), \tag{4.10}$$

with equality if and only if B is an ellipsoid.

This conjecture as well as Petty's conjectured projection inequality would easily solve the following problem (see also [75, 76]).

Problem 2. Let B be the unit ball of $\mathbb{M}^d$. Is it then true that

$$\frac{\mu_B^{\mathrm{HT}}(\partial B)}{\mu_B^{\mathrm{HT}}(B)} \geq d? \tag{4.11}$$

Also in [75] it is shown that $\mu_B^{\mathrm{HT}}(\partial B)/\mu_B^{\mathrm{HT}}(B) = d$ if and only if B is an ellipsoid.

Furthermore, the affirmative answer of the following question would solve this ratio problem as well.

Problem 3. Let B be a centered convex body in $\mathbb{R}^d$. Is it then true that

$$V(B[d-1], \Pi B^\circ) \geq \left(\frac{\epsilon_{d-1}}{\epsilon_d}\right)\lambda(B)\lambda(B^\circ)? \tag{4.12}$$

We should also mention that for $d \geq 3$ the sharp bounds on $\mu_B^{\mathrm{HT}}(\partial B)$ are still unknown, thus yielding a challenging open problem. Thompson (private communication) informed us to have a proof that the sharp lower bound on $\mu_B^{\mathrm{HT}}(\partial B)$ for $d = 3$ equals $36/\pi$ in the case when B is either a rhombic dodecahedron or its dual, that is, a cuboctahedron in $\mathbb{M}^3$.

Since the quantity $\lambda(\Pi K)\lambda^{1-d}(K)$ is not changed under dilation, we obtain, setting $\lambda(K) = \lambda(E_d)$ in Petty's conjectured projection inequality, the following version of this conjecture which is similar to (4.6).

Conjecture 4.3. *If K is a convex body in $\mathbb{R}^d$ with $\lambda(K) = \lambda(E_d)$, then*

$$\lambda(\Pi K) \geq \lambda(\Pi E_d) \tag{4.13}$$

with equality if and only if K is an ellipsoid.

For the class of centered convex bodies this conjecture would follow from the following question which involves Steiner symmetrization. Recall that if u is a unit vector, the *Steiner symmetral* StK of a convex body K with respect to the hyperplane $u^\perp$ is the convex body obtained as union of all translates of chords of K parallel to u, where these chords are translated in their own affine hull such that, in their final position, they intersect $u^\perp$ at their midpoints. The respective procedure is usually called *Steiner symmetrization*.

Problem 4. Let K be a centered convex body in $\mathbb{R}^d$. Is it true that the Steiner symmetral StK of K, created with respect to a given hyperplane through the origin, satisfies the inequality

$$\lambda(\Pi K) \geq \lambda(\Pi(\mathrm{St}K)) \tag{4.14}$$

with equality for each hyperplane through the origin if and only if K is an ellipsoid?

Remark 4.4. Since K is a centered convex body, it suffices to take this hyperplane to be $H_{d-1} = \{x \in \mathbb{R}^d : x_d = 0\}$.

Recall that Steiner symmetrization does not change the volume of a given convex body. In addition we note that there is a sequence of convex bodies obtained from a given convex body by finitely many successive Steiner symmetrizations such that this sequence converges to an ellipsoid (see [9]). Also, the following interesting property of Steiner symmetrization should be noticed (see [9, 77, 78]).

Proposition 4.5. *Let K be a centered convex body in $\mathbb{R}^d$. Then the Steiner symmetral* $\mathrm{St}K$ *of K with respect to a given hyperplane through the origin satisfies the inequality*

$$\lambda\big((\mathrm{St}K)^\circ\big) \geq \lambda(K^\circ). \tag{4.15}$$

From this proposition it follows that $I_B^{\mathrm{HT}} \not\subseteq \mathrm{St}I_B^{\mathrm{HT}}$ and $I_B^{\mathrm{Bus}} \not\subseteq \mathrm{St}I_B^{\mathrm{Bus}}$, unless B is an ellipsoid.

Favard's Theorem (see [79] or [80]) states that $\lambda(K) = V(K[d-1], L)$ holds if and only if K is a $(d-1)$-tangent body of a convex body L. (Recall that a convex body K is a $(d-1)$-*tangent body* of a convex body L if and only if through each boundary point of K there exists a supporting hyperplane of K that also supports L; see [46, page 19] or [42, pages 75-76 and 136], for the definition of tangent bodies.)

Setting $K = B$ and $L = \widehat{I}_B^{\mathrm{Bus}}$ in Favard's Theorem, we obtain $\lambda(B) = V(B[d-1], \widehat{I}_B^{\mathrm{Bus}})$. Hence $\mu_B^{\mathrm{Bus}}(\partial B) = d\epsilon_d$ if and only if B is a $(d-1)$-tangent body of $\widehat{I}_B^{\mathrm{Bus}}$.

In [81], Thompson shows that if the unit ball of a Minkowski space $\mathbb{M}^3$ is an affine regular rhombic dodecahedron, then $\mu_B^{\mathrm{Bus}}(\partial B) = d\epsilon_d = 4\pi$. Thus, if B is an affine regular rhombic dodecahedron in $\mathbb{M}^3$, then $\widehat{I}_B^{\mathrm{Bus}} \subseteq B$ and $r(B, \widehat{I}_B^{\mathrm{Bus}}) = 1$. Furthermore, this is a counterexample to Problem 7.4.2 posed in [38].

Finding the sharp lower bound on $\mu_B^{\mathrm{Bus}}(\partial B)$ is still an open question.

Busemann's intersection inequality states that if K is a convex body in $\mathbb{R}^d$, then

$$\lambda(IK) \leq \frac{\epsilon_{d-1}^d}{\epsilon_d^{d-2}} \lambda^{d-1}(K) \tag{4.16}$$

with equality if and only if K is an ellipsoid; see [82].

In [83] it is also proved that Busemann's intersection inequality cannot be strengthened to

$$\lambda^{d-1}(K)\lambda\big((IK)^\circ\big) \geq \left(\frac{\epsilon_d}{\epsilon_{d-1}}\right)^d \tag{4.17}$$

when K is an affine regular rhombic dodecahedron in $\mathbb{R}^3$.

We should also mention that sharp bounds on $r(B, \widehat{I}_B)$ and $R(B, \widehat{I}_B)$ for some cases are known. Namely, it is known that $2\epsilon_{d-1}/d\epsilon_d \leq r(B, \widehat{I}_B^{\mathrm{HT}}) \leq 1$ with equality on the left if and only if B is a cube or cross polytope, and on the right if and only if B is an ellipsoid; see [38, 84]. In $\mathbb{M}^2$, for $R(B, \widehat{I}_B^{\mathrm{HT}})$ we have $R(B, \widehat{I}_B^{\mathrm{HT}}) \geq 3/\pi$ with equality if and only if B is a regular hexagon (see [38, 76]). In $\mathbb{M}^d$, $R(B, \widehat{I}_B^{\mathrm{HT}}) \leq d\epsilon_d/2\epsilon_{d-1}$ holds with equality if and only if B is a parallelotope (see [84]).

Also, the relations $r(B, \widehat{I}_B^{\mathrm{HT}}) \leq 1$ or $\widehat{I}_B^{\mathrm{HT}} \subseteq B$, with equality if and only if B is an ellipsoid, play an essential role for the proof of a conjecture of Rogers and Shephard (given in [84]), leading to the following theorem.

Theorem 4.6. *If K is a convex body in $\mathbb{R}^d$, then there exists a direction $u \in S^{d-1}$ such that*

$$\frac{\lambda_{d-1}(K \mid u^{\perp})\lambda_1(K,u)}{\lambda(K)} \geq \frac{2\epsilon_{d-1}}{\epsilon_d}. \tag{4.18}$$

Furthermore, equality for each $u \in S^{d-1}$ holds if and only if K is an ellipsoid.

One could also raise the following question.

Problem 5. Does there exist a centered convex body K in $\mathbb{R}^d$ such that

$$\frac{\lambda_{d-1}(K \cap u^{\perp})\lambda_1(K \mid l_u)}{\lambda(K)} > \frac{2\epsilon_{d-1}}{\epsilon_d} \tag{4.19}$$

for each $u \in S^{d-1}$.

Our guess is that such a body does not exist.

Acknowledgments

The authors wish to thank the referees for their valuable comments and suggestions. The second author thanks the Faculty of Mathematics, University of Technology Chemnitz, for hospitality and excellent working conditions. He also thanks the University of Houston-Clear Lake for its support via the FRSF Award no. 970.

References

[1] R. Osserman, "The isoperimetric inequality," *Bulletin of the American Mathematical Society*, vol. 84, no. 6, pp. 1182–1238, 1978.
[2] R. Osserman, "A strong form of the isoperimetric inequality in $\mathbb{R}^n$," *Complex Variables. Theory and Application*, vol. 9, no. 2-3, pp. 241–249, 1987.
[3] R. Osserman, "Isoperimetric inequalities and eigenvalues of the Laplacian," in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pp. 435–442, Academia Scientiarum Fennica, Helsinki, Finland, 1980.
[4] R. Osserman, "Bonnesen-style isoperimetric inequalities," *The American Mathematical Monthly*, vol. 86, no. 1, pp. 1–29, 1979.
[5] V. Blåsjö, "The isoperimetric problem," *American Mathematical Monthly*, vol. 112, no. 6, pp. 526–566, 2005.
[6] H. Gericke, "Zur Geschichte des isoperimetrischen problems," *Mathematische Semesterberichte*, vol. 29, no. 2, pp. 160–187, 1982.
[7] M. Ritoré and A. Ros, "Some updates on isoperimetric problems," *The Mathematical Intelligencer*, vol. 24, no. 3, pp. 9–14, 2002.
[8] K. Leichtweiß, "Das isoperimetrische Problem bei E. Schmidt, A. Dinghas und heute," in *Mathematik aus Berlin*, pp. 217–225, Weidler, Berlin, Germany, 1997.
[9] P. M. Gruber, *Convex and Discrete Geometry*, vol. 336 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 2007.
[10] K. Böröczky Jr., *Finite Packing and Covering*, vol. 154 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.
[11] E. Lutwak, D. Yang, and G. Zhang, "L_P affine isoperimetric inequalities," *Journal of Differential Geometry*, vol. 56, no. 1, pp. 111–132, 2000.

[12] E. Werner and D. Ye, "New L_P affine isoperimetric inequalities," *Advances in Mathematics*, vol. 218, no. 3, pp. 762–780, 2008.

[13] E. Lutwak, "Selected affine isoperimetric inequalities," in *Handbook of Convex Geometry, Vol. A*, P. M. Gruber and J. M. Wills, Eds., pp. 151–176, North-Holland, Amsterdam, The Netherlands, 1993.

[14] W. Yu, "Equivalence of some affine isoperimetric inequalities," *Journal of Inequalities and Applications*, vol. 2009, Article ID 981258, 11 pages, 2009.

[15] S. Lin, X. Bin, and Y. Wuyang, "Dual L_P affine isoperimetric inequalities," *Journal of Inequalities and Applications*, vol. 2006, Article ID 84825, 11 pages, 2006.

[16] V. Ferone, "Isoperimetric inequalities and applications," *Bollettino della Unione Matematica Italiana*, vol. 1, no. 3, pp. 539–557, 2008 (Italian).

[17] V. I. Diskant, "Sharpenings of the isoperimetric inequality," *Sibirskiĭ Matematičeskiĭ Žurnal*, vol. 14, pp. 873–877, 1973 (Russian).

[18] H. T. Ku, M. C. Ku, and X. M. Zhang, "Analytic and geometric isoperimetric inequalities," *Journal of Geometry*, vol. 53, no. 1-2, pp. 100–121, 1995.

[19] A. Siegel, "An isoperimetric theorem in plane geometry," *Discrete & Computational Geometry*, vol. 29, no. 2, pp. 239–255, 2003.

[20] L. Dalla and N. K. Tamvakis, "An isoperimetric inequality in the class of simplicial polytopes," *Mathematica Japonica*, vol. 44, no. 3, pp. 569–572, 1996.

[21] A. P. Burton and P. Smith, "Isoperimetric inequalities and areas of projections in $\mathbb{R}^n$," *Acta Mathematica Hungarica*, vol. 62, no. 3-4, pp. 395–402, 1993.

[22] K. Ball, "Volume ratios and a reverse isoperimetric inequality," *Journal of the London Mathematical Society*, vol. 44, no. 2, pp. 351–359, 1991.

[23] K. Ball, "An elementary introduction to modern convex geometry," in *Flavors of Geometry*, vol. 31 of *Mathematical Sciences Research Institute Publications*, pp. 1–58, Cambridge University Press, Cambridge, UK, 1997.

[24] F. Barthe, "On a reverse form of the Brascamp-Lieb inequality," *Inventiones Mathematicae*, vol. 134, no. 2, pp. 335–361, 1998.

[25] U. Betke and W. Weil, "Isoperimetric inequalities for the mixed area of plane convex sets," *Archiv der Mathematik*, vol. 57, no. 5, pp. 501–507, 1991.

[26] H. Busemann, "The isoperimetric problem in the Minkowski plane," *American Journal of Mathematics*, vol. 69, pp. 863–871, 1947.

[27] W. Süss, "Affine und Minkowskische Geometrie eines ebenen Variationsproblems," *Archiv der Mathematik*, vol. 5, pp. 441–446, 1954.

[28] L. Koschmieder, "Äusserstwerte an Lamëschen Kurven und Flächen," *Gaceta Matemática*, vol. 1, pp. 53–59, 1949.

[29] I. Kocayusufoğlu, "Isoperimetric inequality in taxicab geometry," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 269–272, 2006.

[30] G. D. Chakerian, "The isoperimetric problem in the Minkowski plane," *The American Mathematical Monthly*, vol. 67, pp. 1002–1004, 1960.

[31] G. Bognár, "A geometric approach to the half-linear differential equation," in *Colloquium on Differential and Difference Equations—CDDE 2006*, vol. 16 of *Folia Facultatis Scientiarium Naturalium Universitatis Masarykiana Brunensis. Mathematica*, pp. 51–58, Masaryk University, Brno, Czech Republic, 2007.

[32] R. F. Constantin, "Isoperimetric inequalities in Minkowski space $\mathbb{M}_2$," *Scientific Bulletin-University Politehnica of Bucharest. Series A*, vol. 70, no. 2, pp. 29–36, 2008.

[33] B. Fuglede, "Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$," *Transactions of the American Mathematical Society*, vol. 314, no. 2, pp. 619–638, 1989.

[34] I. Fáry and E. Makai Jr., "Isoperimetry in variable metric," *Studia Scientiarum Mathematicarum Hungarica*, vol. 17, no. 1–4, pp. 143–158, 1982.

[35] V. A. Sorokin, "Certain questions of a Minkowski geometry with a non-symmetric indicatrix," *Orekhovo-Zuevskii Pedagogicheskii Institut*, vol. 22, no. 3, pp. 138–147, 1964 (Russian).

[36] H. Martini and S. Wu, "Geometric dilation of closed curves in normed planes," *Computational Geometry. Theory and Applications*, vol. 42, no. 4, pp. 315–321, 2009.

[37] C. He, H. Martini, and S. Wu, "Halving closed curves in normed planes and related inequalituis," *Mathematical Inequalities and Applications*, vol. 12, no. 4, pp. 719–731, 2009.

[38] A. C. Thompson, *Minkowski Geometry*, vol. 63 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1996.

[39] V. I. Diskant, "Stability of the solution of an isoperimetric problem in Minkowski geometry," *Matematicheskaya Fizika, Analiz, Geometriya*, vol. 3, no. 3-4, pp. 261–266, 1996.

[40] V. I. Diskant, “A remark on a theorem on the stability of the solution of an isoperimetric problem in Minkowski geometry,” *Chebyshevskiĭ Sbornik*, vol. 7, no. 2(18), pp. 95–98, 2006 (Russian).
[41] V. I. Diskant, “Improvements of the isoperimetric inequality in Minkowski geometry,” *Matematicheskaya Fizika, Analiz, Geometriya*, vol. 10, no. 2, pp. 147–155, 2003.
[42] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, vol. 44 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1993.
[43] S. Kesavan, *Symmetrization and Applications*, vol. 3 of *Series in Analysis*, World Scientific, Hackensack, NJ, USA, 2006.
[44] F. Barthe, “Isoperimetric inequalities, probability measures and convex geometry,” in *European Congress of Mathematics*, pp. 811–826, European Mathematical Society, Zürich, Switzerland, 2005.
[45] R. J. Gardner, “The Brunn-Minkowski inequality,” *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355–405, 2002.
[46] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, USA, 1987.
[47] R. J. Gardner, *Geometric Tomography*, vol. 58 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2nd edition, 2006.
[48] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, vol. 285 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1988.
[49] E. Lutwak, “Intersection bodies and dual mixed volumes,” *Advances in Mathematics*, vol. 71, no. 2, pp. 232–261, 1988.
[50] A. Koldobsky, *Fourier Analysis in Convex Geometry*, vol. 116 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2005.
[51] R. D. Holmes and A. C. Thompson, “N-dimensional area and content in Minkowski spaces,” *Pacific Journal of Mathematics*, vol. 85, no. 1, pp. 77–110, 1979.
[52] R. Schneider, “Intrinsic volumes in Minkowski spaces,” *Rendiconti del Circolo Matematico di Palermo*, no. 50, pp. 355–373, 1997.
[53] A. C. Thompson, “On Benson’s definition of area in Minkowski space,” *Canadian Mathematical Bulletin*, vol. 42, no. 2, pp. 237–247, 1999.
[54] H. Busemann, “The foundations of Minkowskian geometry,” *Commentarii Mathematici Helvetici*, vol. 24, pp. 156–187, 1950.
[55] H. Martini and K. J. Swanepoel, “Antinorms and Radon curves,” *Aequationes Mathematicae*, vol. 72, no. 1-2, pp. 110–138, 2006.
[56] G. Strang, “Maximum area with Minkowski measures of perimeter,” *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 138, no. 1, pp. 189–199, 2008.
[57] D. Alonso-Gutiérrez, “On an extension of the Blaschke-Santaló inequality and the hyperplane conjecture,” *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 292–300, 2008.
[58] M. Fradelizi and M. Meyer, “Some functional forms of Blaschke-Santaló inequality,” *Mathematische Zeitschrift*, vol. 256, no. 2, pp. 379–395, 2007.
[59] S. Reisner, “Zonoids with minimal volume-product,” *Mathematische Zeitschrift*, vol. 192, no. 3, pp. 339–346, 1986.
[60] J. Saint-Raymond, “Sur le volume des corps convexes symétriques,” in *Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981*, vol. 46 of *Publications mathématiques de l’Université Pierre et Marie Curie*, p. Exp. No. 11, 25, University of Paris VI, Paris, France, 1981.
[61] J. Bourgain and V. D. Milman, “New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$,” *Inventiones Mathematicae*, vol. 88, no. 2, pp. 319–340, 1987.
[62] G. Kuperberg, “The bottleneck conjecture,” *Geometry and Topology*, vol. 3, pp. 119–135, 1999.
[63] G. Kuperberg, “From the Mahler conjecture to Gauss linking integrals,” *Geometric and Functional Analysis*, vol. 18, no. 3, pp. 870–892, 2008.
[64] G. Y. Zhang, “Restricted chord projection and affine inequalities,” *Geometriae Dedicata*, vol. 39, no. 2, pp. 213–222, 1991.
[65] E. Lutwak, D. Yang, and G. Zhang, “A new affine invariant for polytopes and Schneider’s projection problem,” *Transactions of the American Mathematical Society*, vol. 353, no. 5, pp. 1767–1779, 2001.
[66] E. Makai Jr. and H. Martini, “The cross-section body, plane sections of convex bodies and approximation of convex bodies. I,” *Geometriae Dedicata*, vol. 63, no. 3, pp. 267–296, 1996.
[67] M. Schmuckenschläger, “Petty’s projection inequality and Santalo’s affine isoperimetric inequality,” *Geometriae Dedicata*, vol. 57, no. 3, pp. 285–295, 1995.
[68] M. Schmuckenschläger, “The distribution function of the convolution square of a convex symmetric body in $\mathbb{R}^n$,” *Israel Journal of Mathematics*, vol. 78, no. 2-3, pp. 309–334, 1992.

[69] E. Makai Jr. and H. Martini, "The cross-section body, plane sections of convex bodies and approximation of convex bodies. II," *Geometriae Dedicata*, vol. 70, no. 3, pp. 283–303, 1998.

[70] C. M. Petty, "Isoperimetric problems," in *Proceedings of the Conference on Convexity and Combinatorial Geometry (Univ. Oklahoma, Norman, Okla., 1971)*, pp. 26–41, Department of Mathematics, University of Oklahoma, Norman, Okla, USA, 1971.

[71] E. Lutwak, "On a conjectured projection inequality of Petty," in *Integral Geometry and Tomography (Arcata, CA, 1989)*, vol. 113 of *Contemporary Mathematics*, pp. 171–182, American Mathematical Society, Providence, RI, USA, 1990.

[72] R. Schneider, "Geometric inequalities for Poisson processes of convex bodies and cylinders," *Results in Mathematics*, vol. 11, no. 1-2, pp. 165–185, 1987.

[73] N. S. Brannen, "Volumes of projection bodies," *Mathematika*, vol. 43, no. 2, pp. 255–264, 1996.

[74] N. S. Brannen, "Three-dimensional projection bodies," *Advances in Geometry*, vol. 5, no. 1, pp. 1–13, 2005.

[75] H. Martini and Z. Mustafaev, "Extensions of a Bonnesen-style inequality to Minkowski spaces," *Mathematical Inequalities & Applications*, vol. 11, no. 4, pp. 739–748, 2008.

[76] Z. Mustafaev, "The ratio of the length of the unit circle to the area of the unit disc in Minkowski planes," *Proceedings of the American Mathematical Society*, vol. 133, no. 4, pp. 1231–1237, 2005.

[77] M. Meyer and A. Pajor, "On the Blaschke-Santaló inequality," *Archiv der Mathematik*, vol. 55, no. 1, pp. 82–93, 1990.

[78] M. Meyer and A. Pajor, "On Santaló's inequality," in *Geometric Aspects of Functional Analysis (1987-1988)*, vol. 1376 of *Lecture Notes in Mathematics*, pp. 261–263, Springer, Berlin, Germany, 1989.

[79] J. Favard, "Sur les corps convexes," *Journal de Mathématiques Pures et Appliquées*, vol. 12, no. 9, pp. 219–282, 1933.

[80] J. R. Sangwine-Yager, "Bonnesen-style inequalities for Minkowski relative geometry," *Transactions of the American Mathematical Society*, vol. 307, no. 1, pp. 373–382, 1988.

[81] A. C. Thompson, "Applications of various inequalities to Minkowski geometry," *Geometriae Dedicata*, vol. 46, no. 2, pp. 215–231, 1993.

[82] H. Busemann, "Volume in terms of concurrent cross-sections," *Pacific Journal of Mathematics*, vol. 3, pp. 1–12, 1953.

[83] Z. Mustafaev, "On Busemann surface area of the unit ball in Minkowski spaces," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 1, article 19, pp. 1–10, 2006.

[84] H. Martini and Z. Mustafaev, "Some applications of cross-section measures in Minkowski spaces," *Periodica Mathematica Hungarica*, vol. 53, no. 1-2, pp. 185–197, 2006.