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Research Article

Generalized q-Euler Numbers and Polynomials of Higher Order and Some Theoretic Identities

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We give a new construction of the q-Euler numbers and polynomials of higher order attached to Dirichlet's character χ. We derive some theoretic identities involving the generalized q-Euler numbers and polynomials of higher order.

1. Introduction

Let $\mathbb{C}$ be the complex number field. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and the q-number is defined by $[x]_q = (1 - q^x)/(1 - q)$ in this paper. The q-factorial is given by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and the q-binomial formulae are known as

$$(x : q)_n = \prod_{i=1}^{n}\left(1 - xq^{i-1}\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-x)^i,$$
$$\frac{1}{(x : q)_n} = \prod_{i=1}^{n}\left(\frac{1}{1 - xq^{i-1}}\right) = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q x^i, \tag{1.1}$$

where $\binom{n}{i}_q = [n]_q!/[n-i]_q![i]_q! = [n]_q[n-1]_q \cdots [n-i+1]_q/[i]_q!$ (see [1–3]).

After Carlitz had constructed the q-Bernoulli numbers and polynomials, many mathematicians have studied for q-Bernoulli and q-Euler numbers and polynomials (see [1–29]). Since the q-extensions of Euler numbers and polynomials contain interesting properties to study various fields of mathematical physics and number theory, many researchers considered and investigated the q-Euler numbers and polynomials, and derived some identities from them (see [2–5, 8–19]). The purpose of this paper is to give a new approach to the q-Euler numbers and polynomials of higher order attached to Dirichlet's character χ. From this, we will derive some theoretic identities involving generalized q-Euler numbers and polynomials of higher order.

In Section 2, we present new generating functions which are related to q-Euler numbers and polynomials of higher order attached to χ. We obtain distribution relations for the q-Euler polynomials attached to χ, and have some identities involving these q-Euler polynomials. Using the Cauchy residue theorem, we show that these q-extensions of the q-l-function of order r attached to χ interpolate the q-Euler polynomials of order r at negative integers.

2. q-Euler Polynomials of Higher Order Attached to χ

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod} 2)$, let χ be Dirichlet's character with conductor d. For $r \in \mathbb{Z}$ and $h \in \mathbb{Z}$, we will study the generalized q-Euler and (h, q)-Euler polynomials and numbers of order r attached to χ, respectively.

It is known that the Euler polynomials are defined by $(2/(e^t + 1))e^{xt} = \sum_{n=0}^{\infty} E_n(x)(t^n/n!)$, for $|t| < \pi$. In the special case $x = 0, E_n = E_n(0)$ are called the nth Euler numbers (see [28, 29]).

First, we define the generalized q-Euler polynomials attached to χ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m e^{[m+x]_q t}, \tag{2.1}$$

where $E_{n,\chi,q}(x)$ are called the nth generalized q-Euler polynomials attached to χ. In the special case $x = 0$, $E_{n,\chi,q}(= E_{n,\chi,q}(0))$ are called the nth generalized q-Euler numbers attached to χ.

By (2.1), we see that

$$\begin{aligned} E_{n,\chi,q}(x) &= 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m [m+x]_q^n \\ &= \frac{2}{(1-q)^n} \sum_{a=0}^{d-1} \chi(a)(-1)^a \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{l(x+a)}}{1+q^{ld}}. \end{aligned} \tag{2.2}$$

Now we consider the q-Euler polynomials of order r attached to χ as follows:

$$\begin{aligned} F_{q,\chi}^{(r)}(t,x) &= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r} \chi(m_i) \right) (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}, \end{aligned} \tag{2.3}$$

where $E_{n,\chi,q}^{(r)}(x)$ are called the nth generalized q-Euler polynomials of order r attached to χ. In the special case $x = 0$, $E_{n,\chi,q}^{(r)}(= E_{n,\chi,q}^{(r)}(0))$ are called the nth generalized q-Euler numbers of order r attached to χ.

From (2.3), we note that

$$E_{n,\chi,q}^{(r)}(x) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n$$
$$= \frac{2^r}{(1-q)^n}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{l(x+\sum_{j=1}^{r}a_j)}}{(1+q^{ld})^r}. \tag{2.4}$$

Thus we have

$$E_{n,\chi,q}^{(r)}(x) = 2^r \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}$$
$$\times\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[x+a_1+\cdots+a_r+dm]_q^n. \tag{2.5}$$

That is,

$$F_{q,\chi}^{(r)}(t,x) = 2^r \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}\times\sum_{m=0}^{\infty}\binom{m+r-1}{m}e^{[x+a_1+\cdots+a_r+dm]_q t}. \tag{2.6}$$

In the viewpoint of h-extension of $E_{n,\chi,q}^{(r)}(x)$, we can define the generalized (h,q)-Euler polynomials of order r attached to χ as follows:

$$F_{q,\chi}^{(h,r)}(t,x) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{\sum_{j=1}^{r}(h-j)m_j}e^{[m_1+\cdots+m_r+x]_q t}$$
$$= \sum_{n=0}^{\infty}E_{n,\chi,q}^{(h,r)}(x)\frac{t^n}{n!}, \tag{2.7}$$

where $E_{n,\chi,q}^{(h,r)}(x)$ are called the nth generalized (h,q)-Euler polynomials of order r attached to χ. In the special case $x=0$, $E_{n,\chi,q}^{(h,r)}(=E_{n,\chi,q}^{(h,r)}(0))$ are called the nth generalized (h,q)-Euler numbers of order r attached to χ.

By (2.7), we see that

$$E_{n,\chi,q}^{(h,r)}(x) = 2^r[d]_q^n\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-1)^m q^{d(h-r)m}$$
$$\times\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}q^{\sum_{j=1}^{r}(h-j)a_j}\left[m+\frac{x+a_1+\cdots+a_r}{d}\right]_{q^d}^n. \tag{2.8}$$

That is,

$$F_{q,\chi}^{(h,r)}(t,x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{d(h-r)m} \times \sum_{a_1,\dots,a_r=0}^{d-1} \left(\prod_{i=1}^{r} \chi(a_i)\right) (-1)^{a_1+\cdots+a_r} q^{\sum_{j=1}^{r}(h-j)a_j} e^{[x+a_1+\cdots+a_r+dm]_q t}. \tag{2.9}$$

Let $h = r$. Then we have

$$\begin{aligned} E_{n,\chi,q}^{(r,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{a_1,\dots,a_r=0}^{d-1} \left(\prod_{i=1}^{r} \chi(a_i)\right) (-1)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r} a_j (r-j)} \times \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{l(x+\sum_{j=1}^{r} a_j)}}{(-q^{ld} : q^d)_r} \\ &= 2^r [d]_q^n \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m \\ &\quad \times \sum_{a_1,\dots,a_r=0}^{d-1} \left(\prod_{i=1}^{r} \chi(a_i)\right) (-1)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r}(r-j)a_j} \left[m + \frac{x+a_1+\cdots+a_r}{d}\right]_{q^d}^n. \end{aligned} \tag{2.10}$$

By (2.3), (2.9), and (2.10), we obtain the following equations:

$$\begin{gathered} \frac{2^r q^{mx} \sum_{a_1,\dots,a_r=0}^{d-1} \left(\prod_{i=1}^{r} \chi(a_i)\right) q^{\sum_{j=1}^{r}(m-j)a_j} (-1)^{\sum_{j=1}^{r} a_j}}{(-q^{d(m-r)} : q^d)_r} = \sum_{l=0}^{m} \binom{m}{l} (q-1)^l E_{l,\chi,q}^{(0,r)}(x), \\ q^{d(h-1)} E_{n,\chi,q}^{(h,r)}(x+d) + E_{n,\chi,q}^{(h,r)}(x) = 2 \sum_{l=0}^{d-1} \chi(l)(-1)^l E_{n,q}^{(h-1,r-1)}(x). \end{gathered} \tag{2.11}$$

In the special case $r = 1$, we note that

$$F_{q,\chi}^{(h,1)}(t,x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,1)}(x) \frac{t^n}{n!}. \tag{2.12}$$

By (2.12), we see that

$$F_{q,\chi}^{(h,1)}(t,x) = 2 \sum_{n=0}^{\infty} \chi(n) q^{(h-1)n} (-1)^n e^{[n+x]_q t}. \tag{2.13}$$

Hence

$$\begin{aligned} E_{n,\chi,q}^{(h,1)}(x) &= 2 \sum_{m=0}^{\infty} \chi(m) q^{(h-1)m} (-1)^m [m+x]_q^n \\ &= \frac{2}{(1-q)^n} \sum_{a=0}^{d-1} \chi(a)(-1)^a \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{l(x+a)}}{1+q^{ld}}. \end{aligned} \tag{2.14}$$

For $s \in \mathbb{R}$ and $x \in \mathbb{C}$ with $\Re(x) > 0$, we have

$$\frac{1}{\Gamma(s)} \int_0^\infty F_{q,\chi}^{(r)}(-t, x) t^{s-1} dt = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right)}{[m_1+\cdots+m_r+x]_q^s}. \tag{2.15}$$

By (2.15), we can define the following q-l-function of order r.

Definition 2.1. For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $\Re(x) > 0$, we define the q-l-function as

$$l_q^{(h,r)}(s, x \mid \chi) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right)}{[m_1+\cdots+m_r+x]_q^s}. \tag{2.16}$$

Note that $l_q^{(h,r)}(s, x \mid \chi)$ is analytic in whole complex s-plane. By (2.7), (2.15), and the Cauchy residue theorem, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$l_q^{(h,r)}(-n, x \mid \chi) = E_{n,\chi,q}^{(h,r)}(x). \tag{2.17}$$

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