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*Research Article*

# Improvement and Reversion of Slater's Inequality and Related Results

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We use an inequality given by Matić and Pečarić (2000) and obtain improvement and reverse of Slater's and related inequalities.

## 1. Introduction

In 1981 Slater has proved an interesting companion inequality to Jensen's inequality [1].

**Theorem 1.1.** *Suppose that $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is increasing convex function on interval $I$, for $x_1, x_2, \ldots, x_n \in I^\circ$ (where $I^\circ$ is the interior of the interval $I$) and for $p_1, p_2, \ldots, p_n \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0$, if $\sum_{i=1}^n p_i \phi'_+(x_i) > 0$, then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i) \leq \phi\left(\frac{\sum_{i=1}^n p_i \phi'_+(x_i) x_i}{\sum_{i=1}^n p_i \phi'_+(x_i)}\right). \tag{1.1}$$

*When $\phi$ is strictly convex on $I$, inequality* (1.1) *becomes equality if and only if $x_i = c$ for some $c \in I^\circ$ and for all $i$ with $p_i > 0$.*

It was noted in [2] that by using the same proof the following generalization of Slater's inequality (1981) can be given.

**Theorem 1.2.** *Suppose that $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex function on interval $I$, for $x_1, x_2, \ldots, x_n \in I^\circ$ (where $I^\circ$ is the interior of the interval $I$) and for $p_1, p_2, \ldots, p_n \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0$. Let*

$$\sum_{i=1}^n p_i \phi'_+(x_i) \neq 0, \quad \frac{\sum_{i=1}^n p_i \phi'_+(x_i) x_i}{\sum_{i=1}^n p_i \phi'_+(x_i)} \in I^\circ, \tag{1.2}$$

*then inequality* (1.1) *holds.*

*When $\phi$ is strictly convex on $I$, inequality* (1.1) *becomes equality if and only if $x_i = c$ for some $c \in I^\circ$ and for all $i$ with $p_i > 0$.*

*Remark 1.3.* For multidimensional version of Theorem 1.2 see [3].

Another companion inequality to Jensen's inequality is a converse proved by Dragomir and Goh in [4].

**Theorem 1.4.** *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function defined on interval $I$. If $x_i \in I, i = 1, 2, \ldots, n$ $(n \geq 2)$ are arbitrary members and $p_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $P_n = \sum_{i=1}^n p_i > 0$, and let*

$$\overline{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i, \qquad \overline{y} = \frac{1}{P_n} \sum_{i=1}^n p_i \phi(x_i). \tag{1.3}$$

*Then the inequalities*

$$0 \leq \overline{y} - \phi(\overline{x}) \leq \frac{1}{P_n} \sum_{i=1}^n p_i \phi'(x_i)(x_i - \overline{x}) \tag{1.4}$$

*hold.*

*In the case when $\phi$ is strictly convex, one has equalities in* (1.4) *if and only if there is some $c \in I$ such that $x_i = c$ holds for all $i$ with $p_i > 0$.*

Matić and Pečarić in [5] proved more general inequality from which (1.1) and (1.4) can be obtained as special cases.

**Theorem 1.5.** *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function defined on interval $I$ and let $x_i$, $p_i$, $P_n$, $\overline{x}$, and $\overline{y}$ be stated as in Theorem 1.4. If $d \in I$ is arbitrary chosen number, then one has*

$$\overline{y} \leq \phi(d) + \frac{1}{P_n} \sum_{i=1}^n p_i (x_i - d) \phi'(x_i). \tag{1.5}$$

*Also, when $\phi$ is strictly convex, one has equality in* (1.5) *if and only if $x_i = d$ holds for all $i$ with $p_i > 0$.*

*Remark 1.6.* If $\phi$, $x_i$, $p_i$, $P_n$, and $\overline{x}$ are stated as in Theorem 1.4 and we let $\sum_{i=1}^n p_i \phi'(x_i) \neq 0$, also if $\overline{\overline{x}} = \sum_{i=1}^n p_i x_i \phi'(x_i) / \sum_{i=1}^n p_i \phi'(x_i) \in I$, then by setting $d = \overline{\overline{x}}$ in (1.5), we get Slater's inequality (1.1) and similarly by setting $d = \overline{x}$ in (1.5), we get (1.4).

The following refinement of (1.4) is also valid [5].

**Theorem 1.7.** *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be strictly convex differentiable function defined on interval I and let $x_i$, $p_i$, $P_n$, $\overline{x}$, and $\overline{y}$ be stated as in Theorem 1.4 and $\overline{d} = (\phi')^{-1}((1/P_n)\sum_{i=1}^{n} p_i\phi'(x_i))$, then the inequalities*

$$\overline{y} \le \phi\left(\overline{d}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)\left(x_i - \overline{d}\right), \tag{1.6}$$

$$0 \le \overline{y} - \phi(\overline{x}) \le \phi\left(\overline{d}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)\left(x_i - \overline{d}\right) - \phi(\overline{x}) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)(x_i - \overline{x}) \tag{1.7}$$

*hold.*

*The equalities hold in* (1.6) *and in* (1.7) *if and only if* $x_1 = x_2 = \cdots = x_n$.

*Remark 1.8.* In [6] Dragomir has also proved Theorem 1.7.

In this paper, we use an inequality given in [5] and derive two mean value theorems, exponential convexity, log-convexity, and Cauchy means. As applications, such results are also deduce for related inequality. We use some log-convexity criterion and prove improvement and reverse of Slater's and related inequalities. We also prove some determinantal inequalities.

## 2. Mean Value Theorems

**Theorem 2.1.** *Let $\phi \in C^2(I)$, where I is closed interval in $\mathbb{R}$, and let $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$, $x_i, d \in I$ with $x_i \neq d$ $(i = 1, 2, \ldots, n)$ and $\overline{y} = (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)$. Then there exists $\xi \in I$ such that*

$$\phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \overline{y} = \frac{\phi''(\xi)}{2P_n}\sum_{i=1}^{n} p_i(x_i - d)^2. \tag{2.1}$$

*Proof.* Since $\phi''(x)$ is continuous on $I$, $m \le \phi''(x) \le M$ for $x \in I$, where $m = \min_{x\in I}\phi''(x)$ and $M = \max_{x\in I}\phi''(x)$.

Consider the functions $\phi_1$, $\phi_2$ defined as

$$\begin{aligned} \phi_1(x) &= \frac{Mx^2}{2} - \phi(x), \\ \phi_2(x) &= \phi(x) - \frac{mx^2}{2}. \end{aligned} \tag{2.2}$$

Since

$$\begin{aligned} \phi_1''(x) &= M - \phi''(x) \ge 0, \\ \phi_2''(x) &= \phi''(x) - m \ge 0, \end{aligned} \tag{2.3}$$

$\phi_i(x)$ for $i = 1, 2$ are convex.

Now by applying $\phi_1$ for $\phi$ in inequality (1.5), we have

$$\frac{Md^2}{2}-\phi(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)\left(Mx_i-\phi'(x_i)\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\left(\frac{Mx_i^2}{2}-\phi(x_i)\right)\geq 0. \tag{2.4}$$

From (2.4) we get

$$\phi(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)\phi'(x_i)-\overline{y}\leq \frac{M}{2P_n}\sum_{i=1}^{n}p_i(x_i-d)^2, \tag{2.5}$$

and similarly by applying $\phi_2$ for $\phi$ in (1.5), we get

$$\phi(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)\phi'(x_i)-\overline{y}\geq \frac{m}{2P_n}\sum_{i=1}^{n}p_i(x_i-d)^2. \tag{2.6}$$

Since

$$\sum_{i=1}^{n}p_i(x_i-d)^2>0 \quad \text{as } x_i\neq d,\ p_i>0\ (i=1,2,\ldots,n), \tag{2.7}$$

by combining (2.5) and (2.6), we have

$$m\leq \frac{2P_n\left[\phi(d)+(1/P_n)\sum_{i=1}^{n}p_i(x_i-d)\phi'(x_i)-\overline{y}\right]}{\sum_{i=1}^{n}p_i(x_i-d)^2}\leq M. \tag{2.8}$$

Now using the fact that for $m\leq\rho\leq M$ there exists $\xi\in I$ such that $\phi''(\xi)=\rho$, we get (2.1). □

**Corollary 2.2.** *Let $\phi\in C^2(I)$, where $I$ is closed interval in $\mathbb{R}$, and let $x_i$, $\overline{x}$, $\overline{y}$, and $P_n$ be stated as in Theorem 1.4 with $p_i>0$ and $x_i\neq\overline{x}$ $(i=1,2,\ldots,n)$. Then there exists $\xi\in I$ such that*

$$\phi(\overline{x})+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-\overline{x})\phi'(x_i)-\overline{y}=\frac{\phi''(\xi)}{2P_n}\sum_{i=1}^{n}p_i(x_i-\overline{x})^2. \tag{2.9}$$

*Proof.* By setting $d=\overline{x}$ in Theorem 2.1, we get (2.9). □

**Theorem 2.3.** *Let $\phi,\psi\in C^2(I)$, where $I$ is closed interval in $\mathbb{R}$, and let $P_n=\sum_{i=1}^{n}p_i$, $p_i>0$ and $x_i,d\in I$ with $x_i\neq d$ $(i=1,2,\ldots,n)$. Then there exists $\xi\in I$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)}=\frac{\phi(d)+(1/P_n)\sum_{i=1}^{n}p_i(x_i-d)\phi'(x_i)-(1/P_n)\sum_{i=1}^{n}p_i\phi(x_i)}{\psi(d)+(1/P_n)\sum_{i=1}^{n}p_i(x_i-d)\psi'(x_i)-(1/P_n)\sum_{i=1}^{n}p_i\psi(x_i)}, \tag{2.10}$$

*provided that the denominators are nonzero.*

*Proof.* Let the function $k \in C^2(I)$ be defined by

$$k = c_1\phi - c_2\psi, \tag{2.11}$$

where $c_1$ and $c_2$ are defined as

$$\begin{aligned} c_1 &= \psi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\psi'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi(x_i), \\ c_2 &= \phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi(x_i). \end{aligned} \tag{2.12}$$

Then, using Theorem 2.1 with $\phi = k$, we have

$$0 = \left(\frac{c_1\phi''(\xi)}{2P_n} - \frac{c_2\psi''(\xi)}{2P_n}\right)\sum_{i=1}^{n} p_i(x_i - d)^2, \tag{2.13}$$

because $k(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)k'(d) - (1/P_n)\sum_{i=1}^{n} p_i k(x_i) = 0$.

Since $(1/P_n)\sum_{i=1}^{n} p_i(x_i - d)^2 > 0$ as $x_i \neq d$ and $p_i > 0$ $(i = 1, 2, \ldots, n)$, therefore, (2.13) gives us

$$\frac{c_2}{c_1} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{2.14}$$

After putting the values of $c_1$ and $c_2$, we get (2.10). □

**Corollary 2.4.** *Let $\phi, \psi \in C^2(I)$, where $I$ is closed interval in $\mathbb{R}$, and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ and let $x_i \in I$, $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$ with $x_i \neq \overline{x}$ $(i = 1, 2, \ldots, n)$. Then there exists $\xi \in I$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\phi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)}{\psi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\psi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\psi(x_i)}, \tag{2.15}$$

*provided that the denominators are nonzero.*

*Proof.* By setting $d = \overline{x}$ in Theorem 2.3, we get (2.15). □

**Corollary 2.5.** *Let $x_i, d \in I$ with $x_i \neq d$ and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$. Then for $u, v \in \mathbb{R} \setminus \{0, 1\}$, $u \neq v$, there exists $\xi \in I$, where $I$ is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)\left[d^u + (u/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{u-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^u\right]}{u(u-1)\left[d^v + (v/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{v-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^v\right]}. \tag{2.16}$$

*Proof.* By setting $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in I$, in Theorem 2.3, we get (2.16). □

**Corollary 2.6.** *Let $x_i \in I$, $P_n = \sum_{i=1}^n p_i$, $p_i > 0$ $(i = 1,2,\dots,n)$, and $\overline{x} = (1/P_n)\sum_{i=1}^n p_i x_i$ with $x_i \neq \overline{x}$. Then for $u, v \in \mathbb{R} \setminus \{0,1\}$, $u \neq v$, there exists $\xi \in I$, where $I$ is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)\left[\overline{x}^u + (u/P_n)\sum_{i=1}^n p_i(x_i - \overline{x})x_i^{u-1} - (1/P_n)\sum_{i=1}^n p_i x_i^u\right]}{u(u-1)\left[\overline{x}^v + (v/P_n)\sum_{i=1}^n p_i(x_i - \overline{x})x_i^{v-1} - (1/P_n)\sum_{i=1}^n p_i x_i^v\right]}. \tag{2.17}$$

*Proof.* By setting $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in I$, in (2.15), we get (2.17). □

*Remark 2.7.* Note that we can consider the interval $I = [m_x, M_x]$, where $m_x = \min_i\{x_i, d\}$, $M_x = \max_i\{x_i, d\}$.

Since the function $\xi \to \xi^{u-v}$ with $u \neq v$ is invertible, then from (2.16) we have

$$m_x \leq \left\{\frac{v(v-1)\left[d^u + (u/P_n)\sum_{i=1}^n p_i(x_i - d)x_i^{u-1} - (1/P_n)\sum_{i=1}^n p_i x_i^u\right]}{u(u-1)\left[d^v + (v/P_n)\sum_{i=1}^n p_i(x_i - d)x_i^{v-1} - (1/P_n)\sum_{i=1}^n p_i x_i^v\right]}\right\}^{1/(u-v)} \leq M_x. \tag{2.18}$$

We will say that the expression in the middle is a mean of $x_i, d$.

From (2.17) we have

$$\min_i\{x_i\} \leq \left\{\frac{v(v-1)\left[\overline{x}^u + (u/P_n)\sum_{i=1}^n (x_i - \overline{x})x_i^{u-1} - (1/P_n)\sum_{i=1}^n p_i x_i^u\right]}{u(u-1)\left[\overline{x}^v + (v/P_n)\sum_{i=1}^n p_i(x_i - \overline{x})x_i^{v-1} - (1/P_n)\sum_{i=1}^n p_i x_i^v\right]}\right\}^{1/(u-v)} \leq \max_i\{x_i\}. \tag{2.19}$$

The expression in the middle of (2.19) is a mean of $x_i$.

In fact similar results can also be given for (2.10) and (2.15). Namely, suppose that $\phi''/\psi''$ has inverse function, then from (2.10) and (2.15) we have

$$\begin{aligned}
\xi &= \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\phi(d) + (1/P_n)\sum_{i=1}^n p_i(x_i - d)\phi'(x_i) - (1/P_n)\sum_{i=1}^n p_i\phi(x_i)}{\psi(d) + (1/P_n)\sum_{i=1}^n p_i(x_i - d)\psi\prime(x_i) - (1/P_n)\sum_{i=1}^n p_i\psi(x_i)}\right), \\
\xi &= \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\phi(\overline{x}) + (1/P_n)\sum_{i=1}^n p_i(x_i - \overline{x})\phi'(x_i) - (1/P_n)\sum_{i=1}^n p_i\phi(x_i)}{\psi(\overline{x}) + (1/P_n)\sum_{i=1}^n p_i(x_i - \overline{x})\psi'(x_i) - (1/P_n)\sum_{i=1}^n p_i\psi(x_i)}\right).
\end{aligned} \tag{2.20}$$

So, we have that the expression on the right-hand side of (2.20) is also means.

## 3. Improvements and Related Results

*Definition 3.1* (see [7, page 2]). A function $\phi : I \to \mathbb{R}$ is convex if

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0 \tag{3.1}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

**Lemma 3.2** (see [8]). *Let one define the function*

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{t(t-1)}, & t \neq 0, 1, \\ -\log x, & t = 0, \\ x \log x, & t = 1. \end{cases} \tag{3.2}$$

*Then $\varphi_t''(x) = x^{t-2}$, that is, $\varphi_t$ is convex for $x > 0$.*

*Definition 3.3* (see [9]). A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(x_k + x_l) \geq 0, \tag{3.3}$$

for all $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, and $x_k \in I$, $k = 1, 2, \ldots, n$ such that $x_k + x_l \in I, 1 \leq k, l \leq n$, or equivalently

$$\sum_{k,l=1}^{n} a_k a_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0. \tag{3.4}$$

**Corollary 3.4** (see [9]). *If $\phi$ is exponentially convex function, then*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{n} \geq 0 \tag{3.5}$$

*for every $n \in \mathbb{N}$ $x_k \in I$, $k = 1, 2, \ldots, n$.*

**Corollary 3.5** (see [9]). *If $\phi : I \to (0, \infty)$ is exponentially convex function, then $\phi$ is a log-convex function that is*

$$\phi(\lambda x + (1-\lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \quad \forall x, y \in I, \ \lambda \in [0, 1]. \tag{3.6}$$

**Theorem 3.6.** *Let $x_i, p_i, d \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^{n} p_i$. Consider $\Gamma_t$ to be defined by*

$$\Gamma_t = \varphi_t(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\varphi_t'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_t(x_i). \tag{3.7}$$

*Then*

(i) *for every $m \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, m\}$, the matrix $[\Gamma_{(s_k+s_l)/2}]_{k,l=1}^{m}$ is a positive semidefinite matrix; particularly*

$$\det\left[\Gamma_{(s_k+s_l)/2}\right]_{k,l=1}^{m} \geq 0; \tag{3.8}$$

(ii) *the function $t \to \Gamma_t$ is exponentially convex;*

(iii) *if $\Gamma_t > 0$, then the function $t \longrightarrow \Gamma_t$ is log-convex, that is, for $-\infty < r < s < t < \infty$, one has*

$$(\Gamma_s)^{t-r} \leq (\Gamma_r)^{t-s} (\Gamma_t)^{s-r}. \tag{3.9}$$

*Proof.* (i) Let us consider the function defined by

$$\mu(x) = \sum_{k,l=1}^{m} a_k a_l \varphi_{s_{kl}}(x), \tag{3.10}$$

where $s_{kl} = (s_k + s_l)/2$, $a_k \in \mathbb{R}$ for all $k \in \{1, 2, 3, \dots, m\}$, $x > 0$

Then we have

$$\mu''(x) = \sum_{k,l=1}^{m} a_k a_l x^{s_{kl}-2} = \left( \sum_{k=1}^{m} a_k x^{(s_k-2)/2} \right)^2 \geq 0. \tag{3.11}$$

Therefore, $\mu(x)$ is convex function for $x > 0$. Using $\mu(x)$ in inequality (1.5), we get

$$\sum_{k,l=1}^{m} a_k a_l \Gamma_{s_{kl}} \geq 0, \tag{3.12}$$

so the matrix $[\Gamma_{(s_k+s_l)/2}]_{k,l=1}^{m}$ is positive semi-definite.

(ii) Since $\lim_{t\to 0} \Gamma_t = \Gamma_0$ and $\lim_{t\to 1} \Gamma_t = \Gamma_1$, so $\Gamma_t$ is continuous for all $t \in \mathbb{R}$, $x > 0$, and we have exponentially convexity of the function $t \longrightarrow \Gamma_t$.

(iii) Let $\Gamma_t > 0$, then by Corollary 3.5 we have that $\Gamma_t$ is log-convex, that is, $t \longrightarrow \log \Gamma_t$ is convex, and by (3.1) for $-\infty < r < s < t < \infty$ and taking $\phi(t) = \log \Gamma_t$, we get

$$(t - s) \log \Gamma_r + (r - t) \log \Gamma_s + (s - r) \log \Gamma_t \geq 0, \tag{3.13}$$

which is equivalent to (3.9). □

**Corollary 3.7.** *Let $x_i, p_i \in \mathbb{R}^+$ $(i = 1, 2, \dots, n)$, $P_n = \sum_{i=1}^{n} p_i$ and $\overline{x} = (1/P_n) \sum_{i=1}^{n} p_i x_i$. Consider $\widetilde{\Gamma}_t$ to be defined by*

$$\widetilde{\Gamma}_t = \varphi_t(\overline{x}) + \frac{1}{P_n} \sum_{i=1}^{n} p_i (x_i - \overline{x}) \varphi_t'(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi_t(x_i). \tag{3.14}$$

*Then*

(i) *for every $m \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \dots, m\}$, the matrix $[\widetilde{\Gamma}_{(s_k+s_l)/2}]_{k,l=1}^{m}$ is a positive semi-definite matrix. Particularly*

$$\det \left[ \widetilde{\Gamma}_{(s_k+s_l)/2} \right]_{k,l=1}^{m} \geq 0, \tag{3.15}$$

(ii) *the function* $t \to \widetilde{\Gamma}_t$ *is exponentially convex;*

(iii) *if* $\widetilde{\Gamma}_t > 0$, *then the function* $t \to \widetilde{\Gamma}_t$ *is log-convex, that is, for* $-\infty < r < s < t < \infty$, *one has*

$$\left(\widetilde{\Gamma}_s\right)^{t-r} \leq \left(\widetilde{\Gamma}_r\right)^{t-s}\left(\widetilde{\Gamma}_t\right)^{s-r}. \tag{3.16}$$

*Proof.* To get the required results, set $d = \overline{x}$ in Theorem 3.6. □

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be positive n-tuple and $p_1, p_2, \ldots, p_n$ positive real numbers, and let $P_n = \sum_{i=1}^{n} p_i$. Let $M_t(\mathbf{x})$ denote the power mean of order $t$ $(t \in \mathbb{R})$, defined by

$$M_t(\mathbf{x}) = \begin{cases} \left(\dfrac{1}{P_n}\displaystyle\sum_{i=1}^{n} p_i x_i^t\right)^{1/t}, & t \neq 0, \\ \left(\displaystyle\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}, & t = 0. \end{cases} \tag{3.17}$$

Let us note that $M_1(\mathbf{x}) = \overline{x}$.

By (2.18) we can give the following definition of Cauchy means.

Let $x_i, d \in I$ with $x_i \neq d$, $I$ is positive closed interval, and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$,

$$M_{u,v} = \left(\frac{\Gamma_u}{\Gamma_v}\right)^{1/(u-v)} \tag{3.18}$$

for $-\infty < u \neq v < +\infty$ are means of $x_i$, $d$. Moreover we can extend these means to the other cases.

So by limit we have

$$\begin{aligned}
&M_{u,u} \\
&= \exp\left(\frac{P_n d^u \log d + (u-1)\sum_{i=1}^{n} p_i x_i^u \log x_i + P_n M_u^u(\mathbf{x}) - d\left(u\sum_{i=1}^{n} p_i x_i^{u-1} \log x_i + P_n M_{u-1}^{u-1}(\mathbf{x})\right)}{P_n\left[d^u + (u-1)M_u^u(\mathbf{x}) - du M_{u-1}^{u-1}(\mathbf{x})\right]}\right. \\
&\qquad \left. - \frac{2u-1}{u(u-1)}\right), \quad u \neq 0, 1, \\
&M_{0,0} = \exp\left(\frac{P_n \log^2 d - P_n M_2^2(\log \mathbf{x}) + 2P_n \log M_0(\mathbf{x}) - 2d\sum_{i=1}^{n} p_i x_i^{-1} \log x_i}{2P_n\left[\log d - \log M_0(\mathbf{x}) + 1 - dM_{-1}^{-1}(\mathbf{x})\right]} + 1\right), \\
&M_{1,1} = \exp\left(\frac{P_n d \log^2 d + 2\sum_{i=1}^{n} p_i x_i \log x_i - dP_n\left(M_2^2(\log \mathbf{x}) - 2\log M_0(\mathbf{x})\right)}{2\left[P_n d(\log d - 1) + P_n \overline{x} - dP_n \log M_0(\mathbf{x})\right]} - 1\right),
\end{aligned} \tag{3.19}$$

where $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)$.

**Theorem 3.8.** *Let $t, s, u, v \in \mathbb{R}$ such that $t \le u, s \le v$, then the following inequality is valid:*

$$M_{t,s} \le M_{u,v}. \tag{3.20}$$

*Proof.* For convex function $\phi$ it holds that ([7, page 2])

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \le \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1} \tag{3.21}$$

with $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$. Since by Theorem 3.6, $\Gamma_t$ is log-convex, we can set in (3.21): $\phi(x) = \log \Gamma_x$, $x_1 = t$, $x_2 = s$, $y_1 = u$, and $y_2 = v$, then we get

$$\frac{\log \Gamma_s - \log \Gamma_t}{s - t} \le \frac{\log \Gamma_v - \log \Gamma_u}{v - u}. \tag{3.22}$$

From (3.22) we get (3.20) for $s \ne t$ and $u \ne v$.

For $s = t$ and $u = v$ we have limiting case. □

Similarly by (2.19) we can give the following definition of Cauchy type means.

Let $x_i \in I$ with $x_i \ne \overline{x}$, $I$ is positive closed interval, and $P_n = \sum_{i=1}^n p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$,

$$\widetilde{M}_{u,v} = \left( \frac{\widetilde{\Gamma}_u}{\widetilde{\Gamma}_v} \right)^{1/(u-v)} \tag{3.23}$$

for $-\infty < u \ne v < +\infty$ are means of $x_i$. Moreover we can extend these means to the other cases.

So by limit we have

$$\begin{aligned}
&\widetilde{M}_{u,u} \\
&= \exp\left( \frac{P_n \overline{x}^u \log \overline{x} + (u-1) \sum_{i=1}^n p_i x_i^u \log x_i + P_n M_u^u(\mathbf{x}) - \overline{x}\left( u \sum_{i=1}^n p_i x_i^{u-1} \log x_i + P_n M_{u-1}^{u-1}(\mathbf{x}) \right)}{P_n \left[ \overline{x}^u + (u-1) M_u^u(\mathbf{x}) - \overline{x} u M_{u-1}^{u-1}(\mathrm{x}) \right]} \right. \\
&\qquad \left. - \frac{2u-1}{u(u-1)} \right), \quad u \ne 0, 1, \\
&\widetilde{M}_{0,0} = \exp\left( \frac{P_n \log^2 \overline{x} - P_n M_2^2(\log \mathbf{x}) + 2 P_n \log M_0(\mathbf{x}) - 2\overline{x} \sum_{i=1}^n p_i x_i^{-1} \log x_i}{2 P_n \left[ \log \overline{x} - \log M_0(\mathbf{x}) + 1 - \overline{x} M_{-1}^{-1}(\mathbf{x}) \right]} + 1 \right), \\
&\widetilde{M}_{1,1} = \exp\left( \frac{P_n \overline{x} \log^2 \overline{x} + 2 \sum_{i=1}^n p_i x_i \log x_i - \overline{x} P_n \left( M_2^2(\log \mathbf{x}) + 2 \log M_0(\mathbf{x}) \right)}{2\left[ P_n \overline{x}(\log \overline{x} - 1) + P_n \overline{x} - \overline{x} P_n \log M_0(\mathbf{x}) \right]} - 1 \right),
\end{aligned} \tag{3.24}$$

where $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)$.

**Theorem 3.9.** *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following inequality is valid:*

$$\widetilde{M}_{t,s} \leq \widetilde{M}_{u,v}. \tag{3.25}$$

*Proof.* The proof is similar to the proof of Theorem 3.8. □

Let $M_t(\mathbf{x})$ be stated as above, define $d_t$ as

$$d_t = \frac{\sum_{i=1}^{n} p_i x_i \varphi_t'(x_i)}{\sum_{i=1}^{n} p_i \varphi_t'(x_i)} = \begin{cases} \dfrac{M_t^t(\mathbf{x})}{M_{t-1}^{t-1}(\mathbf{x})}, & t \neq 0, 1, \\ M_{-1}(\mathbf{x}), & t = 0, \\ \dfrac{P_n \overline{x} + \sum_{i=1}^{n} p_i x_i \log x_i}{P_n\left(1 + \log M_0(\mathbf{x})\right)}, & t = 1. \end{cases} \tag{3.26}$$

The following improvement and reverse of Slater's inequality are valid.

**Theorem 3.10.** *Let $x_i,\ p_i,\ d_t \in \mathbb{R}^+$ $(i = 1, 2, \dots, n)$, $P_n = \sum_{i=1}^{n} p_i$. Let $F_t$ be defined by*

$$F_t = \varphi_t(d_t) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t(x_i). \tag{3.27}$$

*Then*

(i)

$$F_t \geq [H(s;t)]^{(t-r)/(s-r)}[H(r;t)]^{(s-t)/(s-r)}, \tag{3.28}$$

*for $-\infty < r < s < t < \infty$ and $-\infty < t < r < s < \infty$.*

(ii)

$$F_t \leq [H(s;t)]^{(t-r)/(s-r)}[H(r;t)]^{(s-t)/(s-r)}, \tag{3.29}$$

*for $-\infty < r < t < s < \infty$.*

*where,*

$$H(s;t) = \varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i (x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_s(x_i). \tag{3.30}$$

*Proof.* (i) By setting $d = d_t$ in (3.7), $\Gamma_t$ becomes $F_t$, and for $-\infty < r < s < t < \infty$, setting $d = d_t$ in (3.9), we get

$$\begin{aligned}&\left(\varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_s(x_i)\right)^{t-r} \\ &\quad\le \left(\varphi_r(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_r'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_r(x_i)\right)^{t-s} (F_t)^{s-r},\end{aligned} \tag{3.31}$$

that is,

$$\begin{aligned}(F_t)^{s-r} &\ge \left(\varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_s(x_i)\right)^{t-r} \\ &\quad\times \left(\varphi_r(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_r'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_r(x_i)\right)^{s-t}.\end{aligned} \tag{3.32}$$

From (3.32) we get (3.28), and similarly for $-\infty < t < r < s < \infty$ (3.9) becomes

$$(\Gamma_r)^{s-t} \le (\Gamma_t)^{s-r}(\Gamma_s)^{r-t}; \tag{3.33}$$

by the same process we can get (3.28).

(ii) For $-\infty < r < t < s < \infty$ (3.9) becomes

$$(\Gamma_s)^{t-r} \le (\Gamma_r)^{t-s}(\Gamma_t)^{s-r}; \tag{3.34}$$

setting $d = d_t$ in (3.34), we get (3.29). □

**Theorem 3.11.** *Let $x_i, p_i, d_t \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^{n} p_i$.*

*Then for every $m \in \mathbb{N}$ and for every $s_k \in \mathbb{R}, k \in \{1, 2, 3, \ldots, m\}$, the matrices $[H((s_k + s_l)/2, s_1)]_{k,l=1}^{m}$, $[H((s_k + s_l)/2, (s_1 + s_2)/2)]_{k,l=1}^{m}$ are positive semi-definite matrices. Particularly*

$$\det\left[H\left(\frac{s_k + s_l}{2}, s_1\right)\right]_{k,l=1}^{m} \ge 0, \tag{3.35}$$

$$\det\left[H\left(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2}\right)\right]_{k,l=1}^{m} \ge 0, \tag{3.36}$$

*where $H(s, t)$ is defined by* (3.30).

*Proof.* By setting $d = d_{s_1}$ and $d = d_{(s_1+s_2)/2}$ in Theorem 3.6(i), we get the required results. □

*Remark 3.12.* We note that $H(t, t) = F_t$. So by setting $m = 2$ in (3.35), we have special case of (3.28) for $t = s_1$, $s = s_2$, and $r = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $t = s_1$, $r = s_2$, and $s = (s_1 + s_2)/2$ if $s_2 < s_1$. Similarly by setting $m = 2$ in (3.36), we have special case of (3.29) for $r = s_1$, $s = s_2$, $t = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $r = s_2$, $s = s_1$, $t = (s_1 + s_2)/2$ if $s_2 < s_1$.

Let $M_t(\mathbf{x})$ be stated as above, define $\overline{d}_t$ as

$$\overline{d}_t = (\varphi_t')^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t'(x_i)\right) = M_{t-1}(\mathbf{x}), \quad t \in \mathbb{R}. \tag{3.37}$$

The following improvement and reverse of inequality (1.6) are also valid.

**Theorem 3.13.** *Let $x_i, p_i, \overline{d_t} \in \mathbb{R}^+$ for all $i = 1, 2, \ldots, n$, $P_n = \sum_{i=1}^{n} p_i$. Let $G_t$ be defined by*

$$G_t = \varphi_t\left(\overline{d_t}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(x_i - \overline{d_t}\right)\varphi_t'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t(x_i). \tag{3.38}$$

*Then*

(i)

$$G_t \geq [K(s;t)]^{(t-r)/(s-r)}[K(r;t)]^{(s-t)/(s-r)}, \tag{3.39}$$

*for $-\infty < r < s < t < \infty$ and $-\infty < t < r < s < \infty$.*

(ii)

$$G_t \leq [K(s;t)]^{(t-r)/(s-r)}[K(r;t)]^{(s-t)/(s-r)}, \tag{3.40}$$

*for $-\infty < r < t < s < \infty$,*

*where*

$$K(s;t) = \varphi_s\left(\overline{d}_t\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(x_i - \overline{d}_t\right)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_s(x_i). \tag{3.41}$$

*Proof.* (i) By setting $d = \overline{d}_t$ in (3.9), we get (3.39) for $-\infty < r < s < t < \infty$, and similarly we can get (3.39) for the case $-\infty < t < r < s < \infty$.

(ii) For $-\infty < r < t < s < \infty$ (3.9) becomes

$$(\Gamma_s)^{t-r} \leq (\Gamma_r)^{t-s}(\Gamma_t)^{s-r}; \tag{3.42}$$

setting $d = \overline{d}_t$ in (3.42), we get (3.40). □

**Theorem 3.14.** *Let* $x_i, p_i, \overline{d}_t \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$*,* $P_n = \sum_{i=1}^{n} p_i$*.*

*Then for every* $m \in \mathbb{N}$ *and for every* $s_k \in \mathbb{R}$*,* $k \in \{1, 2, 3, \ldots, m\}$*, the matrices* $[K((s_k + s_l)/2, s_1)]_{k,l=1}^{m}$*,* $[K((s_k + s_l)/2, (s_1 + s_2)/2)]_{k,l=1}^{m}$ *are positive semi-definite matrices. Particularly*

$$\det\left[K\left(\frac{s_k + s_l}{2}, s_1\right)\right]_{k,l=1}^{m} \geq 0, \tag{3.43}$$

$$\det\left[K\left(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2}\right)\right]_{k,l=1}^{m} \geq 0, \tag{3.44}$$

*where* $K(s, t)$ *is defined by* (3.41).

*Proof.* By setting $d = \overline{d}_{s_1}$ and $d = \overline{d}_{(s_1+s_2)/2}$ in Theorem 3.6(i), we get the required results. □

*Remark 3.15.* We note that $K(t, t) = G_t$. So by setting $m = 2$ in (3.43), we have special case of (3.39) for $t = s_1,\ s = s_2,\ r = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $t = s_1,\ r = s_2$, and $s = (s_1 + s_2)/2$ if $s_2 < s_1$. Similarly by setting $m = 2$ in (3.44), we have special case of (3.40) for $r = s_1,\ s = s_2$, and $t = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $r = s_2,\ s = s_1$, and $t = (s_1 + s_2)/2$ if $s_2 < s_1$.

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