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Research Article

Gradient Estimates for Weak Solutions of $\mathcal{A}$-Harmonic Equations

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We obtain gradient estimates in Orlicz spaces for weak solutions of $\mathcal{A}$-Harmonic Equations under the assumptions that $\mathcal{A}$ satisfies some proper conditions and the given function satisfies some moderate growth condition. As a corollary we obtain L^p-type regularity for such equations.

1. Introduction

In this paper we consider the following general nonlinear elliptic problem:

$$\operatorname{div} \mathcal{A}(\nabla u, x) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$, $\mathbf{f} = (f^1, \dots, f^n)$ and $\mathcal{A} = \mathcal{A}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are two given vector fields, and $\mathcal{A}$ is measurable in x for each ξ and continuous in ξ for almost everywhere x. Moreover, for given $p \in (1, \infty)$ the structural conditions on the function $\mathcal{A}(\xi, x)$ are given as follows:

$$\left[\mathcal{A}(\xi, x) - \mathcal{A}(\eta, x)\right] \cdot (\xi - \eta) \geq C_1\left|\xi - \eta\right|^p, \tag{1.2}$$

$$|\mathcal{A}(\xi, x)| \leq C_2\left(1 + |\xi|^{p-1}\right), \tag{1.3}$$

$$\mathcal{A}(\xi, x) \cdot \xi \geq C_3|\xi|^p - C_4, \tag{1.4}$$

$$\left|\mathcal{A}(\xi, x) - \mathcal{A}(\xi, y)\right| \leq C_5 w\left(\left|x - y\right|\right)\left(1 + |\xi|^{p-1}\right) \tag{1.5}$$

for all $\xi, \eta \in \mathbb{R}^n$, $x, y \in \Omega$, and some positive constants $C_i > 0$, $i = 1, 2, 3, 4, 5$. Here the modulus of continuity $w(x) : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and satisfies

$$w(r) \longrightarrow 0 \quad \text{as } r \longrightarrow 0. \tag{1.6}$$

Especially when $\mathcal{A}(\xi, x) = |\xi|^{p-2}\xi$, (1.1) is reduced to be quasilinear elliptic equations of p-Laplacian type

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega. \tag{1.7}$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi \in W^{1,p}_0(\Omega)$, one has

$$\int_\Omega \mathcal{A}(\nabla u, x) \cdot \nabla\varphi \, dx = \int_\Omega |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla\varphi \, dx. \tag{1.8}$$

DiBenedetto and Manfredi [1] and Iwaniec [2] obtained L^q, $q \geq p$, gradient estimates for weak solutions of (1.7) while Acerbi and Mingione [3] studied the case that $p = p(x)$. Moreover, the authors [4, 5] obtained L^q, $q \geq p$, gradient estimates for weak solutions of quasilinear elliptic equation of p-Laplacian type

$$\operatorname{div}\left((A\nabla u \cdot \nabla u)^{(p-2)/2} A\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega \tag{1.9}$$

under the different assumptions on the coefficients A and the domain Ω. Boccardo and Gallouët [6, 7] obtained $W^{1,q}$, $q < n(p-1)/(n-1)$, $p \leq n$, regularity for weak solutions of the problem $-\operatorname{div} a(x, u, Du) = f$ with some structural conditions.

Recently, Byun and Wang [8] obtained $W^{1,p}$, $2 \leq p < \infty$, regularity for weak solutions of the general nonlinear elliptic problem

$$\operatorname{div} a(\nabla u, x) = \operatorname{div} \mathbf{f} \quad \text{in } \Omega, \tag{1.10}$$

with $a(\xi, x)$ satisfying (δ, R)-vanishing condition and the following structural conditions:

$$\begin{gathered} \left[a(\xi, x) - a(\eta, x)\right] \cdot (\xi - \eta) \geq C_1' \left|\xi - \eta\right|^2, \\ |a(\xi, x)| \leq C_2'(1 + |\xi|), \\ \left|\nabla_\xi a(\xi, x)\right| \geq C_3'. \end{gathered} \tag{1.11}$$

The purpose of this paper is to extend the L^p-type estimates in [8] to the L^ϕ-type estimates in Orlicz spaces for the more general problem (1.1) with $\mathcal{A}$ satisfying (1.2)–(1.5). In particular, we are interested in estimates like

$$\int_{B_r} \phi(|\nabla u|^p)dx \leq C\left\{\int_{B_{2r}} \phi(|\mathbf{f}|^p)dx + \phi\left(\int_{B_{2r}} |u|^p dx\right) + 1\right\}, \tag{1.12}$$

where C is a constant independent from u and $\mathbf{f}$. Indeed, if $\phi(x) = |x|^{q/p}$ with $q > p$, (1.12) is reduced to the classical L^q estimate.

Orlicz spaces have been studied as a generalization of L^p spaces since they were introduced by Orlicz [9] (see [10–16]). The theory of Orlicz spaces plays a crucial role in a very wide spectrum (see [17]). Here for the reader's convenience, we will give some definitions on the general Orlicz spaces. We denote by Φ the function class that consists of all functions $\phi : [0,+\infty) \to [0,+\infty)$ which are increasing and convex.

Definition 1.2. A function $\phi \in \Phi$ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for every $t > 0$,

$$\phi(2t) \leq K\phi(t). \tag{1.13}$$

Moreover, a function $\phi \in \Phi$ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that for every $t > 0$,

$$\phi(t) \leq \frac{\phi(at)}{2a}. \tag{1.14}$$

Remark 1.3. (1) We remark that the global $\Delta_2 \cap \nabla_2$ condition makes the functions grow moderately. For example, $\phi(t) = |t|^\alpha(1 + |\log|t||) \in \Delta_2 \cap \nabla_2$ for $\alpha > 1$. Examples such as $t\log(1+t)$ are ruled out by ∇_2, and those such as $\exp(t^2)$ are ruled out by Δ_2.

(2) In fact, if $\phi \in \Delta_2 \cap \nabla_2$, then ϕ satisfies for $0 < \theta_2 \leq 1 \leq \theta_1 < \infty$,

$$\phi(\theta_1 t) \leq K\theta_1^{\alpha_1}\phi(t), \qquad \phi(\theta_2 t) \leq 2a\theta_2^{\alpha_2}\phi(t), \tag{1.15}$$

where $\alpha_1 = \log_2 K$ and $\alpha_2 = \log_a 2 + 1$.

(3) Under condition (1.15), it is easy to check that $\phi \in \Phi$ satisfies $\phi(0) = 0$ and

$$\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to +\infty} \frac{t}{\phi(t)} = 0. \tag{1.16}$$

Definition 1.4. Let $\phi \in \Phi$. Then the Orlicz class $K^{\phi}(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_{\Omega} \phi(|g|)dx < \infty. \tag{1.17}$$

The Orlicz space $L^{\phi}(\Omega)$ is the linear hull of $K^{\phi}(\Omega)$.

Remark 1.5. We remark that Orlicz spaces generalize L^q spaces in the sense that if we take $\phi(t) = t^q$, $t \geq 0$, then $\phi \in \Delta_2 \cap \nabla_2$, so for this special case,

$$L^{\phi}(\Omega) = L^q(\Omega). \tag{1.18}$$

Moreover, we give the following lemma.

Lemma 1.6 (see [10, 12, 15]). *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^{\phi}(\Omega)$. Then*

(1) $K^{\phi} = L^{\phi}$ *and* C_0^{∞} *is dense in* L^{ϕ},

(2) $L^{\alpha_1}(\Omega) \subset L^{\phi}(\Omega) \subset L^{\alpha_2}(\Omega) \subset L^1(\Omega)$, *where* α_1 *and* α_2 *are defined in* (1.15),

(3)

$$\int_{\Omega} \phi(|g|)dx = \int_0^{\infty} \left|\{x \in \Omega : |g| > \lambda\}\right| d[\phi(\lambda)], \tag{1.19}$$

(4)

$$\int_0^{\infty} \frac{1}{\mu} \int_{\{x\in\Omega:|g|>a\mu\}} |g| dx\, d[\phi(b\mu)] \leq C \int_{\Omega} \phi(|g|)dx, \tag{1.20}$$

for any $a, b > 0$, *where* $C = C(a, b, \phi)$.

Now we are set to state the main result.

Theorem 1.7. *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and $|\mathbf{f}|^p \in L^{\phi}_{\text{loc}}(\Omega)$. If u is a local weak solution of* (1.1) *with $\mathcal{A}$ satisfying* (1.2)–(1.5), *then one has*

$$|\nabla u|^p \in L^{\phi}_{\text{loc}}(\Omega) \tag{1.21}$$

with the estimate (1.12), *that is,*

$$\int_{B_r} \phi(|\nabla u|^p)dx \leq C\left\{\int_{B_{2r}} \phi(|\mathbf{f}|^p)dx + \phi\left(\int_{B_{2r}} |u|^p dx\right) + 1\right\}, \tag{1.22}$$

where $B_{2r} \subset \Omega$ and C is a constant independent from u and $\mathbf{f}$.

Remark 1.8. We remark that the global $\Delta_2 \cap \nabla_2$ condition is optimal. Actually, the authors in [15] have proved that if u is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right) dx \le C \int_{\mathbb{R}^n} \phi(|f|) dx \tag{1.23}$$

holds if and only if $\phi \in \Delta_2 \cap \nabla_2$.

Our approach is based on the paper [18]. Recently Acerbi and Mingione [18] obtained local L^q, $q \ge p$, gradient estimates for the degenerate parabolic p-Laplacian systems which are not homogeneous if $p \ne 2$. There, they invented a new iteration-covering approach, which is completely free from harmonic analysis, in order to avoid the use of the maximal function operator.

This paper will be organized as follows. In Section 2, we give a new normalization method and the iteration-covering procedure, which are very important to obtain the main result. We finish the proof of Theorem 1.7 in Section 3.

2. Preliminary Materials

2.1. New Normalization

In this paper we will use a new normalization method, which is much influenced by [8, 19], so that the highly nonlinear problem considered here is invariant.

For each $\lambda \ge 1$, we define

$$u_\lambda(x) = \frac{u(x)}{\lambda}, \qquad \mathbf{f}_\lambda(x) = \frac{\mathbf{f}(x)}{\lambda}, \tag{2.1}$$

$$\mathcal{A}_\lambda(\xi, x) = \frac{\mathcal{A}(\lambda\xi, x)}{\lambda^{p-1}}. \tag{2.2}$$

Lemma 2.1 (new normalization). *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution of* (1.1) *and $\mathcal{A}$ satisfies* (1.2)–(1.5)*, then*

(1) *$\mathcal{A}_\lambda$ satisfies* (1.2)–(1.5) *with the same constants C_i $(1 \le i \le 4)$,*

(2) *u_λ is a local weak solution of*

$$\operatorname{div} \mathcal{A}_\lambda(\nabla u_\lambda, x) = \frac{1}{\lambda^{p-1}} \operatorname{div} \mathcal{A}(\nabla u, x) = \operatorname{div}\left(|\mathbf{f}_\lambda|^{p-2}\mathbf{f}_\lambda\right) \quad \text{in } \Omega. \tag{2.3}$$

Proof. We first prove that $\mathcal{A}_\lambda$ satisfies (1.2)–(1.5) with the same constants C_i $(1 \le i \le 4)$. From (1.2) and (2.2) we find that

$$\begin{aligned}\left[\mathcal{A}_\lambda(\xi, x) - \mathcal{A}_\lambda(\eta, x)\right] \cdot (\xi - \eta) &= \frac{1}{\lambda^p}\left[\mathcal{A}(\lambda\xi, x) - \mathcal{A}(\lambda\eta, x)\right] \cdot (\lambda\xi - \lambda\eta) \\ &\ge C_1 \frac{1}{\lambda^p}|\lambda\xi - \lambda\eta|^p = C_1|\xi - \eta|^p\end{aligned} \tag{2.4}$$

for all $\xi, \eta \in \mathbb{R}^n$. That is to say, $\mathcal{A}_\lambda$ satisfies (1.2). Moreover, $\mathcal{A}_\lambda$ satisfies (1.3)-(1.4) since

$$|\mathcal{A}_\lambda(\xi, x)| = \frac{|\mathcal{A}(\lambda\xi, x)|}{\lambda^{p-1}} \leq C_2\left(\frac{1}{\lambda^{p-1}} + |\xi|^{p-1}\right) \leq C_2\left(1 + |\xi|^{p-1}\right),$$
$$\mathcal{A}_\lambda(\xi, x) \cdot \xi = \frac{1}{\lambda^p}\mathcal{A}(\lambda\xi, x) \cdot \lambda\xi \geq \frac{1}{\lambda^p}\left(C_3|\lambda\xi|^p - C_4\right) \geq C_3|\xi|^p - C_4 \tag{2.5}$$

for all $\xi, \eta \in \mathbb{R}^n$ and $\lambda \geq 1$. Furthermore,

$$\begin{aligned}\left|\mathcal{A}_\lambda(\xi, x) - \mathcal{A}_\lambda(\xi, y)\right| &= \frac{1}{\lambda^{p-1}}\left|\mathcal{A}(\lambda\xi, x) - \mathcal{A}(\lambda\xi, y)\right| \\ &\leq \frac{C_5}{\lambda^{p-1}} w(|x-y|)\left(1 + |\lambda\xi|^{p-1}\right) \\ &\leq C_5 w(|x-y|)\left(1 + |\xi|^{p-1}\right)\end{aligned} \tag{2.6}$$

for all $\xi, \eta \in \mathbb{R}^n$ and $\lambda \geq 1$.

Finally we prove (2). Indeed, since u is a local weak solution of (1.1), it follows from Definition 1.1, (2.1), and (2.2) that

$$\begin{aligned}\int_\Omega \mathcal{A}_\lambda(\nabla u_\lambda, x) \cdot \nabla\varphi\, dx &= \frac{1}{\lambda^{p-1}}\int_\Omega \mathcal{A}(\nabla u, x) \cdot \nabla\varphi\, dx \\ &= \frac{1}{\lambda^{p-1}}\int_\Omega |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla\varphi\, dx = \int_\Omega |\mathbf{f}_\lambda|^{p-2}\mathbf{f}_\lambda \cdot \nabla\varphi\, dx.\end{aligned} \tag{2.7}$$

Thus we complete the proof. □

2.2. The Iteration-Covering Procedure

In this subsection we give one important lemma (the iteration-covering procedure), which is much motivated by [18]. To start with, let u be a local weak solution of the problem (1.1). By a scaling argument we may as well assume that $r = 1$ in Theorem 1.7. We write

$$\lambda_0 = \left[⨍_{B_2} |\nabla u|^p dx + \frac{1}{\epsilon}⨍_{B_2} |\mathbf{f}|^p dx\right]^{1/p}, \tag{2.8}$$

where $\epsilon > 0$ is going to be chosen later in (3.47). Moreover, for any $x \in \Omega$ and $\rho > 0$, we write

$$J_\lambda\left[B_\rho(x)\right] = ⨍_{B_\rho(x)} |\nabla u_\lambda|^p dy + \frac{1}{\epsilon}⨍_{B_\rho(x)} |\mathbf{f}_\lambda|^p dy, \tag{2.9}$$

$$E_\lambda(1) = \left\{x \in B_1 : |\nabla u_\lambda|^p > 1\right\}. \tag{2.10}$$

From (1.6), we can choose a proper constant $R_0 = R_0(\epsilon) \in (0,1)$ such that

$$w(R_0) \le \epsilon. \tag{2.11}$$

Lemma 2.2. *Given $\lambda \ge \lambda_* =: (10/R_0)^{n/p}\lambda_0 + 1$, there exists a family of disjoint balls $\{B_{\rho_i}(x_i)\}$, $x_i \in E_\lambda(1)$ such that $0 < \rho_i = \rho(x_i) \le R_0/10$ and*

$$J_\lambda\left[B_{\rho_i}(x_i)\right] = 1, \qquad J_\lambda\left[B_\rho(x_i)\right] < 1 \quad \text{for any } \rho > \rho_i. \tag{2.12}$$

Moreover, one has

$$E_\lambda(1) \subset \bigcup_{i\in\mathbb{N}} B_{5\rho_i}(x_i) \cup \text{negligible set}, \tag{2.13}$$

$$\left|B_{\rho_i}(x_i)\right| \le 3\left(\int_{\{x\in B_{\rho_i}(x_i):|\nabla u_\lambda|^p>1/3\}} |\nabla u_\lambda|^p dx + \frac{1}{\epsilon}\int_{\{x\in B_{\rho_i}(x_i):|\mathbf{f}_\lambda|^p>\epsilon/3\}} |\mathbf{f}_\lambda|^p dx\right). \tag{2.14}$$

Proof. (1) We first claim that

$$\sup_{w\in B_1}\ \sup_{R_0/10\le\rho\le R_0} J_\lambda\left[B_\rho(w)\right] \le 1. \tag{2.15}$$

To prove this, fix any $w \in B_1$ and $R_0/10 \le \rho \le R_0$. Let $\lambda \ge \lambda_* = (10/R_0)^{n/p}\lambda_0 + 1$. Then we have

$$\begin{aligned}
⨍_{B_\rho(w)} |\nabla u_\lambda|^p dx &\le \frac{|B_1|}{\left|B_\rho(w)\right|} ⨍_{B_1} |\nabla u_\lambda|^p dx \\
&\le \left(\frac{10}{R_0}\right)^n ⨍_{B_1} |\nabla u_\lambda|^p dx \\
&= \frac{1}{\lambda^p}\left(\frac{10}{R_0}\right)^n ⨍_{B_1} |\nabla u|^p dx.
\end{aligned} \tag{2.16}$$

Similarly,

$$⨍_{B_\rho(w)} |\mathbf{f}_\lambda|^p dx \le \left(\frac{10}{R_0}\right)^n ⨍_{B_1} |\mathbf{f}_\lambda|^p dx = \frac{1}{\lambda^p}\left(\frac{10}{R_0}\right)^n ⨍_{B_1} |\mathbf{f}|^p dx. \tag{2.17}$$

Consequently, combining the two inequalities above, (2.8) and (2.9), we know that

$$J_\lambda\left[B_\rho(w)\right] \le 1 \tag{2.18}$$

for any $w \in B_1$ and $R_0/10 \le \rho \le R_0$, which implies that (2.15) holds truely.

(2) Now for a.e. $w \in E_\lambda(1)$, a version of Lebesgue's differentiation theorem implies that

$$\lim_{\rho\to 0} J_\lambda\left[B_\rho(w)\right] > 1, \tag{2.19}$$

which implies that there exists some $\rho > 0$ satisfying

$$J_\lambda\left[B_\rho(w)\right] > 1. \tag{2.20}$$

Therefore from (2.15) we can select a radius $\rho_w \in (0, R_0/10]$ such that

$$\rho_w =: \max\left\{\rho \mid J_\lambda\left[B_\rho(w)\right] = 1,\ 0 < \rho \le \frac{R_0}{10}\right\}. \tag{2.21}$$

Then we observe that

$$J_\lambda\left[B_{\rho_w}(w)\right] = 1 \tag{2.22}$$

and that for $\rho_w < \rho \le R_0$,

$$J_\lambda\left[B_\rho(w)\right] < 1. \tag{2.23}$$

From the argument above we know that for a.e. $w \in E_\lambda(1)$ there exists a ball $B_{\rho_w}(w)$ constructed as above. Therefore, applying Vitali's covering lemma, we can find a family of disjoint balls $\{B_{\rho_i}(x_i)\}_{i\in\mathbb{N}}$ with $x_i \in E_\lambda(1)$ and $\rho_i = \rho(x_i) \in (0, R_0/10]$, so that (2.12) and (2.13) hold truely.

(3) From (2.12) we see that

$$J_\lambda\left[B_{\rho_i}(x_i)\right] =: \fint_{B_{\rho_i}(x_i)} |\nabla u_\lambda|^p dx + \frac{1}{\epsilon}\fint_{B_{\rho_i}(x_i)} |\mathbf{f}_\lambda|^p dx = 1. \tag{2.24}$$

That is to say,

$$\left|B_{\rho_i}(x_i)\right| = \int_{B_{\rho_i}(x_i)} |\nabla u_\lambda|^p dx + \frac{1}{\epsilon}\int_{B_{\rho_i}(x_i)} |\mathbf{f}_\lambda|^p dx. \tag{2.25}$$

Therefore, by splitting the right-side two integrals in (2.25) as follows we have

$$\begin{aligned} \left|B_{\rho_i}(x_i)\right| \le& \int_{\{x\in B_{\rho_i}(x_i):|\nabla u_\lambda|^p>1/3\}} |\nabla u_\lambda|^p dx + \frac{\left|B_{\rho_i}(x_i)\right|}{3} \\ &+ \frac{1}{\epsilon}\int_{\{x\in B_{\rho_i}(x_i):|\mathbf{f}_\lambda|>\epsilon/3\}} |\mathbf{f}_\lambda|^p dx + \frac{\left|B_{\rho_i}(x_i)\right|}{3}. \end{aligned} \tag{2.26}$$

Thus we obtain the desired estimate (2.14). This completes our proof. □

3. Proof of Main Result

In the following it is sufficient to consider the proof of Theorem 1.7 as an a priori estimate, therefore assuming a priori that $|\nabla u|^p \in L^{\infty}_{\mathrm{loc}}(\Omega) \subset L^{\phi}_{\mathrm{loc}}(\Omega)$. This assumption can be removed in a standard way via an approximation argument as the one in [12, 15, 18].

We first give the following local L^p estimates for problem (1.1).

Lemma 3.1. *Suppose that* $|\mathbf{f}|^p \in L^{\phi}(B_{2R})$, $B_{2R} \subset \Omega$, *and let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a local weak solution of* (1.1) *with* $\mathcal{A}$ *satisfying* (1.2)–(1.5). *Then one has*

$$\int_{B_R} |\nabla u|^p dx \leq C\left\{\frac{1}{R^p}\int_{B_{2R}} |u|^p dx + \int_{B_{2R}} |\mathbf{f}|^p dx + 1\right\}. \tag{3.1}$$

Proof. We may choose the test function $\varphi = \zeta^p u \in W^{1,p}_0(B_{2R})$ in Definition 1.1, where $\zeta \in C^{\infty}_0(\mathbb{R}^n)$ is a cutoff function satisfying

$$0 \leq \zeta \leq 1, \qquad \zeta \equiv 1 \quad \text{in } B_R, \qquad \zeta \equiv 0 \quad \text{in } \frac{\mathbb{R}^n}{B_{2R}}, \qquad |\nabla \zeta| \leq \frac{C}{R}. \tag{3.2}$$

Then we have

$$\int_{B_{2R}} \mathcal{A}(\nabla u, x) \cdot \nabla(\zeta^p u) dx = \int_{B_{2R}} |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla(\zeta^p u) dx \tag{3.3}$$

and then write the resulting expression as

$$I_1 = I_2 + I_3 + I_4, \tag{3.4}$$

where

$$\begin{aligned} I_1 &= \int_{B_{2R}} \zeta^p \mathcal{A}(\nabla u, x) \cdot \nabla u \, dx, \\ I_2 &= -\int_{B_{2R}} p\zeta^{p-1} u \mathcal{A}(\nabla u, x) \cdot \nabla \zeta \, dx, \\ I_3 &= \int_{B_{2R}} \zeta^p |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla u \, dx, \\ I_4 &= \int_{B_{2R}} p\zeta^{p-1} u |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla \zeta \, dx. \end{aligned} \tag{3.5}$$

Estimate of I_1. Using (1.4), we find that

$$I_1 \geq C\int_{B_{2R}} \left(|\nabla u|^p - 1\right)\zeta^p dx \tag{3.6}$$

Estimate of I_2. From Young's inequality with τ, (1.3), and (3.2) we have

$$\begin{aligned}
I_2 &\le \int_{B_{2R}} p\zeta^{p-1}|\mathcal{A}(\nabla u, x)||u||\nabla\zeta|dx \\
&\le C\int_{B_{2R}} p\zeta^{p-1}\left|\left(1+|\nabla u|^{p-1}\right)\right||u||\nabla\zeta|dx \\
&\le \tau\int_{B_{2R}} \zeta^p\left(|\nabla u|^p+1\right)dx + C(\tau)\int_{B_{2R}}|\nabla\zeta|^p|u|^p dx \\
&\le \tau\int_{B_{2R}} \zeta^p\left(|\nabla u|^p+1\right)dx + \frac{C(\tau)}{R^p}\int_{B_{2R}}|u|^p dx.
\end{aligned} \tag{3.7}$$

Estimate of I_3. From Young's inequality with τ we have

$$I_3 \le \tau\int_{B_{2R}} \zeta^p|\nabla u|^p dx + C(\tau)\int_{B_{2R}}|\mathbf{f}|^p dx. \tag{3.8}$$

Estimate of I_4. From Young's inequality and (3.2) we have

$$\begin{aligned}
I_4 &\le C\left\{\int_{B_{2R}}|\nabla\zeta|^p|u|^p dx + \int_{B_{2R}}|\mathbf{f}|^p dx\right\} \\
&\le C\left\{\frac{1}{R^p}\int_{B_{2R}}|u|^p dx + \int_{B_{2R}}|\mathbf{f}|^p dx\right\}.
\end{aligned} \tag{3.9}$$

Combining the estimates of I_i $(1\le i\le 4)$, we deduce that

$$C\int_{B_{2R}}\zeta^p|\nabla u|^p dx \le 2\tau\int_{B_{2R}}\zeta^p|\nabla u|^p dx + C(\tau)\left\{\frac{1}{R^p}\int_{B_{2R}}|u|^p dx + \int_{B_{2R}}|\mathbf{f}|^p dx + 1\right\} \tag{3.10}$$

and then finish the proof by choosing τ small enough. □

Let v be the weak solution of the following reference equation:

$$\begin{aligned}
\operatorname{div}\mathcal{A}(\nabla v, x^*) &= 0 \quad \text{in } B_r, \\
v &= u \quad \text{on } \partial B_r,
\end{aligned} \tag{3.11}$$

where $x^* \in B_r$ is a fixed point.

We first state the definition of the global weak solutions.

Definition 3.2. Assume that $v \in W^{1,p}(B_r)$. One says that $v \in W^{1,p}(B_r)$ with $v - u \in W_0^{1,p}(B_r)$ is the weak solution of (3.11) in B_r if one has

$$\int_{B_r} \mathcal{A}(\nabla v, x^*) \cdot \nabla \varphi \, dx = 0 \tag{3.12}$$

for any $\varphi \in W_0^{1,p}(B_r)$.

From the definition above we can easily obtain the following lemma.

Lemma 3.3. *If $v \in W^{1,p}(B_{10\rho_i}(x_i))$ is the weak solution of* (3.11) *in $B_{10\rho_i}(x_i)$, where $x_i \in E_\lambda(1)$ and ρ_i are defined in Lemma 2.2, then one has*

$$⨍_{B_{10\rho_i}(x_i)} |\nabla v|^p dx \le C\left\{⨍_{B_{10\rho_i}(x_i)} |\nabla u|^p dx + 1\right\}. \tag{3.13}$$

Proof. Choosing the test function $\varphi = u - v \in W_0^{1,p}(B_{10\rho_i}(x_i))$, from Definition 3.2, we find that

$$\int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*) \cdot \nabla (u - v) dx = 0. \tag{3.14}$$

That is to say,

$$\int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*) \cdot \nabla v \, dx = \int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*) \cdot \nabla u \, dx. \tag{3.15}$$

From (1.4), we conclude that

$$\int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*) \cdot \nabla v \, dx \ge C\left(\int_{B_{10\rho_i}(x_i)} |\nabla v|^p dx - \left|B_{10\rho_i}\right|\right). \tag{3.16}$$

Moreover, from (1.3) and Young's inequality with τ we have

$$\begin{aligned} \int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*) \cdot \nabla u \, dx &\le C \int_{B_{10\rho_i}(x_i)} \left(1 + |\nabla v|^{p-1}\right)|\nabla u| dx \\ &\le \tau \int_{B_{10\rho_i}(x_i)} |\nabla v|^p dx + C(\tau) \int_{B_{10\rho_i}(x_i)} (1 + |\nabla u|^p) dx. \end{aligned} \tag{3.17}$$

Combining the estimates of I_i $(i = 1,2)$ and selecting a small enough constant $\tau > 0$, we deduce that

$$\fint_{B_{10\rho_i}(x_i)} |\nabla v|^p dx \le C\left\{\fint_{B_{10\rho_i}(x_i)} |\nabla u|^p dx + 1\right\} \tag{3.18}$$

and then finish the proof. □

Lemma 3.4. *Suppose that $v \in W^{1,p}(B_{10\rho_i}(x_i))$ is the weak solution of* (3.11) *in $B_{10\rho_i}(x_i)$ with $\mathcal{A}$ satisfying* (1.2)–(1.5). *If*

$$\fint_{B_{10\rho_i}(x_i)} |\nabla u|^p dx \le 1, \qquad \fint_{B_{10\rho_i}(x_i)} |\mathbf{f}|^p dx \le \epsilon, \tag{3.19}$$

then there exists $N_0 > 1$ such that

$$\sup_{B_{5\rho_i}(x_i)} |\nabla v| \le N_0, \tag{3.20}$$

$$\fint_{B_{10\rho_i}(x_i)} |\nabla(u-v)|^p dx \le \epsilon. \tag{3.21}$$

Proof. If the conclusion (3.21) is true, then the conclusion (3.20) can follow from [20, Lemma 5.1].

Next we are set to prove (3.21). We may choose the test function $\varphi = u - v \in W_0^{1,p}(B_{10\rho_i}(x_i))$ in Definitions 1.1 and 3.2 to find that

$$\begin{gathered}\int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla u, x)\cdot\nabla(u-v)dx = \int_{B_{10\rho_i}(x_i)} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla(u-v)dx,\\ \int_{B_{10\rho_i}(x_i)} \mathcal{A}(\nabla v, x^*)\cdot\nabla(u-v)dx = 0,\end{gathered} \tag{3.22}$$

where $x^* \in B_{10\rho_i}(x_i)$ is a fixed point. Then a direct calculation shows the resulting expression as

$$I_1 = I_2 + I_3, \tag{3.23}$$

where

$$I_1 = \int_{B_{10\rho_i}(x_i)} (\mathcal{A}(\nabla u, x) - \mathcal{A}(\nabla v, x)) \cdot \nabla(u - v)dx,$$
$$I_2 = -\int_{B_{10\rho_i}(x_i)} (\mathcal{A}(\nabla v, x) - \mathcal{A}(\nabla v, x^*)) \cdot \nabla(u - v)dx, \tag{3.24}$$
$$I_3 = \int_{B_{10\rho_i}(x_i)} |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla(u - v)dx.$$

Estimate of I_1. Equation (1.2) implies that

$$I_1 \geq C \int_{B_{10\rho_i}(x_i)} |\nabla(u - v)|^p dx. \tag{3.25}$$

Estimate of I_2. From (1.5) and the fact that $\rho_i \in (0, R_0/10]$ we obtain

$$\begin{aligned} I_2 &\leq Cw(10\rho_i) \int_{B_{10\rho_i}(x_i)} \left(1 + |\nabla v|^{p-1}\right)(|\nabla u| + |\nabla v|)dx \\ &\leq Cw(R_0) \int_{B_{10\rho_i}(x_i)} \left[1 + |\nabla v|^p + \left(1 + |\nabla v|^{p-1}\right)|\nabla u|\right]dx, \end{aligned} \tag{3.26}$$

then it follows from (2.11), Young's inequality, and Lemma 3.3 that

$$\begin{aligned} I_2 &\leq C\epsilon\left\{\int_{B_{10\rho_i}(x_i)} (1 + |\nabla v|^p)dx + \int_{B_{10\rho_i}(x_i)} |\nabla u|^p dx\right\} \\ &\leq C\epsilon \int_{B_{10\rho_i}(x_i)} (1 + |\nabla u|^p)dx. \end{aligned} \tag{3.27}$$

Furthermore, using (3.19) we can obtain

$$I_2 \leq C\left|B_{10\rho_i}(x_i)\right|\epsilon. \tag{3.28}$$

Estimate of I_3. Using Young's inequality with τ, we have

$$\begin{aligned} I_3 &\leq \int_{B_{10\rho_i}(x_i)} |\mathbf{f}|^{p-1}|\nabla(u - v)|dx \\ &\leq \tau \int_{B_{10\rho_i}(x_i)} |\nabla(u - v)|^p dx + C(\tau) \int_{B_{10\rho_i}(x_i)} |\mathbf{f}|^p dx. \end{aligned} \tag{3.29}$$

Combing all the estimates of I_i $(1 \le i \le 3)$ and selecting a small enough constant $\tau > 0$, we obtain

$$\int_{B_{10\rho_i}(x_i)} |\nabla(u-v)|^p dx \le C\left|B_{10\rho_i}(x_i)\right|\epsilon + C\int_{B_{10\rho_i}(x_i)} |\mathbf{f}|^p dx, \tag{3.30}$$

then it follows from (3.19) that

$$⨍_{B_{10\rho_i}(x_i)} |\nabla(u-v)|^p dx \le C\varepsilon. \tag{3.31}$$

This completes our proof. □

In view of Lemma 2.2, given $\lambda \ge \lambda_* =: (10/R_0)^{n/p}\lambda_0 + 1$, we can construct the disjoint family of balls $\{B_{\rho_i}(x_i)\}_{i\in\mathbb{N}}$, where $x_i \in E_\lambda(1)$. Fix any $i \in \mathbb{N}$. It follows from Lemma 2.2 that

$$⨍_{B_\rho(x_i)} |\nabla u_\lambda|^p dx \le 1, \quad ⨍_{B_\rho(x_i)} |\mathbf{f}_\lambda|^p dx \le \epsilon \quad \text{for any } \rho > \rho_i. \tag{3.32}$$

Furthermore, from the new normalization in Lemma 2.1, we can easily obtain the following corollary of Lemma 3.4.

Corollary 3.5. *Suppose that $v_\lambda \in W^{1,p}(B_{10\rho_i}(x_i))$ is the weak solution of*

$$\begin{gathered} \operatorname{div} \mathcal{A}_\lambda(\nabla v_\lambda, x^*) = 0 \quad \text{in } Q_r, \\ v_\lambda = u_\lambda \quad \text{on } \partial_p Q_r, \end{gathered} \tag{3.33}$$

with $x^* \in B_{10\rho_i}(x_i)$ and $\mathcal{A}_\lambda$ satisfying (1.2)–(1.5). Then there exists $N_0 > 1$ such that

$$\begin{gathered} \sup_{B_{5\rho_i}(x_i)} |\nabla v_\lambda| \le N_0, \\ ⨍_{B_{10\rho_i}(x_i)} |\nabla(u_\lambda - v_\lambda)|^p dx \le C\epsilon. \end{gathered} \tag{3.34}$$

Now we are ready to prove the main result, Theorem 1.7.

Proof. From Corollary 3.5, for any $\lambda \geq \lambda_* =: (10/R_0)^{n/p}\lambda_0 + 1$ we have

$$\begin{aligned}
\left|\{x \in B_{5\rho_i}(x_i) : |\nabla u| > 2N_0\lambda\}\right| &= \left|\{x \in B_{5\rho_i}(x_i) : |\nabla u_\lambda| > 2N_0\}\right| \\
&\leq \left|\{x \in B_{5\rho_i}(x_i) : |\nabla(u_\lambda - v_\lambda)| > N_0\}\right| \\
&\quad + \left|\{x \in B_{5\rho_i}(x_i) : |\nabla v_\lambda| > N_0\}\right| \\
&= \left|\{x \in B_{5\rho_i}(x_i) : |\nabla(u_\lambda - v_\lambda)| > N_0\}\right| \\
&\leq \frac{1}{N_0^p}\int_{B_{5\rho_i}(x_i)} |\nabla(u_\lambda - v_\lambda)|^p dx \\
&\leq C\epsilon\left|B_{\rho_i}(x_i)\right|,
\end{aligned} \tag{3.35}$$

then it follows from (2.14) in Lemma 2.2 that

$$\begin{aligned}
&\left|\{x \in B_{5\rho_i}(x_i) : |\nabla u| > 2N_0\lambda\}\right| \\
&\quad \leq C\epsilon\left(\int_{\{x\in B_{\rho_i}(x_i):|\nabla u_\lambda|^p>1/3\}} |\nabla u_\lambda|^p dx + \frac{1}{\epsilon}\int_{\{x\in B_{\rho_i}(x_i):|\mathbf{f}_\lambda|^p>\epsilon/3\}} |\mathbf{f}_\lambda|^p dx\right),
\end{aligned} \tag{3.36}$$

where $C = C(n,p)$. Recalling the fact that the balls $\{B_{\rho_i}(x_i)\}$ are disjoint and

$$\bigcup_{i\in\mathbb{N}} B_{5\rho_i}(x_i) \supset E_\lambda(1) = \{x \in B_1 : |\nabla u_\lambda| > 1\} \tag{3.37}$$

for any $\lambda \geq \lambda_* =: (10/R_0)^{n/p}\lambda_0 + 1$ and then summing up on $i \in \mathbb{N}$ in the inequality above, we have

$$\begin{aligned}
\left|\{x \in B_1 : |\nabla u|^p > (2N_0)^p\lambda^p\}\right| &= |\{x \in B_1 : |\nabla u| > 2N_0\lambda\}| \\
&\leq \sum_i \left|\{x \in B_{5\rho_i}(x_i) : |\nabla u| > 2N_0\lambda\}\right| \\
&\leq C\epsilon\left(\int_{\{x\in B_2:|\nabla u_\lambda|^p>1/3\}} |\nabla u_\lambda|^p dx + \frac{1}{\epsilon}\int_{x\in B_2:|\mathbf{f}_\lambda|^p>\epsilon/3} |\mathbf{f}_\lambda|^p dx\right)
\end{aligned} \tag{3.38}$$

for any $\lambda \geq \lambda_* =: (10/R_0)^{n/p}\lambda_0 + 1$. Recalling Lemma 1.6(3), we compute

$$\begin{aligned}
\int_{B_1} \phi(|\nabla u|^p)\,dx &= \int_0^\infty \left|\{x \in B_1 : |\nabla u|^p > \mu\}\right| d[\phi(\mu)] \\
&= \int_0^{(2N_0)^p\lambda_*^p} \left|\{x \in B_1 : |\nabla u|^p > \mu\}\right| d[\phi(\mu)] \\
&\quad + \int_{(2N_0)^p\lambda_*^p}^\infty \left|\{x \in B_1 : |\nabla u|^p > \mu\}\right| d[\phi(\mu)] \\
&= \int_0^{(2N_0)^p\lambda_*^p} \left|\{x \in B_1 : |\nabla u|^p > \mu\}\right| d[\phi(\mu)] \\
&\quad + \int_{\lambda_*}^\infty \left|\{x \in B_1 : |\nabla u|^p > (2N_0)^p\lambda^p\}\right| d\left[(2N_0)^p\lambda^p\right] \\
&=: J_1 + J_2.
\end{aligned} \tag{3.39}$$

Estimate of J_1. From the definition of λ_0 in (2.8) we deduce that

$$\lambda_*^p \leq C\left[\lambda_0^p + 1\right] \leq C\left\{ ⨍_{B_1} u|\nabla|^p dx + \frac{1}{\epsilon} ⨍_{B_1} |\mathbf{f}|^p dx + 1 \right\}, \tag{3.40}$$

then it follows from Lemma 3.1 that

$$\begin{aligned}
\lambda_*^p &\leq C\left\{ ⨍_{B_2} |u|^p dx + ⨍_{B_2} |\mathbf{f}|^p dx + \frac{1}{\epsilon} ⨍_{B_1} |\mathbf{f}|^p dx + 1 \right\} \\
&\leq C\left\{ ⨍_{B_2} |u|^p dx + ⨍_{B_2} |\mathbf{f}|^p dx + 1 \right\},
\end{aligned} \tag{3.41}$$

where $C = C(n, p, \epsilon)$. Therefore, by (1.15) and Jensen's inequality, we conclude that

$$\begin{aligned}
J_1 &\leq \phi\left[(2N_0)^p\lambda_*^p\right]|B_1| \\
&\leq C\left\{ \phi\left(⨍_{B_2} |u|^p dx \right) + \phi\left(⨍_{B_2} |\mathbf{f}|^p dx \right) + 1 \right\} \\
&\leq C\left\{ \phi\left(\int_{B_2} |u|^p dx \right) + \int_{B_2} \phi(|\mathbf{f}|^p)\,dx + 1 \right\},
\end{aligned} \tag{3.42}$$

where $C = C(n, p, \phi, \epsilon)$.

Estimate of J_2. From (3.38) we deduce that

$$J_2 \leq C\epsilon \int_0^\infty \int_{\{x\in B_2:|\nabla u_\lambda|^p>1/3\}} |\nabla u_\lambda|^p dx\, d\left[(2N_0)^p \lambda^p\right] + C\int_0^\infty \int_{\{x\in B_2:|\mathbf{f}_\lambda|^p>\epsilon/3\}} |\mathbf{f}_\lambda|^p dx\, d\left[(2N_0)^p\lambda^p\right]. \tag{3.43}$$

Set $\mu = \lambda^p$. The above inequality and (2.1) imply that

$$J_2 \leq C\epsilon \int_0^\infty \frac{1}{\mu} \int_{\{x\in B_2:|\nabla u|^p>\mu/3\}} |\nabla u|^p dx\, d\left[(2N_0)^p \mu\right] + C\int_0^\infty \frac{1}{\mu}\int_{\{x\in B_2:|\mathbf{f}|^p>\mu\epsilon/3\}} |\mathbf{f}|^p dx\, d\left[(2N_0)^p\mu\right], \tag{3.44}$$

then it follows from Lemma 1.6(4) that

$$J_2 \leq C_1\epsilon \int_{B_2} \phi(|\nabla u|^p)dx + C_2\int_{B_2}\phi(|\mathbf{f}|^p)dx, \tag{3.45}$$

where $C_1 = C(n,p,\phi)$ and $C_2 = C(n,p,\phi,\epsilon)$.

Combining the estimates of J_1 and J_2, we obtain

$$\int_{B_1}\phi(|\nabla u|^p)dx \leq C_1\epsilon\int_{B_2}\phi(|\nabla u|^p)dx + C_3\int_{B_2}\phi(|\mathbf{f}|^p)dx + C_4\phi\left(\int_{B_2}|u|^p dx\right) + 1, \tag{3.46}$$

where $C_3 = C_3(n,p,\phi,\epsilon)$ and $C_4 = C_4(n,p,\phi,\epsilon)$. Selecting suitable ϵ such that

$$C_1\epsilon = \frac{1}{2} \tag{3.47}$$

and reabsorbing at the right-side first integral in the inequality above by a covering and iteration argument (see [21, Lemma 4.1, Chapter 2], or [22, Lemma 2.1, Chapter 3]), we have

$$\int_{B_1}\phi(|\nabla u|^p)dx \leq C\left\{\int_{B_2}\phi(|\mathbf{f}|^p)dx + \phi\left(\int_{B_2}|u|^p dx\right) + 1\right\}. \tag{3.48}$$

Then by an elementary scaling argument, we can finish the proof of the main result. □

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