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Research Article

Optimal Interior Partial Regularity for Nonlinear Elliptic Systems for the Case $1 < m < 2$ under Natural Growth Condition

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We consider the interior regularity for weak solutions of second-order nonlinear elliptic systems with subquadratic growth under natural growth condition. We obtain a general criterion for a weak solution to be regular in the neighborhood of a given point. In particularly the regularity we obtained is optimal.

1. Introduction

In this paper we consider optimal interior partial regularity for the weak solutions of nonlinear elliptic systems with subquadratic growth under natural growth condition of the following type:

$$-\sum_{\alpha=1}^{n} D_\alpha A_i^\alpha(x,u,Du) = B_i(x,u,Du), \quad i = 1,\dots,N \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in R^n, u and B_i taking values in R^N, and $A_i^\alpha(\cdot,\cdot,\cdot)$ has value in R^{nN}. $N > 1$, $u : \Omega \mapsto R^N Du = \{D_\alpha u^i\}$, $1 \le \alpha \le n$, $1 \le i \le N$ stand for the div of u and $1 < m < 2$. To define weak solution to (1.1), one needs to impose certain structural and regularity conditions on A_i^α and the inhomogeneity B_i, as well as to restrict u to a particular class of functions as follows, for $1 < m < 2$,

(E1) $A_i^\alpha(x,u,p)$ are differentiable functions in p and there exists $L > 0$ such that

$$\left|\frac{\partial A_i^\alpha(x,u,p)}{\partial p_\beta^j}\right| \le L\left(1+|p|^2\right)^{(m-2)/2}, \quad \forall (x,u,p) \in \Omega \times R^N \times R^{nN}, \tag{1.2}$$

(E2) A_i^α is uniformly strongly elliptic, that is, for some $\lambda > 0$, we have

$$\frac{\partial A_i^\alpha(x,u,p)}{\partial p_\beta^j}\xi_\alpha^i \xi_\beta^j \geq \lambda\left(1+|p|^2\right)^{(m-2)/2}|\xi|^2, \quad \forall x \in \Omega,\ u \in R^N,\ p,\xi \in R^{nN}, \tag{1.3}$$

(E3) There exists $\beta \in (0,1)$ and $K : [0,\infty) \to [0,\infty)$ monotone nondecreasing such that

$$\left|A_i^\alpha(x,\xi,p) - A_i^\alpha\left(\tilde{x},\tilde{\xi},p\right)\right| \leq K(|\xi|)\left(|x-\tilde{x}|^m + \left|\xi-\tilde{\xi}\right|^m\right)^{\beta/m}(1+|p|)^{m/2} \tag{1.4}$$

for all $x,\tilde{x} \in \Omega$, $\xi,\tilde{\xi} \in R^N$, and $p \in R^{nN}$; without loss of generality, we take $K \geq 1$.

Furthermore (E1) allows us to deduce the existence of a function $\omega(t,s) : [0,\infty) \times [0,\infty) \mapsto [0,\infty)$ with $\omega(t,0) = 0$ for all t such that $t \mapsto \omega(t,s)$ is monotone nondecreasing for fixed s, $s \mapsto \omega(t,s)$ is concave and monotone nondecreasing for fixed t, and such that for all $(x,u) \in \Omega \times R^N$ and $p,q \in R^{nN}$, we have

$$\left|A^\alpha_{ip_\beta^j}(x,u,p) - A^\alpha_{ip_\beta^j}(x,u,q)\right| \leq C\left(1+|p|^2+|q|^2\right)^{(m-2)/2}\omega(|p|,|p-q|), \tag{1.5}$$

(E4) there exist constants a and b, such that

$$|B_i(x,u,p)| \leq a|p|^m + b, \tag{1.6}$$

or

(E4′)

$$|B_i(x,u,p)| \leq C\left(|p|^{m-\varepsilon} + b\right), \quad \varepsilon > 0. \tag{1.7}$$

Definition 1.1. By a weak solution of (1.1) with structure assumptions (E1)–(E4) (or (E4′)), we mean a vector valued function $u \in W^{1,m}(\Omega,R^N) \cap L^\infty(\Omega,R^N)$ such that

$$\int_\Omega A_i^\alpha(x,u,Du)D_\alpha\varphi^i dx = \int_\Omega B_i(x,u,Du)\varphi^i dx, \tag{1.8}$$

for all $\varphi \in C_0^\infty(\Omega,R^N)$.

Even under reasonable assumptions on A_i^α and B_i, in the case of systems (i.e., $N > 1$) one cannot, in general, expect that weak solutions of (1.1) will be classical, that is, C^2-solutions. This was first shown by De Giorgi [1, 2]. The goal, then, is to establish partial regularity theory. We refer the reader to monographs of Giaquinta [3, 4] for an extensive treatment of partial regularity theory for systems of the form (1.1), as well as more general elliptic systems.

In the class direct proofs, one "freezes the coefficients" with constant coefficients. The solution of the Dirichlet problem associated to these coefficients with boundary data u and the solution itself can then be compared. This procedure was first carried out by Giaquinta and Modica [5].

But the technique of harmonic approximation is to show that a function which is "approximately-harmonic" lies L^2close to some harmonic function. This technique has its origins in Simon's proof [6] of the regularity theorem of Allard [7]. Which also be used in [8] to find a so-called ε-regularity theorem for energy minimizing harmonic maps. The technique of harmonic approximation allows the author to simplify the original ε-regularity theorem due to Schoen and Uhlenbeck [9].

In the remarkable proof when $m \equiv 2$ given by Duzaar and Grotowski in [10], the key difference is that the solution is compared not to the solution of the Dirichlet problem for the system with frozen coefficients, but rather to an A-harmonic function which is close to w in L^2, where w is a function corresponding from weak solutions. In particular, the optimal regularity result can be obtained. In [11, 12], we deal with the optimal partial regularity of the weak solution to (1.1) for the case $m > 2$ by the method of A-harmonic approximation technique, which is advantage to the result of [13]. The extension of A-harmonic approximation technique also can be found in [14, 15].

The purpose of this paper is to establish the optimal partial regularity of weak solution to (1.1) under natural growth condition with subquadratic growth, that is, the case of $1 < m < 2$, directly. Indeed the main difficulty in our setting is that the exponent of the integral function is negative ($-1/2 < (m-2)/2 < 0$), which means we cannot use the amplify technique as usually. Motivated by the technique used in [16], where the authors considered the minimizers of nonquadratic functional, we removed the hinder at last. And then with the help of $\mathcal{A}$-harmonic approximation technique, one can find a $(\partial A_i^\alpha/\partial p_\beta^j)(x_0, u_{x_0,\rho}, (Du)_{x_0,\rho})$-harmonic function, which is close to a function w in sense of L^2, the function w is which we defined in Lemma 4.2 and which is a corresponding function from the weak solution u. Thanks to the standard results of linear theory presented in Section 2 and the elementary inequalities, we obtain the decay estimate of

$$\Phi(x_0,\rho,p_0) = ⨍_{B_\rho(x_0)} |V(Du) - V(p_0)|^2 dx \tag{1.9}$$

and the optimal regularity. Now we may state the main result.

Theorem 1.2. *Let* $u \in W^{1,m}(\Omega, R^N) \cap L^\infty(\Omega, R^N)$ $(m \in (1,2))$ *be a weak solution of* (1.1) *with* $\sup_\Omega|u| = M$. *Suppose that the natural growth conditions (E1)–(E4) (or (E4′)) and* $2aM < \lambda$ *hold. Then there exists* Ω_0 *that is open in* Ω *and* $u \in C^{1,\beta}(\Omega_0, R^N)$ *for* β *is defined in (E3). Furthermore,*

$$\Omega \setminus \Omega_0 = \Sigma_1 \bigcup \Sigma_2, \tag{1.10}$$

where

$$\begin{aligned} \Sigma_1 &= \left\{ x_0 \in \Omega : \liminf_{\rho\to 0^+} ⨍_{B_\rho(x_0)} \left|Du - (Du)_{x_0,\rho}\right|^m dx > 0 \right\}, \\ \Sigma_2 &= \left\{ x_0 \in \Omega : \limsup_{\rho\to 0^+} \left(\left|(Du)_{x_0,\rho}\right|\right) = \infty \right\}. \end{aligned} \tag{1.11}$$

In particular, $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$.

2. The $\mathscr{A}$-Harmonic Approximation Technique and Preliminary Lemmas

In this section, we present the $\mathscr{A}$-harmonic approximation lemma, the key ingredient in proving our regularity result, and some useful preliminaries will be need in later. At first, we introduce two new functions.

Throughout the paper we will use the functions $V = V_p : R^n \to R^n$ and $W = W_p : R^n \to R^n$ defined by

$$V(\xi) = \frac{\xi}{\left(1+|\xi|^2\right)^{(2-m)/4}}, \qquad W(\xi) = \frac{\xi}{\sqrt{1+|\xi|^{2-m}}}, \tag{2.1}$$

for each $\xi \in R^n$ and for any $m > 1$. From the elementary inequality

$$\|x\|_{2/(2-m)} \le \|x\|_1 \le 2^{1-((2-m)/2)}\|x\|_{2/(2-m)}, \tag{2.2}$$

applied to the vector $x = (1, |\xi|^{2-m}) \in R^2$ we deduce that

$$\left(1+|\xi|^2\right)^{(2-m)/2} \le 1+|\xi|^{2-m} \le 2^{m/2}\left(1+|\xi|^2\right)^{(2-m)/2}, \tag{2.3}$$

which immediately yields

$$|W(\xi)| \le |V(\xi)| \le C(m)|W(\xi)|. \tag{2.4}$$

The purpose of introducing W is the fact that in contrast to $|V|^{2/m}$, the function $|W|^{2/m}$ is a convex function on R^k. This can easily be shown as follows. Firstly a direct computation yields that $t \to W^{2/m}(t) = t^{2/m}(1+t^{2-m})^{-1/m}$ is convex and monotone increasing on $[0,\infty)$ with $W^{2/m}(0) = 0$. Secondly we have

$$\begin{aligned} \left|W\left(\frac{\xi+\eta}{2}\right)\right|^{2/m} &= W\left(\frac{|\xi+\eta|}{2}\right)^{2/m} \\ &\le W\left(\frac{|\xi|+|\eta|}{2}\right)^{2/m} \le \frac{W(|\xi|)^{2/m}+W(|\eta|)^{2/m}}{2} = \frac{|W(\xi)|^{2/m}+|W(\eta)|^{2/m}}{2}, \end{aligned} \tag{2.5}$$

for any $\xi, \eta \in R^n$.

We use a number of properties of $V = V_p$ which can be found in [17, Lemma 2.1].

Lemma 2.1. *Let $m \in (1,2)$ and $V, W : R^n \to R^n$ be the functions defined in* (2.1)*. Then for any $\xi, \eta \in R^n$ and $t > 0$ there holds:*

(i) $1/\sqrt{2}\min(|\xi|, |\xi|^{m/2}) \le |V(\xi)| \le \min(|\xi|, |\xi|^{m/2})$;

(ii) $|V(t\xi)| \le \max(t, t^{m/2})|V(\xi)|$;

(iii) $|V(\xi+\eta)| \leq c(m)(|V(\xi)|+|V(\eta)|)$;

(iv) $(m/2)|\xi-\eta| \leq |V(\xi)-V(\eta)|/(1+|\xi|^2+|\eta|^2)^{(m-2)/4} \leq C(k,m)|\xi-\eta|$;

(v) $|V(\xi)-V(\eta)| \leq c(k,m)|V(\xi-\eta)|$;

(vi) $|V(\xi-\eta)| \leq C(m,M)|V(\xi)-V(\eta)|$,*for all* η *with* $|\eta| \leq M$.

The inequalities (i)–(iii) also hold if we replace V by W.

For later purposes we state the following two simple estimates which can easily be deduced from Lemma 2.1(i) and (vi). For $\xi, \eta \in R^n$ with $|\eta| \leq M$ we have for $|\xi-\eta| \leq 1$ the estimate

$$|\xi-\eta|^2 \leq C(m,M)|V(\xi)-V(\eta)|^2; \tag{2.6}$$

as for $|\xi-\eta| > 1$ we have

$$|\xi-\eta|^m \leq C(m,M)|V(\xi)-V(\eta)|^2. \tag{2.7}$$

The next result we would state is the $\mathcal{A}$-harmonic approximation lemma, which is prove in [18].

Lemma 2.2 ($\mathcal{A}$-harmonic approximation lemma). *Let κ, K be positive constants. Then for any $\varepsilon > 0$ there exist $\delta = \delta(n,N,\kappa,K,\varepsilon) \in (0,1]$ with the following property. For any bilinear form $\mathcal{A}$ on R^{nN} which is elliptic in the sense of Legendre-Hadamard with ellipticity constant κ and upper bound K, for any $v \in W^{1,m}(B_\rho(x_0), R^N)$ satisfying*

$$\begin{gathered}
⨍_{B_\rho(x_0)} |W(Dv)|^2 dx \leq \gamma^2 \leq 1, \\
⨍_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi)dx \leq \gamma\delta \sup_{B_\rho(x_0)} |D\varphi|,
\end{gathered} \tag{2.8}$$

for all $\varphi \in C_0^1(B_\rho(x_0), R^N)$, there exists an $\mathcal{A}$-harmonic function h satisfying

$$⨍_{B_\rho(x_0)} |W(Dh)|^2 dx \leq 1, \qquad ⨍_{B_\rho(x_0)} \left|W\left(\frac{v-\gamma h}{\rho}\right)\right|^2 dx \leq \gamma^2\varepsilon. \tag{2.9}$$

Definition 2.3. Here a function h is called $\mathcal{A}$-harmonic if it satisfies

$$\int_{B_\rho(x_0)} \mathcal{A}(Dh, D\varphi)dx = 0, \tag{2.10}$$

for all $\varphi \in C_0^1(B_\rho, R^N)$.

Then we would recall a simple consequence of the a prior estimates for solutions of linear elliptic systems of second order with constant coefficients; see [17, Proposition 2.10] for a similar result.

Lemma 2.4. *Let $h \in W^{1,1}(B_\rho(x_0), R^N)$ be such that*

$$\int_{B_\rho(x_0)} \mathcal{A}(Dh, D\varphi)\,dx = 0, \tag{2.11}$$

for any $\varphi \in C_0^1(B_\rho(x_0), R^N)$, where $\mathcal{A} \in R^{nN}$ is elliptic in the sense of Legendre-Hadamard with ellipticity constant κ and upper bound K. Then $h \in C^\infty(B_\rho(x_0), R^N)$ and

$$\rho \sup_{B_{\rho/2}(x_0)} \left|D^2 h\right| + \sup_{B_{\rho/2}(x_0)} |Dh| \le C_a \fint_{B_\rho} |Dh|\,dx, \tag{2.12}$$

where the constant C_a depends only on n, N, κ, and K.

The next lemma is a more general version of [17, Lemma 2.7], which itself is an extension of [3, Lemma 3.1, Chapter V]. The proof in which can easily be adapted to the present situation by replacing the condition of homogeneity by Lemma 2.1(ii).

Lemma 2.5. *Let $0 \le \nu < 1$, $a, b \ge 0$, $v \in L^p(B_\rho(x_0))$, and g be a nonnegative bounded function satisfying*

$$g(t) \le \nu g(s) + a \int_{B_\rho(x_0)} \left|V\left(\frac{v}{s-t}\right)\right|^2 dx + b, \tag{2.13}$$

for all $\rho/2 \le t < s \le \rho$. Then there exists a constant $C = C(\nu)$ such that

$$g\left(\frac{\rho}{2}\right) \le C(\nu)\left(a \int_{B_\rho(x_0)} \left|V\left(\frac{v}{\rho}\right)\right|^2 dx + b\right). \tag{2.14}$$

And then we state a Poincare type inequality involving the function V, which have been found in [17] and, in a sharp way, in [18].

Lemma 2.6 (Poincare-type inequality). *Let $m \in (1,2)$ and $u \in W^{1,m}(B_\rho, R^N)$, $B_\rho \subset \Omega$, then*

$$\left(\fint_{B_\rho} \left|V\left(\frac{u - u_{x_0,\rho}}{\rho}\right)\right|^{m'} dx\right)^{1/m'} \le C_P(n, N, m)\left(\fint_{B_\rho} |V(Du)|^2 dx\right)^{1/2}, \tag{2.15}$$

where $m' = 2n/(n-m)$. In particular, the previous inequality is valid with m' replaced by 2.

We conclude the section with an algebraic fact can be retrieved again from [16], Lemma 2.1.

Lemma 2.7. *For every $t \in (-1/2, 0)$ and $\mu \geq 0$, one has*

$$1 \leq \frac{\int_0^1 \left(\mu^2 + \left|A + s\left(\widetilde{A} - A\right)\right|^2\right)^t ds}{\left(\mu^2 + |A|^2 + \left|\widetilde{A}\right|^2\right)^t} \leq \frac{8}{2t+1}, \tag{2.16}$$

for any $A, \widetilde{A} \in R^{nN}$, not both zero if $\mu = 0$.

3. A Caccioppoli Second Inequality

For $x_0 \in \Omega,\ u_0 \in R^N,\ p_0 \in R^{nN}$, we define $P = \{p_i(x)\},\ i = 1, \cdots, N,\ p_i = u_0^i + p_{0\alpha}^i(x_\alpha - x_{0\alpha})$ and we simply write $P = u_0 + p_0(x - x_0)$.

In order to prove the main result, our first aim is to establish a suitable Caccioppoli inequality.

Lemma 3.1 (Caccioppoli second inequality). *Let $u \in W^{1,m}(\Omega, R^N) \cap L^\infty(\Omega, R^N)$ $(1 < m < 2)$ be a weak solution of (1.1) with $\sup_\Omega |u| = M$ and $2aM < \lambda$ hold under natural growth conditions (E1)–(E4) (or (E4′)). Then for every $x_0 \in \Omega,\ u_0 \in R^N,\ p_0 \in R^{nN}$, and arbitrary ρ with $0 < \rho < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$, one has*

$$⨍_{B_{\rho/2(x_0)}} |V(Du - p_0)|^2 dx \leq C_c \left[⨍_{B_\rho(x_0)} \left| V\left(\frac{u - u_0 - p_0(x - x_0)}{\rho}\right)\right|^2 dx + G \right], \tag{3.1}$$

for

$$G = \left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^\sigma \rho^{2\beta} + \max\left\{ \left[a|p_0|^m + b\right]^2,\ \left[a|p_0|^m + b\right]^{m/(m-1)} \right\}\rho^2. \tag{3.2}$$

where $\sigma = \max\{2m/(m - 2\beta),\ (m + 2\beta)/(m-1)\} > 2$ and the constant $C_c = C_c(n, N, m, L, \lambda, M)$.

Proof. Let $B_\rho(x_0) \subset \Omega$. Choose $\rho/2 \leq t < s \leq \rho$ and a standard cut off function $\eta \in C_0^1(B_\rho(x_0), [0,1])$ with $\eta \equiv 1$ on $B_t(x_0)$, which satisfies $|\nabla\eta| \leq 1/(s-t)$. For $u_0 \in R^N$ and $p_0 \in R^{nN}$, let

$$v = u - u_0 - p_0(x - x_0) \tag{3.3}$$

and define

$$\varphi = \eta v, \qquad \psi = (1 - \eta)v. \tag{3.4}$$

Then

$$D\varphi + D\psi = Dv = Du - p_0, \tag{3.5}$$

and further there holds

$$|D\varphi|^m \le C(m)\left(|Dv|^m + \left|\frac{v}{s-t}\right|^m\right); \qquad |D\psi|^m \le C(m)\left(|Dv|^m + \left|\frac{v}{s-t}\right|^m\right). \tag{3.6}$$

Using hypothesis (E2), from Lemma 2.7, and as the elementary inequality

$$\frac{1}{3}\left(1+|b|^2+|a|^2\right) \le 1+|a|^2+|b-a|^2 \le 3\left(1+|a|^2+|b|^2\right), \tag{3.7}$$

we can get

$$\begin{aligned}
&\int_{B_s(x_0)} \left[A_i^\alpha(x,u,p_0+D\varphi) - A_i^\alpha(x,u,p_0)\right] D_\alpha\varphi^i dx \\
&\quad = \int_{B_s(x_0)} \int_0^1 \frac{\partial A_i^\alpha(x,u,p_0+\theta D\varphi)}{\partial P_\beta^j} d\theta D_\beta\varphi^j D_\alpha\varphi^i dx \\
&\quad \ge \lambda \int_{B_s(x_0)} \int_0^1 \left(1+|p_0+\theta D\varphi|^2\right)^{(m-2)/2} d\theta |D\varphi|^2 dx \\
&\quad \ge 3^{(m-2)/2}\lambda \int_{B_s(x_0)} \left(1+|p_0|^2+|D\varphi|^2\right)^{(m-2)/2} |D\varphi|^2 dx,
\end{aligned} \tag{3.8}$$

A simple calculation yields

$$\begin{aligned}
&3^{(m-2)/2}\lambda \int_{B_s(x_0)} \left(1+|p_0|^2+|D\varphi|^2\right)^{(m-2)/2} |D\varphi|^2 dx \\
&\quad \le -\int_{B_s(x_0)} \int_0^1 \frac{\partial A_i^\alpha(x,u,Du-\theta D\psi)}{\partial P_\beta^j} d\theta D_\beta\psi^j D_\alpha\varphi^i dx \\
&\qquad - \int_{B_s(x_0)} \left(A_i^\alpha(x,u,p_0) - A_i^\alpha(x,P,p_0)\right) D_\alpha\varphi^i dx \\
&\qquad - \int_{B_s(x_0)} \left(A_i^\alpha(x,P,p_0) - A_i^\alpha(x_0,u_0,p_0)\right) D_\alpha\varphi^i dx \\
&\qquad + \int_{B_s(x_0)} B_i(x,u,Du)\varphi^i dx \\
&\quad \le I + II + III + IV.
\end{aligned} \tag{3.9}$$

By (E1), Lemma 2.7 and (3.7), there holds

$$I \le C \int_{B_s(x_0)} \left(1 + |Du|^2 + \left|Du - D\varphi\right|^2\right)^{(m-2)/2} \left|D\psi\right|\left|D\varphi\right| dx. \tag{3.10}$$

Noting that supp$D\psi \subset B_t \setminus B_s$ and $-1/2 < (m-2)/2 < 0$, one can take the domain $B_s(x_0)$ into $B_s(x_0) \cap \{|D\psi| > 1\} \cap \{|D\varphi| > 1\}$, $B_s(x_0) \cap \{|D\psi| > 1\} \cap \{|D\varphi| \le 1\}$, $B_s(x_0) \cap \{|D\psi| \le 1\} \cap \{|D\varphi| > 1\}$, and $B_s(x_0) \cap \{|D\psi| \le 1\} \cap \{|D\varphi| \le 1\}$, four parts, and then by Young inequality and the estimations (2.6) and (2.7), thus there is

$$I \le C_1 \left(\int_{B_s(x_0) \setminus B_t(x_0)} \left|V(Du - p_0)\right|^2 dx + \int_{B_s(x_0)} \left|V\left(\frac{v}{\rho}\right)\right|^2 dx \right). \tag{3.11}$$

From the structure condition (E3) yields

$$II \le \int_{B_s(x_0)} K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} |v|^{\beta} \left|D\varphi\right| dx. \tag{3.12}$$

Similar to I, we split the domain of integration into four parts as follows. And on the part $B_s(x_0) \cap \{|v/s| > 1\} \cap \{|D\varphi| \le 1\}$, we see

$$\begin{aligned} &K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} |v|^{\beta} \left|D\varphi\right| \\ &\quad \le \varepsilon \left|D\varphi\right|^2 + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} |v|^{\beta}\right]^2 \\ &\quad \le \varepsilon \left|D\varphi\right|^2 + C(\varepsilon)\left|\frac{v}{s}\right|^m + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} s^{\beta}\right]^{2m/(m-2\beta)} \\ &\quad \le \varepsilon C |V(Dv)|^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2 + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{2m/(m-2\beta)} s^{2\beta}, \end{aligned} \tag{3.13}$$

as on the set $B_s(x_0) \cap \{|v/s| \le 1\} \cap \{|D\varphi| > 1\}$, there are

$$\begin{aligned} &K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} |v|^{\beta} \left|D\varphi\right| \\ &\quad \le K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} s^{\beta} \left|D\varphi\right| \\ &\quad \le \varepsilon \left|D\varphi\right|^m + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} s^{\beta}\right]^{m/(m-1)} \\ &\quad \le \varepsilon C |V(Dv)|^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2 + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{m/(m-1)} s^{2\beta}, \end{aligned} \tag{3.14}$$

and on the case $B_s(x_0) \cap \{|v/s| \le 1\} \cap \{|D\varphi| \le 1\}$, one can get

$$\begin{aligned} &K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}|v|^{\beta}|D\varphi| \\ &\quad \le K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}s^{\beta}|D\varphi| \\ &\quad \le \varepsilon|D\varphi|^2 + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}s^{\beta}\right]^2 \\ &\quad \le \varepsilon C|V(Dv)|^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2 + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^2 s^{2\beta}, \end{aligned} \tag{3.15}$$

Finally, noting that $\sup_{\Omega}|u| = M$, then for the case $B_s(x_0) \cap \{|v/s| > 1\} \cap \{|D\varphi| > 1\}$, there exists a constant $0 < 2(m-1)/m + 2\beta < 1$ such that

$$\begin{aligned} &K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}|v|^{\beta}|D\varphi| \\ &\quad \le K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\left(|v|^{\beta}\right)^{2(m-1)/(m+2\beta)}(2M + p_0 s)^{(2-m+2\beta)/(m+2\beta)}|D\varphi| \\ &\quad \le \varepsilon|D\varphi|^m + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{m/(m-1)} \\ &\qquad \times |v|^{2m\beta/(m+2\beta)}(2M + p_0 s)^{(2-m+2\beta)/(m+2\beta)\cdot(m/(m-1))} \le \varepsilon|D\varphi|^2 + C(\varepsilon)\left|\frac{v}{s}\right|^m \\ &\qquad + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{(m+2\beta)/(m-1)} s^{2\beta}(2M + p_0 s)^{(2-m+2\beta)/(m-1)} \\ &\quad \le \varepsilon|V(Dv)|^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2 \\ &\qquad + C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{(m+2\beta)/(m-1)}(2M + p_0 s)^{(2-m+2\beta)/(m-1)} s^{2\beta}. \end{aligned} \tag{3.16}$$

Combining these estimations on II, we have

$$\begin{aligned} II \le{}& C\varepsilon\int_{B_s(x_0)}|V(Dv)|^2dx + C(\varepsilon)\int_{B_s(x_0)}\left|V\left(\frac{v}{s}\right)\right|^2dx \\ &+ C(\varepsilon)\left[1 + (2M + p_0 s)^{(2-m+2\beta)/(m-1)}\right]\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2}\right]^{\sigma}\alpha_n s^{n+2\beta}, \end{aligned} \tag{3.17}$$

for $\sigma = \max\{2, m/(m-1),\ (m+2\beta)/(m-1),\ 2m/(m-2\beta)\} = \max\{(m+2\beta)/(m-1),\ 2m/(m-2\beta)\}$.

And noting that $K \ge 1$, and that $m/(m-1) \ge 2$, and similarly as II, we see

$$\begin{aligned} III \le{}& C\varepsilon\int_{B_s(x_0)}|V(Dv)|^2dx + C(\varepsilon)\int_{B_s(x_0)}\left|V\left(\frac{v}{s}\right)\right|^2dx \\ &+ C(\varepsilon)\left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2+\beta}\right]^{m/(m-1)}\alpha_n s^{n+2\beta}, \end{aligned} \tag{3.18}$$

and for μ positive to be fixed later, we have

$$IV = \int_{B_s(x_0)} a|Du|^m \left|u - u_0 - p_0(x - x_0)\right| \eta dx + \int_{B_s(x_0)} \left|\frac{v}{s}\right| (bs\eta) dx. \tag{3.19}$$

On the part $\{B_s(x_0)\} \cap \{|Du - p_0| \geq 1\} \cap \{|v/s| \leq 1\}$, argue anginous as II and III, by Young's inequality and (2.6) and (2.7), we have

$$\begin{aligned}
& a|Du|^m \left|u - u_0 - p_0(x - x_0)\right| \eta + \left|\frac{v}{s}\right| (bs\eta) \\
&\quad \leq a\left[(1+\mu)\left|Du - p_0\right|^m + \left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m\right] \left|u - u_0 - p_0(x - x_0)\right| \eta \\
&\qquad + \varepsilon b^2 s^2 \eta^2 + C(\varepsilon)\left|\frac{v}{s}\right|^2 \\
&\quad \leq a(1+\mu)(2M + p_0 s)|V(Dv)|^2 + a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m |v| \eta + \varepsilon b^2 s^2 \eta^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2 \\
&\quad \leq a(1+\mu)(2M + p_0 s)|V(Dv)|^2 + \varepsilon a^2 \left(1 + \frac{1}{\mu}\right)^2 \left|p_0\right|^{2m} s^2 \eta^2 + \varepsilon b^2 s^2 \eta^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2.
\end{aligned} \tag{3.20}$$

Similarly, on the part $\{B_s(x_0)\} \cap \{|Du - p_0| \geq 1\} \cap \{|v/s| \leq 1\}$, we see

$$\begin{aligned}
& a|Du|^m \left|u - u_0 - p_0(x - x_0)\right| \eta + \left|\frac{v}{s}\right| (bs\eta) \\
&\quad \leq a(1+\mu)(2M + p_0 s)|V(Dv)|^2 + \varepsilon s^{m/(m-1)} \eta^{m/(m-1)} \\
&\qquad \times \left[b^{m/(m-1)} + a^{m/(m-1)} \left(1 + \frac{1}{\mu}\right)^{m/(m-1)} \left|p_0\right|^{m^2/(m-1)}\right] \\
&\qquad + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2,
\end{aligned} \tag{3.21}$$

and on the part $\{B_s(x_0)\} \cap \{|Du - p_0| \geq 1\} \cap \{|v/s| \leq 1\}$,

$$\begin{aligned}
& a|Du|^m \left|u - u_0 - p_0(x - x_0)\right| \eta + \left|\frac{v}{s}\right| (bs\eta) \\
&\quad \leq a(1+\mu)\left|Du - p_0\right|^m |v| \eta + \left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right] |v| \eta
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon s^2 \left|Du - p_0\right|^2 + C(\varepsilon)(2M + p_0 s)^{2(m-1)/(2-m)} (a(1+\mu))^{2/(2-m)} \left|\frac{v}{s}\right|^2 \\
&\quad + \varepsilon\left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^2 s^2\eta^2 + C(\varepsilon)\left|\frac{v}{s}\right|^2 \\
&\leq \varepsilon|V(Dv)|^2 + C(\varepsilon)(2M + p_0 s)^{2(m-1)/(2-m)} (a(1+\mu))^{2/(2-m)} \left|V\left(\frac{v}{s}\right)\right|^2 \\
&\quad + \varepsilon\left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^2 s^2\eta^2 + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2,
\end{aligned} \tag{3.22}$$

and on the part $\{B_s(x_0)\} \cap \{|Du - p_0| \geq 1\} \cap \{|v/s| \geq 1\}$,

$$\begin{aligned}
&a|Du|^m \left|u - u_0 - p_0(x - x_0)\right|\eta + \left|\frac{v}{s}\right|(bs\eta) \\
&\quad \leq \varepsilon s^2 \left|Du - p_0\right|^2 + C(\varepsilon)(2M + p_0 s)^{(2-2m+m^2)/(2-m)} (a(1+\mu))^{2/(2-m)} \left|\frac{v}{s}\right|^m \\
&\qquad + \varepsilon\left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^{m/(m-1)} s^{m/(m-1)}\eta^{m/(m-1)} + C(\varepsilon)\left|\frac{v}{s}\right|^m \\
&\quad \leq \varepsilon|V(Dv)|^2 + C(\varepsilon)(2M + p_0 s)^{(2-2m+m^2)/(2-m)} (a(1+\mu))^{2/(2-m)} \left|V\left(\frac{v}{s}\right)\right|^2 \\
&\qquad + \varepsilon\left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^{m/(m-1)} s^{m/(m-1)}\eta^{m/(m-1)} + C(\varepsilon)\left|V\left(\frac{v}{s}\right)\right|^2.
\end{aligned} \tag{3.23}$$

Combining these estimates in IV, and noting that $m/(m-1) > 2$ and $s \leq 1$, $\eta \leq 1$, we have

$$\begin{aligned}
IV &\leq C(\varepsilon, M, a)\int_{B_s(x_0)} |V(Dv)|^2 dx + C(\varepsilon, M, m, a)\int_{B_s(x_0)} \left|V\left(\frac{v}{s}\right)\right|^2 dx \\
&\quad + \max\left\{\left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^2, \left[a\left(1 + \frac{1}{\mu}\right)\left|p_0\right|^m + b\right]^{m/(m-1)}\right\}\alpha_n s^{n+2}.
\end{aligned} \tag{3.24}$$

Finally, on $B_t(x_0)$ we use Lemma 2.1(iv) and (vi) to bound the integrand of the left-hand side of (3.9) from below:

$$\begin{aligned}
\lambda\left(1 + \left|p_0\right|^2 + \left|D\varphi\right|^2\right)^{(m-2)/2}\left|D\varphi\right|^2 &= \lambda\left(1 + \left|p_0\right|^2 + |Dv|^2\right)^{(m-2)/2}|Dv|^2 \\
&\geq C(m)\lambda\left(1 + \left|p_0\right|^2 + |Du|^2\right)^{(m-2)/2}\left|Du - p_0\right|^2 \\
&\geq C(n, N, m, \lambda)\left|V(Du) - V(p_0)\right|^2 \\
&\geq C(n, N, m, \lambda, M)|V(Dv)|^2.
\end{aligned} \tag{3.25}$$

Using this in (3.9) together with the estimates I, II, III, and IV we finally arrive at

$$
\begin{aligned}
C_1 \int_{B_t(x_0)} |V(Dv)|^2 dx \le{} & C_2 \int_{B_s(x_0)\setminus B_t(x_0)} \left(|V(Dv)|^2 + \left| V\left(\frac{v}{s-t}\right)\right|^2 \right) dx \\
& + C_3 \int_{B_s(x_0)} \left(|V(Dv)|^2 + \left| V\left(\frac{v}{s-t}\right)\right|^2 \right) dx + C_4 \left[K(|u_0| + |p_0|)(1 + |p_0|)^{m/2} \right]^{\sigma} \alpha_n s^{n+2\beta} \\
& + C_5 \max\left\{ \left[a\left(1 + \frac{1}{\mu}\right)|p_0|^m + b \right]^2, \left[a\left(1 + \frac{1}{\mu}\right)|p_0|^m + b \right]^{m/(m-1)} \right\} \alpha_n s^{n+2}.
\end{aligned}
\tag{3.26}
$$

The proof is now completed by applying Lemma 2.5. □

4. The Proof of the Main Theorem

In this section we proceed to the proof of the partial regularity result and hence consider $u \in W^{1,m}(\Omega, R^N) \cap L^{\infty}(\Omega, R^N)$ $(1 < m < 2)$ to be a weak solution of (1.1). Then we have the following.

Lemma 4.1. *Consider $\rho < 1$ and $\varphi \in C_0^{\infty}(B_\rho(x_0), R^N)$ with $\sup_{B_\rho(x_0)}|D\varphi| \le 1$. Furthermore fixed p_0 in R^{nN} and set $u_0 = u_{x_0,\rho} = ⨍_{B_\rho(x_0)} u\,dx$ and $\Phi(x_0, \rho, p_0) \le 1$. Then for the weak solution $u \in W^{1,m}(\Omega, R^N) \cap L^{\infty}(\Omega, R^N)(1 < m < 2)$ to systems* (1.1) *with $\sup_{\Omega}|u| = M$ and $2aM < \lambda$ being hold, there holds*

$$
\begin{aligned}
& \int_{B_\rho(x_0)} \left[\frac{\partial A_i^{\alpha}}{\partial p_\beta^j}(x_0, u_0, p_0)(Du - p_0) \right] \cdot D_\alpha \varphi^i dx \\
& \quad \le C_e \alpha_n \rho^n \left[\omega^{1/2}\left(|p_0|, \Phi^{1/2}(x_0, \rho, p_0)\right)\Phi^{1/2}(x_0, \rho, p_0) + \Phi(x_0, \rho, p_0) + \rho^{\beta} H(|p_0|) \right]
\end{aligned}
\tag{4.1}
$$

for $C_e = C_e(C_p, N, n, L)$ and where one defines

$$
\begin{aligned}
\Phi(x_0, \rho, p_0) &= ⨍_{B_\rho(x_0)} |V(Du) - V(p_0)|^2 dx, \\
H(t) &= \left[\widetilde{K}(M + t)(1 + M + t)^{m/2} \right]^{\widetilde{\sigma}},
\end{aligned}
\tag{4.2}
$$

for $\widetilde{\sigma} = \max\{\sigma, 2m/(m-1)\}$ and $\widetilde{K}(M + t) = \max\{K(M + t), a, b, a^2, b^2, a^{m/(m-1)}, b^{m/(m-1)}\}$.

Proof. We assume initially that $\sup_{B_\rho(x_0)}|D\varphi| \le 1$. Applying Lemma 2.7 and noting that the definition of the weak solution of (1.1), for $0 \le t \le 1$, we deduce

$$\begin{aligned}
&\int_{B_\rho(x_0)} \left[\int_0^1 \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x_0,u_0,p_0+t(Du-p_0))\,dt(Du-p_0)\right]\cdot D_\alpha\varphi^i\,dx \\
&= \int_{B_\rho(x_0)} \left(A_i^\alpha(x_0,u_0,Du) - A_i^\alpha(x_0,u_0,p_0)\right)\cdot D_\alpha\varphi^i\,dx \\
&= \int_{B_\rho(x_0)} \left(A_i^\alpha(x_0,u_0,Du) - A_i^\alpha(x,u,Du)\right)\cdot D_\alpha\varphi^i\,dx + \int_{B_\rho(x_0)} B_i(\cdot,u,Du)\cdot\varphi^i\,dx.
\end{aligned} \tag{4.3}$$

Rearranging this, we find

$$\begin{aligned}
&\int_{B_\rho(x_0)} \left[\frac{\partial A_i^\alpha}{\partial p_\beta^j}(x_0,u_0,p_0)(Du-p_0)\right]\cdot D_\alpha\varphi^i\,dx \\
&= \int_{B_\rho(x_0)} \left[\int_0^1 \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x_0,u_0,p_0)\,dt(Du-p_0)\right]\cdot D_\alpha\varphi^i\,dx \\
&= \int_{B_\rho(x_0)} \left[\int_0^1 \left(\frac{\partial A_i^\alpha}{\partial p_\beta^j}(x_0,u_0,p_0) - \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x_0,u_0,p_0+t(Du-p_0))\right)dt(Du-p_0)\right] \\
&\quad \cdot D_\alpha\varphi^i\,dx + \int_{B_\rho(x_0)} \left[A_i^\alpha(x_0,u_0,Du) - A_i^\alpha(x,u_0+p_0(x-x_0),Du)\right]\cdot D_\alpha\varphi^i\,dx \\
&\quad + \int_{B_\rho(x_0)} \left[A_i^\alpha(x,u_0+p_0(x-x_0),Du) - A_i^\alpha(x,u,Du)\right]\cdot D_\alpha\varphi^i\,dx \\
&\quad + \int_{B_\rho(x_0)} B_i(\cdot,u,Du)\cdot\varphi^i\,dx \\
&= I + II + III + IV.
\end{aligned} \tag{4.4}$$

Using the structure condition (E1) and the estimate (1.5) for the modulus of continuity of $\partial A_i^\alpha/\partial p_\beta^j$, by Lemma 2.7 and let

$$s_1 = B_\rho(x_0)\cap\{|Du-p_0|\le 1\}, \qquad s_2 = B_\rho(x_0)\cap\{|Du-p_0|>1\}, \tag{4.5}$$

we can derive

$$I = \int_{B_\rho(x_0)} \int_0^1 \left[L(1+|p_0|^2)^{(m-2)/2} + L\left(1+|p_0+t(Du-p_0)|^2\right)^{(m-2)/2} \right]^{1/2}$$
$$\cdot \left[L\left(1+|p_0|^2+|p_0+t(Du-p_0)|^2\right)^{(m-2)/2} \omega(|p_0|,|t(Du-p_0)|) \right]^{1/2} dt|Du-p_0|dx$$
$$\le C\int_{S_1} \omega^{1/2}(|p_0|,|Du-p_0|)|Du-p_0|dx + C\int_{S_2} \omega^{1/2}(|p_0|,|Du-p_0|)|Du-p_0|^{m/2}dx. \tag{4.6}$$

Noting that the estimates (2.6) and (2.7), using first Hölder's inequality and then Jensen's inequality:

$$I \le C\alpha_n \rho^n \Phi^{1/2}(x_0,\rho,p_0)\omega^{1/2}\left(|p_0|,\Phi^{1/2}(x_0,\rho,p_0)\right), \tag{4.7}$$

here we have used $\Phi^{1/m}(x_0,\rho,p_0) \le \Phi^{1/2}(x_0,\rho,p_0)$ for $\Phi(x_0,\rho,p_0) \le 1$.

By (E3), Young inequality, (2.6), (2.7), and noting the function K monotone nondecreasing and $K(M+|p_0|) \ge 1$ and that $\rho \le 1$, we can estimate II as follows

$$\begin{aligned} II &\le \int_{B_\rho(x_0)} K(|u_0|+|p_0|)\rho^\beta(1+|p_0|)^\beta(1+|Du|)^{m/2}dx \\ &\le K(M+|p_0|)\rho^\beta(1+|p_0|)^{\beta+m/2}\alpha_n\rho^{n+\beta} + \left[K(M+|p_0|)(1+|p_0|)^\beta\right]^2\alpha_n\rho^{n+2\beta} \\ &\quad + \Phi(x_0,\rho,p_0)\alpha_n\rho^n + \left[K(M+|p_0|)(1+|p_0|)^\beta\right]^{4/(4-m)}\alpha_n\rho^{n+(4\beta/(4-m))} \\ &\le \Phi(x_0,\rho,p_0)\alpha_n\rho^n + 2\left[K(M+|p_0|)(1+|p_0|)^{\beta+m/2}\right]^\sigma\alpha_n\rho^{n+\beta}, \end{aligned} \tag{4.8}$$

for $1 < 4/(4-m) \le 2 < \sigma$.

Similar to (3.11), to estimate III, one can divide the domain $B_\rho(x_0)$ as previously mentioned. On the set $B_\rho(x_0)\cap\{|v/\rho|>1\}\cap\{|Du-p_0|\le 1\}$, for $m/(m-\beta) < 2m/(m-2\beta) \le \sigma$,

$$\begin{aligned} &K(|u_0|+|p_0|)(1+|Du|)^{m/2}|v|^\beta \\ &\quad\le K(M+|p_0|)(1+|p_0|)^{m/2}|v|^\beta + K(M+|p_0|)|v|^\beta \\ &\quad\le 2\varepsilon\left|\frac{v}{\rho}\right|^m + C(\varepsilon)\left[K(M+|p_0|)(1+|p_0|)^{m/2}\right]^{m/(m-\beta)}\rho^{m\beta/(m-\beta)} \\ &\qquad + C(\varepsilon)\left[K(M+|p_0|)\right]^{m/(m-\beta)}\rho^{m\beta/(m-\beta)} \\ &\quad\le 2\varepsilon C\left|V\left(\frac{v}{\rho}\right)\right|^2 + 2C(\varepsilon)\left[K(M+|p_0|)(1+|p_0|)^{m/2}\right]^\sigma\rho^\beta, \end{aligned} \tag{4.9}$$

while on the part $B_s(x_0) \cap \{|v/\rho| \le 1\} \cap \{|Du - p_0| > 1\}$ and noting that $1 < 2/(2-\beta) < 2 < \sigma$,

$$\begin{aligned}
&K(|u_0| + |p_0|)(1 + |Du|)^{m/2}|v|^{\beta} \\
&\quad \le K(M + |p_0|)(1 + |p_0|)^{m/2}\rho^{\beta}\left|\frac{v}{\rho}\right|^{\beta} + K(M + |p_0|)\rho^{\beta}|Du - p_0|^{m/2} \\
&\quad \le \varepsilon\left|\frac{v}{\rho}\right|^{2} + C(\varepsilon)\left[K(M + |p_0|)(1 + |p_0|)^{m/2}\right]^{2/(2-\beta)}\rho^{2\beta/(2-\beta)} \\
&\qquad + \varepsilon|Du - p_0|^{m} + C(\varepsilon)[K(M + |p_0|)]^{2}\rho^{2\beta} \\
&\quad \le \varepsilon C\left|V\left(\frac{v}{\rho}\right)\right|^{2} + \varepsilon C|V(Dv)|^{2} + C(\varepsilon)\left[K(M + |p_0|)(1 + |p_0|)^{m/2}\right]^{\sigma}\rho^{\beta}.
\end{aligned} \tag{4.10}$$

On $B_s(x_0) \cap \{|v/\rho| \le 1\} \cap \{|Du - p_0| \le 1\}$

$$\begin{aligned}
&K(|u_0| + |p_0|)(1 + |Du|)^{m/2}|v|^{\beta} \\
&\quad \le K(M + |p_0|)(1 + |p_0|)^{m/2}|v|^{\beta} + K(M + |p_0|)|v|^{\beta} \\
&\quad \le \varepsilon\left|\frac{v}{\rho}\right|^{2} + C(\varepsilon)\left[K(M + |p_0|)(1 + |p_0|)^{m/2}\right]^{2/(2-\beta)}\rho^{2\beta/(2-\beta)} \\
&\quad \le \varepsilon C\left|V\left(\frac{v}{\rho}\right)\right|^{2} + C(\varepsilon)\left[K(M + |p_0|)(1 + |p_0|)^{m/2}\right]^{\sigma}\rho^{\beta}.
\end{aligned} \tag{4.11}$$

Finally, on the case $B_s(x_0) \cap \{|v/\rho| > 1\} \cap \{|Du - p_0| > 1\}$, there exists a constant $0 < m/(m+\beta) < 1$ such that

$$\begin{aligned}
&K(|u_0| + |p_0|)(1 + |Du|)^{m/2}|v|^{\beta} \\
&\quad \le K(M + |p_0|)(1 + |p_0|)^{m/2}\left(|v|^{\beta}\right)^{m/(m+\beta)}(2M + p_0\rho)^{\beta^2/(m+\beta)} \\
&\qquad + K(M + |p_0|)|Du - p_0|^{m/2}\left(|v|^{\beta}\right)^{m/(m+\beta)}(2M + p_0\rho)^{\beta^2/(m+\beta)} \\
&\quad \le C(\varepsilon)\left|\frac{v}{\rho}\right|^{m} + C(\varepsilon)|Du - p_0|^{m} + C(\varepsilon)\left[K(M + |p_0|)(1 + |p_0|)^{m/2}\right]^{(m+\beta)/m}(2M + p_0\rho)^{\beta^2/m}\rho^{\beta} \\
&\qquad + C(\varepsilon)[K(M + |p_0|)]^{2(m+\beta)/(m-\beta)}(2M + p_0\rho)^{\beta^2/(m-\beta)}\rho^{2m\beta/(m-\beta)} \\
&\quad \le C(\varepsilon)\left|V\left(\frac{v}{\rho}\right)\right|^{2} + C(\varepsilon)|V(Dv)|^{2} + C(\varepsilon, n, N)\left[K(M + |p_0|)(1 + M + |p_0|)^{m/2}\right]^{\sigma}\rho^{\beta},
\end{aligned} \tag{4.12}$$

for $1 < 2(m+\beta)/(m-\beta) + \beta^2/(m-\beta) < 2m/(m-2\beta) \le \sigma$.

Whereas, Lemma 2.1 yields

$$
\begin{aligned}
III &\leq C(\varepsilon)\int_{B_\rho(x_0)} \left|V(Du)-V(p_0)\right|^2 dx + C(\varepsilon)\int_{B_S(x_0)} \left|V\left(\frac{v}{\rho}\right)\right|^2 dx \\
&\quad + C(\varepsilon,n,N)\left[K\left(M+|p_0|\right)\left(1+M+|p_0|\right)^{m/2}\right]^{\sigma}\alpha_n\rho^{n+\beta},
\end{aligned}
\tag{4.13}
$$

where σ is defined in Lemma 3.1.

Noting that $\sup_{B_\rho(x_0)}|\varphi| \leq \rho \leq 1$, and by Young's inequality, we see

$$
\begin{aligned}
IV &\leq C\int_{B_\rho(x_0)} \left(a|Du|^m + b\right)|\varphi|\,dx \\
&\leq C\int_{B_\rho(x_0)} a\left|Du-p_0\right|^m|\varphi|\,dx + C\int_{B_\rho(x_0)} \left(b+|p_0|^m\right)\rho\,dx.
\end{aligned}
\tag{4.14}
$$

On $D_1 = \{B_\rho(x_0)\cap\{|Du-p_0|>1\}\}$, by (2.7) and Young inequality, we have

$$
(4.14) \leq C\int_{D_1} a\left|V(Du)-V(p_0)\right|^2 dx + C\alpha_n\rho^{n+1}\left(b+|p_0|^m\right). \tag{4.15}
$$

On the other hand, on $D_2 = \{B_\rho(x_0)\cap\{|Du-p_0|\leq 1\}\}$, using (2.6) and Young inequality, we have

$$
\left|Du-p_0\right|^m \leq \left|Du-p_0\right|^2 + 1 \leq \left|V(Du)-V(p_0)\right|^2 + 1. \tag{4.16}
$$

Thus

$$
(4.14) \leq C\int_{D_2} a\left|V(Du)-V(p_0)\right|^2 dx + C\alpha_n\rho^{n+1}\left(a+b+|p_0|^m\right). \tag{4.17}
$$

Combining these estimates and noting that definition of $H(t)$, we derive

$$
IV \leq C\int_{B_\rho(x_0)} \left|V(Du)-V(p_0)\right|^2 dx + C\alpha_n\rho^{n+1}H\left(|p_0|\right). \tag{4.18}
$$

By Lemma 2.6, there is

$$
III \leq C(\varepsilon, C_P, n, N)\int_{B_\rho(x_0)} \left|V(Du)-V(p_0)\right|^2 dx + C(\varepsilon)H\left(|p_0|\right)\alpha_n\rho^{n+\beta}. \tag{4.19}
$$

Combining the above of I, II, III, IV with (4.4) and noting the definition of $H(t)$, we can get the lemma immediately. □

We next establish an initial excess-improvement estimate, assuming that the excess $\Phi(\rho)$ is initially sufficient small. We also define $\Gamma(\rho) = \sqrt{\Phi(\rho) + 4\delta^{-2}H^2\rho^{2\beta}}$, $w(x) = u(x) - (u_{x_0,\rho} - \gamma h(x_0)) - (Du)_{x_0,\rho}(x - x_0)$, and $\gamma = C_6 C_e \Gamma(\rho)$, where C_6 stands for the constants $C(m, M)$ form Lemma 2.1(vi). The precise statement is the following.

Lemma 4.2 (excess-improvement). *Consider weak solution $u \in W^{1,m}(\Omega, R^N) \cap L^\infty(\Omega, R^N)$ $(1 < m < 2)$ satisfying the conditions of Theorem 1.2 and β fixed in (E3). Then we can find positive constants C_i, C_k, and δ, and $\theta \in (0, 1/4]$ (with C_i depends only on n, N, m, λ, and L and with C_k, δ and θ depending only on these quantities as well as β) such that the smallness condition $\rho \in (0, \rho]$:*

$$\begin{gathered} 2\sqrt{2}C_a\gamma \le 1, \\ \omega^{1/2}\left(\left|(Du)_{x_0,\rho}\right|, \Phi^{1/2}(\rho)\right) + \Phi^{1/2}(\rho) \le \frac{\delta}{2}, \\ \left|(Du)_{x_0,\rho}\right| \le M_1, \quad \textit{for given constant } 0 \le M_1 < \infty \\ 2C_i\rho^\beta H\left(1 + \left|(Du)_{x_0,\rho}\right|\right) \le \frac{\delta}{2}, \\ C_e C_a \Phi(\rho) \le 1, \end{gathered} \tag{4.20}$$

together imply the growth condition

$$\Phi(\theta\rho) \le \theta^{2\beta}\left[\Phi(\rho) + C_k\rho^{2\beta}H^2\left(1 + \left|(Du)_{x_0,\rho}\right|\right)\right]. \tag{4.21}$$

Here one uses the abbreviate $\Phi(\rho) = \Phi(x_0, \rho, (Du)_{x_0,\rho})$.

Proof. For $\varepsilon > 0$ to be determined later, we take $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon) \in (0, 1)$ to be corresponding constant from the $\mathcal{A}$-harmonic approximation lemma, that is, Lemma 2.2, and set

$$\begin{gathered} w(x) = u(x) - (u_{x_0,\rho} - \gamma h(x_0)) - (Du)_{x_0,\rho}(x - x_0), \\ \Gamma(\rho) = \sqrt{\Phi(\rho) + 4\delta^{-2}H^2\rho^{2\beta}}, \quad \gamma = C_6 C_e \Gamma(\rho). \end{gathered} \tag{4.22}$$

where C_6 stands for the constant $C(m, M)$ from Lemma 2.1(vi).

Then, from (2.4) and Lemma 2.1(vi), we have

$$⨍_{B_\rho(x_0)} |W(Dw)|^2 dx \le ⨍_{B_\rho(x_0)} |V(Dw)|^2 dx \le C_6\Phi^2(\rho) \le \gamma^2. \tag{4.23}$$

And by Lemma 4.1 and the smallness condition

$$\omega^{1/2}\left(\left|(Du)_{x_0,\rho}\right|, \Phi^{1/2}(\rho)\right) + \Phi^{1/2}(\rho) \le \frac{\delta}{2}, \tag{4.24}$$

we can deduce

$$\begin{aligned}
&\left| \fint_{B_\rho(x_0)} \left[\frac{\partial A_i^\alpha}{\partial p_\beta^j}\Big(x_0, u_{x_0,\rho}, (Du)_{x_0,\rho}\Big) Dw \right] \cdot D_\alpha \varphi^i \, dx \right| \\
&\quad \le \gamma \frac{\omega^{1/2}\Big(\big|(Du)_{x_0,\rho}\big|, \Phi^{1/2}(\rho)\Big)\Phi^{1/2}(\rho) + \Phi(\rho) + \rho^\beta H\Big(\big|(Du)_{x_0,\rho}\big|\Big)}{C_1 \Gamma(\rho)} \sup_{B_\rho(x_0)} |D\varphi| \\
&\quad \le \gamma \left[\omega^{1/2}\Big(\big|(Du)_{x_0,\rho}\big|, \Phi^{1/2}(\rho)\Big) + \Phi^{1/2}(\rho) + \frac{\delta}{2} \right] \sup_{B_\rho(x_0)} |D\varphi| \\
&\quad \le \gamma \delta \sup_{B_\rho(x_0)} |D\varphi|.
\end{aligned} \tag{4.25}$$

Inequalities (4.23) and (4.25) fulfill the condition of $\mathcal{A}$-harmonic approximation lemma, which allow us to apply Lemma 2.2. Therefore we can find a function $h \in W^{1,m}(B_\rho(x_0), R^N)$ which is $(\partial A_i^\alpha / \partial p_\beta^j)(x_0, u_{x_0,\rho}, (Du)_{x_0,\rho})$-harmonic such that

$$\fint_{B_\rho(x_0)} |W(Dh)|^2 dx \le 1, \qquad \fint_{B_\rho(x_0)} \left| W\left(\frac{w - \gamma h}{\rho} \right) \right|^2 dx \le \gamma^2 \varepsilon. \tag{4.26}$$

With the help of Lemma 2.1(iii) and (v), we have

$$\begin{aligned}
\Phi^2(\theta\rho) &= \fint_{B_{\theta\rho}(x_0)} \Big| V(Du) - V\Big((Du)_{x_0,\theta\rho}\Big) \Big|^2 dx \\
&\le C \fint_{B_{\theta\rho}(x_0)} \Big| V\Big(Du - (Du)_{x_0,\theta\rho}\Big) \Big|^2 dx \\
&\le C \fint_{B_{\theta\rho}(x_0)} \Big| V\Big(Du - (Du)_{x_0,\rho} - \gamma Dh(x_0)\Big) \Big|^2 dx \\
&\quad + C \Big| V\Big((Du)_{x_0,\theta\rho} - (Du)_{x_0,\rho} - \gamma Dh(x_0)\Big) \Big|^2,
\end{aligned} \tag{4.27}$$

where the constant C depends only on n, N, and m.

We proceed to estimate the right-hand side of (4.27). Decomposing $B_{\theta\rho}(x_0)$ into the set with $|Du - (Du)_{x_0,\rho} - \gamma Dh(x_0)| \le 1$ and that with $|Du - (Du)_{x_0,\rho} - \gamma Dh(x_0)| > 1$, that using Lemma 2.1(i) and Hölder inequality, we obtain

$$\Big| (Du)_{x_0,\theta\rho} - (Du)_{x_0,\rho} - \gamma Dh(x_0) \Big| \le \fint_{B_{\theta\rho}(x_0)} \Big| Du - (Du)_{x_0,\rho} - \gamma Dh(x_0) \Big| dx = \sqrt{2}\Big(I^{1/2} + I^{1/m} \Big), \tag{4.28}$$

where we have abbreviated

$$I = \fint_{B_{\theta\rho}(x_0)} \left| V\left(Du - (Du)_{x_0,\rho} - \gamma Dh(x_0) \right) \right|^2 dx. \tag{4.29}$$

Now, since $|V(A)| = V(|A|)$ and $t \to V(t)$ is monotone increasing, we deduce from (4.27), also by using Lemma 2.1(i) and (ii), that there holds

$$\Phi^2(\theta\rho) \le C\left(I + V^2\left(I^{1/2} + I^{1/m} \right) \right) \le C\left(I + I^{2/m} \right), \tag{4.30}$$

where C depends only on n, N, and m. Therefore it remains for us to estimate the quantity I. By considering the cases $|Dh| \le 1$ and $|Dh| > 1$ seperately and keeping in mind (4.26), we have (using Lemma 2.1(i)):

$$\fint_{B_\rho(x_0)} |Dh| dx \le 2\sqrt{2}. \tag{4.31}$$

Using the assumption $|(Du)_{x_0,\rho}| \le M_1$ and Lemma 2.4, this shows

$$\begin{aligned} \left| (Du)_{x_0,\rho} \right| + \gamma |Dh(x_0)| &\le M_1 + \gamma |Dh(x_0)| \\ &\le M_1 + \gamma C_a \fint_{B_\rho(x_0)} |Dh| dx \\ &\le M_1 + 2\sqrt{2}\gamma C_a \\ &\le M_1 + 1. \end{aligned} \tag{4.32}$$

Lemma 3.1 applied on $B_{\theta\rho}(x_0)$ with $u_{x_0,\rho}$, respectively $(Du)_{x_0,\rho} + \gamma Dh(x_0)$, instead of u_0, respectively, p_0; note that the constant C_c depends only on n, N, m, L, λ, M:

$$I \le C_c \left[\fint_{B_{2\theta\rho}(x_0)} \left| V\left(\frac{u - u_{x_0,\rho} - \left((Du)_{x_0,\rho} + \gamma Dh(x_0) \right)(x - x_0)}{2\theta\rho} \right) \right|^2 dx + G \right], \tag{4.33}$$

for

$$\begin{aligned} G = {} & \left[K\left(|u_{x_0,\rho}| + \left| (Du)_{x_0,\rho} + \gamma Dh(x_0) \right| \right) \left(1 + \left| (Du)_{x_0,\rho} + \gamma Dh(x_0) \right| \right)^{m/2} \right]^{\sigma} (2\theta\rho)^{2\beta} \\ & + \max\left\{ \left[a\left| (Du)_{x_0,\rho} + \gamma Dh(x_0) \right|^m + b \right]^2, \left[a\left| (Du)_{x_0,\rho} + \gamma Dh(x_0) \right|^m + b \right]^{m/(m-1)} \right\} (2\theta\rho)^2. \end{aligned} \tag{4.34}$$

Lemma 2.1(iii) yields

$$\begin{aligned}
&\fint_{B_{2\theta\rho}(x_0)} \left| V\left(\frac{u - u_{x_0,\rho} - \left((Du)_{x_0,\rho} + \gamma Dh(x_0) \right)(x - x_0)}{2\theta\rho} \right) \right|^2 dx \\
&\quad \le \fint_{B_{2\theta\rho}(x_0)} \left| V\left(\frac{u - (u_{x_0,\rho} - \gamma h(x_0)) - (Du)_{x_0,\rho}(x - x_0) - \gamma h(x) + \gamma h(x)}{2\theta\rho} \right.\right. \\
&\qquad\qquad \left.\left. + \frac{-\gamma h(x_0) - \gamma Dh(x_0)(x - x_0)}{2\theta\rho} \right) \right|^2 dx \\
&\quad \le C\left[\fint_{B_{2\theta\rho}(x_0)} \left(\left| V\left(\frac{w - \gamma h(x)}{2\theta\rho} \right) \right|^2 + \left| V\left(\gamma \frac{h - h_0 - Dh(x_0)(x - x_0)}{2\theta\rho} \right) \right|^2 \right) dx \right],
\end{aligned} \tag{4.35}$$

where the constant C is given by $C(m)C_c$. To estimate the right-hand side of (4.33) we use (2.4), Lemma 2.1(ii) (note that $1/2\theta \ge 1$) and (4.26) to infer

$$\begin{aligned}
\fint_{B_{2\theta\rho}(x_0)} \left| V\left(\frac{w - \gamma h}{2\theta\rho} \right) \right|^2 dx &\le C(m) \fint_{B_{2\theta\rho}(x_0)} \left| W\left(\frac{w - \gamma h}{2\theta\rho} \right) \right|^2 dx \\
&\le C(m)(2\theta)^{-n} \fint_{B_\rho(x_0)} \left| W\left(\frac{w - \gamma h}{2\theta\rho} \right) \right|^2 dx \\
&\le C(m)(2\theta)^{-n-2} \fint_{B_\rho(x_0)} \left| W\left(\frac{w - \gamma h}{\rho} \right) \right|^2 dx \\
&\le C(m) 2^{-n-2} \theta^{-n-2} \gamma^2 \varepsilon.
\end{aligned} \tag{4.36}$$

Using Lemma 2.1(i), Taylor's theorem applied to h on $B_{2\theta\rho}(x_0)$, Lemma 2.4 and (4.31), we obtain

$$\begin{aligned}
&\fint_{B_{2\theta\rho}(x_0)} \left| V\left(\gamma \frac{h - h_0 - Dh(x_0)(x - x_0)}{2\theta\rho} \right) \right|^2 dx \\
&\quad \le \gamma^2 \fint_{B_{2\theta\rho}(x_0)} \left| \frac{h - h_0 - Dh(x_0)(x - x_0)}{2\theta\rho} \right|^2 dx \\
&\quad \le \frac{\gamma^2}{4\theta^2\rho^2} \sup_{B_{2\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^2 \\
&\quad \le 8 C_a^2 \theta^2 \gamma^2.
\end{aligned} \tag{4.37}$$

Using the smallness condition $2\sqrt{2}C_a\gamma \leq 1$ and (4.32) together with the definition of H yields

$$\begin{aligned}
&\left[K\left(\left|u_{x_0,\rho}\right| + \left|(Du)_{x_0,\rho} + \gamma Dh(x_0)\right|\right)\left(1 + \left|(Du)_{x_0,\rho} + \gamma Dh(x_0)\right|\right)^{m/2}\right]^{\sigma}(2\theta\rho)^{2\beta} \\
&\quad \leq \left[K\left(M + \left|(Du)_{x_0,\rho}\right| + 1\right)\left(2 + \left|(Du)_{x_0,\rho}\right|\right)^{m/2}\right]^{\sigma}(2\theta\rho)^{2\beta} \\
&\quad \leq H\left(1 + \left|(Du)_{x_0,\rho}\right|\right)(2\theta\rho)^{2\beta}, \\
&\max\left\{\left[a\left|(Du)_{x_0,\rho} + \gamma Dh(x_0)\right|^{m} + b\right]^2, \left[a\left|(Du)_{x_0,\rho} + \gamma Dh(x_0)\right|^{m} + b\right]^{m/(m-1)}\right\}(2\theta\rho)^2 \\
&\quad \leq H\left(1 + \left|(Du)_{x_0,\rho}\right|\right)(2\theta\rho)^{2\beta}.
\end{aligned} \tag{4.38}$$

Combining all the above estimates with (4.33), and let $\varepsilon = \theta^{n+4}$ for $\theta \in (0, 1/4]$, we get

$$I \leq C_i\left[\theta^2\gamma^2 + H\left(1 + \left|(Du)_{x_0,\rho}\right|\right)(2\theta\rho)^{2\beta}\right], \tag{4.39}$$

where the constant C_i depends only on n, N, L, m, λ, M, and θ (the dependency from θ occurs due to the fact that δ depends on θ). Choose $\theta \in (0, 1/4)$ suitable such that $C_i\theta^2 \leq \theta^{2\beta}$, and inserting this into (4.30) we easily find (recalling also that $\Phi(\rho) \leq 1$):

$$\Phi(\theta\rho) \leq \theta^{2\beta}\left[\Phi(\rho) + C_k H^2\left(1 + \left|(Du)_{x_0,\rho}\right|\right)\rho^{2\beta}\right], \tag{4.40}$$

where the constant C_k has the same dependencies as C_i. □

The regularity result then follows from the fact that this excess-decay estimate for any x in a neighborhood of x_0. From this estimate we conclude (by Campanato's characterization of Hölder continuous functions [19, 20]) that $V(Du)$ has the modulus of continuity $\rho \mapsto \Phi(x_0, \rho)$ by a constant times $\rho^{2\beta}$. By Lemma 2.1(iv) this modulus of continuity carries over to Du.

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