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Research Article

Boundedness of Littlewood-Paley Operators Associated with Gauss Measures

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Modeled on the Gauss measure, the authors introduce the locally doubling measure metric space $(\mathcal{X}, d, \mu)_\rho$, which means that the set $\mathcal{X}$ is endowed with a metric d and a locally doubling regular Borel measure μ satisfying doubling and reverse doubling conditions on admissible balls defined via the metric d and certain admissible function ρ. The authors then construct an approximation of the identity on $(\mathcal{X}, d, \mu)_\rho$, which further induces a Calderón reproducing formula in $L^p(\mathcal{X})$ for $p \in (1, \infty)$. Using this Calderón reproducing formula and a locally variant of the vector-valued singular integral theory, the authors characterize the space $L^p(\mathcal{X})$ for $p \in (1, \infty)$ in terms of the Littlewood-Paley g-function which is defined via the constructed approximation of the identity. Moreover, the authors also establish the Fefferman-Stein vector-valued maximal inequality for the local Hardy-Littlewood maximal function on $(\mathcal{X}, d, \mu)_\rho$. All results in this paper can apply to various settings including the Gauss measure metric spaces with certain admissible functions related to the Ornstein-Uhlenbeck operator, and Euclidean spaces and nilpotent Lie groups of polynomial growth with certain admissible functions related to Schrödinger operators.

1. Introduction

The Littlewood-Paley theory on $\mathbb{R}^n$ nowadays becomes a very important tool in harmonic analysis, partial differential equations, and other related fields. Especially, the extent to which the Littlewood-Paley theory characterizes function spaces is very remarkable; see, for example, Stein [1], Frazier, et al. [2], and Grafakos [3, 4]. Moreover, Han and Sawyer [5] established a Littlewood-Paley theory essentially on the Ahlfors 1-regular metric measure space with a quasimetric, which means that the measure of any ball is comparable with its radius. This theory was further generalized to the RD-space in [6], namely, a space of homogeneous type in the sense of Coifman and Weiss [7, 8] with an additional property that the measure satisfies the reverse doubling condition. Tolsa [9] established a Littlewood-Paley theory with the nondoubling measure μ on $\mathbb{R}^n$, which means that μ is a Radon measure on $\mathbb{R}^n$ and satisfies that $\mu(B(x,r)) \le Cr^d$ for all $x \in \mathbb{R}^n$, $r > 0$, and some fixed $d \in (0,n]$. Furthermore, these Littlewood-Paley theories were used to establish the corresponding Besov and Triebel-Lizorkin spaces on these different underlying spaces; see [5, 6, 10].

Let $(\mathbb{R}^n, |\cdot|, d\gamma)$ be the Gauss measure metric space, namely, the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the Euclidean norm $|\cdot|$ and the Gauss measure $d\gamma(x) \equiv \pi^{-n/2}e^{-|x|^2}dx$ for all $x \in \mathbb{R}^n$. Such an underlying space naturally appears in the study of the Ornstein-Uhlenbeck operator; see, for example, [11–18]. In particular, via introducing some local BMO (γ) space and Hardy space $H^1(\gamma)$ associated to admissible balls defined via the Euclidean metric and the admissible function $\rho(x) \equiv \min\{1, 1/|x|\}$ for $x \in \mathbb{R}^n$, Mauceri and Meda [12] developed a theory of singular integrals on $(\mathbb{R}^n, |\cdot|, d\gamma)_\rho$, which plays for the Ornstein-Uhlenbeck operator the same role as that the theory of classical Calderón-Zygmund operators plays for the Laplacian on classical Euclidean spaces. The results of [12] are further generalized to some kind of nondoubling measure metric spaces by Carbonaro et al. in [18, 19].

It is well known that the Gauss measure metric space is beyond the space of homogeneous type in the sense of Coifman and Weiss, a fortiori, the RD-space. To be precise, the Gauss measure is known to be only locally doubling (see [12]). In this paper, modeled on the Gauss measure, we introduce the locally doubling measure metric space $(\mathcal{X}, d, \mu)_\rho$, which means that the set $\mathcal{X}$ is endowed with a metric d and a locally doubling regular Borel measure μ satisfying the doubling and reverse doubling conditions on admissible balls defined via the metric d and certain admissible function ρ. An interesting phenomenon is that even in such a weak setting, we are able to construct an approximation of the identity on $(\mathcal{X}, d, \mu)_\rho$, which further induces a Calderón reproducing formula in $L^p(\mathcal{X})$ for $p \in (1,\infty)$. Using this Calderón reproducing formula and a locally variant of the vector-valued singular integral theory, we then characterize the space $L^p(\mathcal{X})$ for $p \in (1,\infty)$ in terms of the Littlewood-Paley g-function which is defined by the aforementioned constructed approximation of the identity. As a byproduct, we establish the Fefferman-Stein vector-valued maximal inequality for the local Hardy-Littlewood maximal function on $(\mathcal{X}, d, \mu)_\rho$, which together with the Calderón reproducing formula paves the way for further developing a theory of local Besov and Triebel-Lizorkin spaces on $(\mathcal{X}, d, \mu)_\rho$.

To be precise, motivated by [12], in Section 2, we introduce locally doubling measure metric space $(\mathcal{X}, d, \mu)_\rho$; see Definition 2.1 below. The reasonabilities of Definition 2.1 are given by Propositions 2.3 and 2.5. Some geometric properties of these spaces are also presented in Section 2.

To develop a Littlewood-Paley theory on the space $(\mathcal{X}, d, \mu)_\rho$, one of the main difficulties is the construction of appropriate approximations of the identity. In Section 3, by subtly modifying Coifman's idea in [20] (see (3.2) through (3.4) below), for any given $\ell_0 \in \mathbb{Z}$, we construct an approximation of the identity, $\{S_k\}_{k=\ell_0}^\infty$, associated to ρ; see Proposition 3.2 below. Indeed, we not only modify the operators appearing in the construction of Coifman to the setting associated with the given admissible function ρ, but also use an adjoint operator in our construction as in Tolsa [9]. Some basic estimates on such approximations of the identity are given in Lemma 3.4 and Proposition 3.5 below. We remark that, although the Gauss measure is a nondoubling measure considered by Tolsa [9], due to its advantage-locally doubling property, the construction of the corresponding approximation of the identity here does not appeal to the complicated constructions of some special doubling cubes and associated "dyadic" cubes as in [9].

In Section 4, invoking some ideas of [3, 7, 11], we establish the $L^p(\mathcal{X})$-boundedness for $p \in (1,\infty)$ and weak-$(1,1)$ estimate of local vector-valued singular integral operators on $(\mathcal{X},d,\mu)_\rho$; see Theorem 4.1 below. As a consequence, in Theorem 4.4 below, we also obtain the Fefferman-Stein vector-valued maximal function inequality with respect to the noncentered local Hardy-Littlewood maximal operator (see (2.20)).

The existence of the approximation of the identity guarantees that we obtain some Calderón reproducing formulae in $L^p(\mathcal{X})$ for $p \in (1,\infty)$ in (5.2) and Corollary 5.4, by using the methods developed in [20]. Applying such formula, we then establish the Littlewood-Paley characterization for $L^p(\mathcal{X})$ with $p \in (1,\infty)$ on $(\mathcal{X},d,\mu)_\rho$ in terms of Littlewood-Paley g-function; see Theorem 5.6 below.

Some typical examples of locally doubling measure metric spaces in Definition 2.1 are presented in Section 6. These typical examples include the aforementioned Gauss measure metric spaces with certain admissible functions related to the Ornstein-Uhlenbeck operator, and Euclidean spaces and nilpotent Lie groups of polynomial growth with certain admissible functions related to Schrödinger operators; see [21–25]. All results, especially, Theorems 4.4 and 5.6, are new even for these typical examples.

It should be pointed out that all results in Section 2 through Section 4 are exempt from using the reverse locally doubling condition (2.3); see Remark 2.2(iii) below.

We make the following conventions on notation. Let $\mathbb{N} \equiv \{1,2,\ldots\}$. For any $p \in [1,\infty]$, denote by p' the conjugate index, namely, $1/p + 1/p' = 1$. In general, we use $\mathfrak{B}$ to denote a Banach space, and $\mathcal{B}_a$ with $a > 0$ to denote a collection of admissible balls. For any set $E \subset \mathcal{X}$, denote by χ_E the characteristic function of E, and by $\#(E)$ the cardinality of E, and set $E^{\complement} \equiv \mathcal{X} \setminus E$. For any operator T, denote by T^* its dual operator. For any $a,b \in \mathbb{R}$, set $a \wedge b \equiv \min\{a,b\}$ and $a \vee b \equiv \max\{a,b\}$. Denote by C a positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ and $f \gtrsim g$ to denote $f \le Cg$ and $f \ge Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Locally Doubling Measure Metric Spaces

Let $(\mathcal{X},d,\mu)$ be a set $\mathcal{X}$ endowed with a regular Borel measure μ such that all balls defined by the metric d have finite and positive measures. Here, the regular Borel measure μ means that open sets are measurable and every set is contained in a Borel set with the same measure; see, for example, [26]. For any $x \in \mathcal{X}$ and $r > 0$, set $B(x,r) \equiv \{y \in \mathcal{X} : \ d(x,y) < r\}$. For a ball $B \subset \mathcal{X}$, we use c_B and r_B to denote its center and radius, respectively, and for $\kappa > 0$, we set $\kappa B \equiv B(c_B, \kappa r_B)$. Now we introduce the precise definition of locally doubling measure metric spaces.

Definition 2.1. A function $\rho : \mathcal{X} \to (0,\infty)$ is called admissible if for any given $\tau \in (0,\infty)$, there exists a constant $\Theta_\tau \ge 1$ such that for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \le \tau\rho(x)$,

$$[\Theta_\tau]^{-1}\rho(y) \le \rho(x) \le \Theta_\tau \rho(y). \tag{2.1}$$

For each $a > 0$, denote by $\mathcal{B}_a$ the set of all balls $B \subset \mathcal{X}$ such that $r_B \le a\rho(c_B)$. Balls in $\mathcal{B}_a$ are referred to as admissible balls with scale a. The triple $(\mathcal{X},d,\mu)_\rho$ is called a locally doubling metric space associated with admissible function ρ if for every $a > 0$, there exist constants $D_a, K_a, R_a \in (1,\infty)$ such that for all $B \in \mathcal{B}_a$,

$$\mu(2B) \le D_a \mu(B) \quad \text{(locally doubling condition)}, \tag{2.2}$$

and

$$\mu(K_a B) \ge R_a \mu(B) \quad \text{(locally reverse doubling condition)}. \tag{2.3}$$

Remark 2.2. (i) Another notion of admissible functions was introduced in [25] in the following way: a function $\rho : \mathcal{X} \to (0,\infty)$ is called admissible if there exist positive constants C and ν such that for all $x, y \in \mathcal{X}$,

$$\rho(y) \le C[\rho(x)]^{1/(1+\nu)}[\rho(x) + d(x,y)]^{\nu/(1+\nu)}. \tag{2.4}$$

By [25, Lemma 2.1], any ρ satisfying (2.4) also satisfies (2.1), while the converse may be not true; see Example 6.5 below.

(ii) Obviously, any constant function is admissible. When $\rho \equiv 1$, if $\{D_a\}_{a>0}$ has upper bound, then $(\mathcal{X}, d, \mu)_\rho$ is the space of homogeneous type in the sense of Coifman and Weiss [7, 8]; furthermore, if $\{K_a\}_{a>0}$ has upper bound and $\{R_a\}_{a>0}$ has lower bound away from 1, then $(\mathcal{X}, d, \mu)_\rho$ is just the RD-space in [6]. Conversely, any RD-space is obviously a locally doubling measure metric space with $\rho \equiv 1$.

(iii) We remark that the locally reverse doubling condition (2.3) is a mild requirement of the underlying space. Indeed, if $a > 0$ and $\mathcal{X}$ is path connected on all balls contained in $\mathcal{B}_{2a}$ and (2.2) holds for certain $\tilde{a} > 0$, then (2.3) holds; see Proposition 2.3(vi) below. Moreover, (2.3) is required only in Section 5, that is, all results in Section 2 through Section 4 are true by only assuming that ρ is an admissible function satisfying (2.1) and that $(\mathcal{X}, d, \mu)_\rho$ satisfies (2.2).

(iv) Let d be a quasimetric, which means that there exists $A_0 \ge 1$ such that for all $x, y, z \in \mathcal{X}, d(x,y) \le A_0(d(x,z) + d(z,y))$. Recall that Macías and Segovia [27, Theorem 2] proved that there exists an equivalent quasimetric $\tilde{d}$ such that all balls corresponding to $\tilde{d}$ are open in the topology induced by $\tilde{d}$, and there exist constants $\tilde{A}_0 > 0$ and $\theta \in (0,1)$ such that for all x, y, $z \in \mathcal{X}$,

$$\left|\tilde{d}(x,z) - \tilde{d}(y,z)\right| \le \tilde{A}_0\left[\tilde{d}(x,y)\right]^\theta \left[\tilde{d}(x,z) + \tilde{d}(y,z)\right]^{1-\theta}. \tag{2.5}$$

If the metric d in Definition 2.1 is replaced by $\tilde{d}$, then all results in this paper have corresponding generalization on the space $(\mathcal{X}, \tilde{d}, \mu)_\rho$. To simplify the presentation, we always assume d to be a metric in this paper.

Proposition 2.3. *Fix $a \in (0,\infty)$. Then the following hold:*

(i) *the condition* (2.2) *is equivalent to the following: there exist $K > 1$ and $\tilde{D}_a > 1$ such that for all $B \in \mathcal{B}_{(2/K)a}$, $\mu(KB) \le \tilde{D}_a \mu(B)$;*

(ii) *the condition* (2.2) *is equivalent to the following: there exist* $C_a > 1$ *and* $n_a > 0$, *which depend on* a, *such that for all* $\lambda \in (1, \infty)$ *and* $\lambda B \in \mathcal{B}_{2a}$, $\mu(\lambda B) \le C_a \lambda^{n_a} \mu(B)$;

(iii) *the following two statements are equivalent:*

(a) *there exists* $R_a > 1$ *such that for all* $B \in \mathcal{B}_a$, $\mu(2B) \ge R_a \mu(B)$;

(b) *there exist* $K_1 \in (1, 2]$ *and* $\widetilde{R}_a > 1$ *such that* $\mu(K_1 B) \ge \widetilde{R}_a \mu(B)$ *for all* $B \in \mathcal{B}_{(2/K_1)a}$;

(iv) *if* (2.3) *holds, then there exist* $\widetilde{C}_a \in (0, 1]$ *and* $\kappa_a > 0$ *such that for all* $\lambda > 1$ *and* $\lambda B \in \mathcal{B}_{aK_a}$, $\mu(\lambda B) \ge \widetilde{C}_a \lambda^{\kappa_a} \mu(B)$;

(v) *if* (2.3) *holds, then* $K_a B \setminus B \neq \emptyset$ *for all* $B \in \mathcal{B}_a$;

(vi) *if there exists* $a_0 > 1$ *such that* $a_0 B \setminus B \neq \emptyset$ *for all* $B \in \mathcal{B}_{2a}$, *and* (2.2) *holds for all* $B \in \mathcal{B}_{\widetilde{a}}$ *with* $\widetilde{a} \equiv (a/2)[1 + 4a_0]\Theta_{2a_0 a}$, *then for any given* $a_1 > a_0$, *there exists a positive constant* $\widetilde{C}$ *depending on* a_0 *and* $\widetilde{a}$ *such that for all* $B \in \mathcal{B}_a$, $\mu(a_1 B) \ge \widetilde{C} \mu(B)$.

Proof. The sufficiency of (i) follows from letting $K = 2$. To see its necessity, we consider $K \in (1, 2)$ and $K \in [2, \infty)$, respectively. When $K \in (1, 2)$, there exists a unique $N \in \mathbb{N}$ such that $K^N < 2 \le K^{N+1}$, which implies that for all $B \in \mathcal{B}_a$,

$$\mu(2B) = \mu\left(K^{N+1} \frac{2}{K^{N+1}} B\right) \le \left(\widetilde{D}_a\right)^{N+1} \mu\left(\frac{2}{K^{N+1}} B\right) \le \left(\widetilde{D}_a\right)^{1+\log_2 K} \mu(B). \tag{2.6}$$

When $K \in [2, \infty)$, for any $B \in \mathcal{B}_a$, we have $(2/K)B \in \mathcal{B}_{(2/K)a}$ and $\mu(2B) \le \widetilde{D}_a \mu((2/K)B) \le \widetilde{D}_a \mu(B)$, thus, (2.2) holds. Therefore, we obtain (i).

Now we assume (2.2) and prove the sufficiency of (ii). For any $\lambda > 1$, choose $N \in \mathbb{N}$ such that $2^{N-1} < \lambda \le 2^N$. Then, for all $\lambda B \in \mathcal{B}_{2a}$, we have $(\lambda/2^j)B \in \mathcal{B}_a$ for all $1 \le j \le N$; we therefore apply (2.2) N times and obtain $\mu(\lambda B) \le (D_a)^N \mu((\lambda/2^N)B) \le D_a \lambda^{n_a} \mu(B)$, where $n_a \equiv \log_2 D_a$. The necessity of (ii) is obvious.

Next we prove (iii). If (a) holds, then (b) follows from setting $K_1 = 2$. Conversely, if (b) holds, then for any $B \in \mathcal{B}_a$, we have $(2/K_1)B \in \mathcal{B}_{(2/K_1)a}$ and

$$\mu(2B) = \mu\left(K_1 \frac{2}{K_1} B\right) \ge \widetilde{R}_a \mu\left(\frac{2}{K_1} B\right) \ge \widetilde{R}_a \mu(B), \tag{2.7}$$

which implies (a).

To prove (iv), for any $\lambda > 1$, there exists a unique $N \in \mathbb{N}$ such that $(K_a)^{N-1} < \lambda \le (K_a)^N$. This combined with the fact that $(\lambda/K_a)B \in \mathcal{B}_a$ implies that

$$\mu(\lambda B) = \mu\left((K_a)^{N-1} \frac{\lambda}{(K_a)^{N-1}} B\right) \ge (R_a)^{N-1} \mu\left(\frac{\lambda}{(K_a)^{N-1}} B\right) \ge (R_a)^{\log_{K_a} \lambda - 1} \mu(B) \equiv \widetilde{C}_a \lambda^{\kappa_a} \mu(B), \tag{2.8}$$

where $\widetilde{C}_a \equiv (R_a)^{-1}$ and $\kappa_a \equiv \log_{K_a} R_a$. Thus, (iv) holds.

Notice that (v) is obvious. To show (vi), without loss of generality, we may assume that $a_1 \in (a_0, 2a_0]$. Set $\sigma \equiv (a_1 - a_0)/(1 + a_0)$. Observe that $0 < \sigma < 1$. Thus, for any $B \in \mathcal{B}_a$, we have $(1 + \sigma)B \in \mathcal{B}_{2a}$ and $a_0(1 + \sigma)B \setminus (1 + \sigma)B \neq \emptyset$. Choose $y \in a_0(1 + \sigma)B \setminus (1 + \sigma)B$. It is easy to check that $B(y,\sigma r_B)\cap B=\emptyset$ and $B(y,\sigma r_B)\subset a_1B\subset B(y,[\sigma+2a_0(1+\sigma)]r_B)$. Notice that $r_B\le a\rho(c_B)\le a\Theta_{2a_0a}\rho(y)$ and $B(y,[\sigma+2a_0(1+\sigma)]r_B)\in\mathcal{B}_{2\tilde{a}}$. This combined with (2.2) and (i) of Proposition 2.3 yields that

$$\begin{aligned}\mu(a_1B)&\ge\mu(B)+\mu(B(y,\sigma r_B)\ )\\&\ge\mu(B)+[C_{\tilde{a}}]^{-1}\left[\frac{\sigma}{\sigma+2a_0(1+\sigma)}\right]^{n_{\tilde{a}}}\mu(B(y,[\sigma+2a_0(1+\sigma)]r_B))\\&\ge\mu(B)+[C_{\tilde{a}}]^{-1}\left[\frac{\sigma}{\sigma+2a_0(1+\sigma)}\right]^{n_{\tilde{a}}}\mu(a_1B),\end{aligned}\tag{2.9}$$

which further implies that $\mu(a_1B)\ge\widetilde{C}\mu(B)$ with $\widetilde{C}\equiv\left\{1-[C_{\tilde{a}}]^{-1}[\sigma/(\sigma+2a_0(1+\sigma))]^{n_{\tilde{a}}}\right\}^{-1}>1$. This finishes the proof of (vi), and hence the proof of Proposition 2.3. □

Remark 2.4. (i) By Proposition 2.3(i), there is no essential difference whether we define the locally doubling condition (2.2) by using $2B$ or KB for some constant $K>0$.

(ii) The assumption $K_1\in(1,2]$ in (b) of Proposition 2.3(iii) cannot be replaced by $K_1\in(1,\infty)$; see Proposition 2.5 below. Therefore, in Definition 2.1, it is more reasonable to require (2.3) rather than (a) of Proposition 2.3(iii).

In the following Proposition 2.5, we temporarily consider the Gauss measure space $(\mathbb{R}^n,|\cdot|,\gamma)_\rho$, where ρ is given by $\rho(x)\equiv\min\{1,1/|x|\}$ and $d\gamma(x)\equiv\pi^{-n/2}e^{-|x|^2}dx$ for all $x\in\mathbb{R}^n$. In this case, for any ball B centered at c_B and is of radius r_B, we have $B\equiv\{x\in\mathbb{R}^n:\ |x-c_B|<r_B\}$, and moreover, $B\in\mathcal{B}_a$ if and only if $r_B\le a\rho(c_B)$; see [12].

Proposition 2.5. *Let $a\in(0,\infty)$ and $(\mathbb{R}^n,|\cdot|,\gamma)_\rho$ be the Gauss measure space. Then,*

(a) *there exist positive constants $K_a>1$ and $C_a>1$, which depend on a, such that for all $B\in\mathcal{B}_a$, $\gamma(K_aB)\ge C_a\gamma(B)$;*

(b) *there exists a sequence of balls, $\{B_j\}_{j\in\mathbb{N}}\subset\mathcal{B}_a$, such that $\lim_{j\to\infty}(\gamma(2B_j)/\gamma(B_j))=1$.*

Proof. Recall that for all $B\in\mathcal{B}_a$ and $x\in B$, it was proved in [12, Proposition 2.1], that $e^{-2a-a^2}\le e^{|c_B|^2-|x|^2}\le e^{2a}$. From this, it follows that for any $K_a>0$,

$$\begin{aligned}\gamma(B)&=\int_B\pi^{-n/2}e^{-|x|^2}dx\le\pi^{-n/2}e^{-|c_B|^2+2a}|B|,\\\gamma(K_aB)&=\int_{K_aB}\pi^{-n/2}e^{-|x|^2}dx\ge\pi^{-n/2}e^{-|c_B|^2-2a-a^2}(K_a)^n|B|,\end{aligned}\tag{2.10}$$

where and in what follows, we denote by $|B|$ the Lebesgue measure of the ball B. Thus, $\gamma(K_aB)\ge(K_a)^ne^{-4a-a^2}\gamma(B)$. Hence, (a) holds by choosing $K_a>e^{(4a+a^2)/n}$.

To show (b), for simplicity, we may assume $n = 1$. Consider the ball $B_y \equiv B(y, e^{-y})$, where $y \geq 1$ such that $e^{-y} \leq a/y$. Thus, $B_y \in \mathcal{B}_a$ for any such chosen y. A simple calculation yields that $\lim_{y\to\infty}\gamma(B_y) = 0$. Therefore, using the L$'$-Hospital rule, we obtain

$$\begin{aligned}
\lim_{y\to\infty}\frac{\gamma(2B_y)}{\gamma(B_y)} &= \lim_{y\to\infty}\frac{\int_{y-2e^{-y}}^{y+2e^{-y}} e^{-|x|^2}dx}{\int_{y-e^{-y}}^{y+e^{-y}} e^{-|x|^2}dx}\\
&= \lim_{y\to\infty}\frac{(1-2e^{-y})e^{-(y+2e^{-y})^2} - (1+2e^{-y})e^{-(y-2e^{-y})^2}}{(1-e^{-y})e^{-(y+e^{-y})^2} - (1+e^{-y})e^{-(y-e^{-y})^2}}\\
&= \lim_{y\to\infty} e^{-3e^{-2y}-2ye^{-y}}\frac{(1-2e^{-y}) - (1+2e^{-y})e^{8ye^{-y}}}{(1-e^{-y}) - (1+e^{-y})e^{4ye^{-y}}} = 1,
\end{aligned} \tag{2.11}$$

which implies the desired result of (b). This finishes the proof of Proposition 2.5. □

Next we present some properties concerning the underlying space $(\mathcal{X}, d, \mu)_\rho$. In what follows, for any $x, y \in \mathcal{X}$ and $\delta > 0$, set $V_\delta(x) \equiv \mu(B(x,\delta))$ and $V(x,y) \equiv \mu(B(x, d(x,y)))$.

Proposition 2.6. *Let $\tau > 0$, $\eta > 0$, $a > 0$, and $B \in \mathcal{B}_a$. Then the following hold:*

(a) *for any given $\tau' \in (0,\tau]$, if $x, y \in \mathcal{X}$ satisfy $d(x,y) \leq \tau'\rho(x)$, then $d(x,y) \leq \tau'\Theta_\tau\rho(y)$,*

$$V_{\tau'\rho(x)}(x) \sim V_{\tau'\rho(y)}(y) \sim V_{\tau'\rho(y)}(x) \sim V_{\tau'\rho(x)}(y), \tag{2.12}$$

and $V(x,y) \sim V(y,x)$ with equivalent constants depending only on τ;

(b) *for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \leq \eta\rho(x)$,*

$$V_{\tau\rho(x)}(x) + V(x,y) \sim V_{\tau\rho(y)}(y) + V(x,y) \sim \mu(B(x, \tau\rho(x) + d(x,y))), \tag{2.13}$$

with equivalent constants depending on η and τ;

(c) *$\int_{d(z,x)<r} d(z,x)^a (1/V(z,x))\, d\mu(z) \leq Cr^a$ uniformly in $x \in \mathcal{X}$ and $r \in (0, \tau\rho(x)]$;*

(d) *for any ball B' satisfying $B' \cap B \neq \emptyset$ and $r_{B'} \leq \tau r_B$, $B' \in \mathcal{B}_{\tau a\Theta_{(1+\tau)a}}$;*

(e) *there exists a positive constant $D_{a,\tau}$ depending only on a and τ such that if $\mathcal{B}' \cap B \neq \emptyset$ and $r_{B'} \leq \tau r_B$, then $\mu(B') \leq D_{a,\tau}\mu(B)$.*

Proof. We first show (a). For all $\tau' \in (0,\tau]$, if $d(x,y) \leq \tau'\rho(x)$, then $d(x,y) \leq \tau\Theta_\tau\rho(y)$ by (2.1). Since

$$B(x, \tau'\rho(x)) \subset B(y, 2\tau'\rho(x)) \subset B(y, 2\tau'\Theta_\tau(\rho(y))), \tag{2.14}$$

by (2.2), we obtain $V_{\tau'\rho(x)}(x) \leq D_{\Theta_\tau} V_{\tau'\rho(y)}(y)$. A similar argument together with (2.1) and (2.2) shows the rest estimates of (a) as well (b). The details are omitted.

To prove (c), by (a) and (2.2), we obtain

$$\int_{d(z,x)<r} \frac{d(z,x)^a}{V(z,x)} d\mu(z) \sim \int_{d(z,x)<r} \frac{d(z,x)^a}{\mu(B(x,d(z,x)))} d\mu(z)$$
$$\leq \sum_{j=0}^{\infty} \int_{2^{-j-1}r \leq d(z,x) < 2^{-j}r} \frac{(2^{-j}r)^a}{\mu(B(x,2^{-j-1}r))} d\mu(z) \tag{2.15}$$
$$\leq \sum_{j=0}^{\infty} 2^{-ja} D_\tau r^a \lesssim r^a,$$

which implies (c).

To see (d), by $B \cap B' \neq \emptyset$ and $r_{B'} \leq \tau r_B$, we have $d(c_{B'}, c_B) < r_B + r_{B'} < (1+\tau) r_B$, which combined with (2.1) and the fact $B \in \mathcal{B}_a$ implies that

$$r_{B'} \leq \tau r_B \leq \tau a \rho(c_B) \leq \tau a \Theta_{(1+\tau)a} \rho(c_{B'}). \tag{2.16}$$

Thus, (d) holds.

To show (e), notice that $B' \subset B(c_B, (2\tau+1) r_B) \in \mathcal{B}_{(2\tau+1)a}$. Choose $N \in \mathbb{N}$ such that $2^{N-1} < 2\tau+1 \leq 2^N$. Then, by (2.2), we obtain $\mu(B') \leq \mu(2^N B) \leq [D_{(2\tau+1)a}]^N \mu(B)$, which implies (e) by setting $D_{a,\tau} \equiv [D_{(2\tau+1)a}]^{1+\log_2(2\tau+1)}$. This finishes the proof of Proposition 2.6. □

A geometry covering lemma on $(\mathcal{X}, d, \mu)_\rho$ is as follows.

Lemma 2.7. *Let ρ be an admissible function. For any $\lambda > 0$, there exists a sequence of balls, $\{B(x_j, \lambda\rho(x_j))\}_j$, such that*

(i) *$\mathcal{X} = \bigcup_j B_j$, where $B_j \equiv B(x_j, \lambda\rho(x_j))$;*

(ii) *the balls $\{\widetilde{B}_j\}_j$ are pairwise disjoint, where $\widetilde{B}_j \equiv B(x_j, ([\Theta_\lambda]^2+1)^{-1}\lambda\rho(x_j))$;*

(iii) *for any $\tau > 0$, there exists a positive constant M depending on τ and λ such that any point $x \in \mathcal{X}$ belongs to no more than M balls of $\{\tau B_j\}_j$.*

Proof. Let $\mathcal{O}$ be the maximal set of balls, $\widetilde{B}_j \equiv B(x_j, ([\Theta_\lambda]^2+1)^{-1}\lambda\rho(x_j)) \subset \mathcal{X}$, such that for all $k \neq j$, $\widetilde{B}_j \cap \widetilde{B}_k = \emptyset$. The existence of such a set is guaranteed by the Zorn lemma. We claim that $\mathcal{O}$ is at most countable.

Indeed, we choose $x_0 \in \mathcal{X}$, and set $\mathcal{X}_N \equiv B(x_0, N\rho(x_0))$ and $J_N \equiv \{j : \widetilde{B}_j \cap \mathcal{X}_N \neq \emptyset\}$. For any $j \in J_N$, denote by w_j an arbitrary point in $\widetilde{B}_j \cap \mathcal{X}_N$. From (2.1), it follows that $\rho(x_j) \sim \rho(w_j) \sim \rho(x_0)$ with constants depending only on N and λ; thus, for all $z \in \widetilde{B}_j$,

$$d(z, x_0) \leq d(z, x_j) + d(x_j, w_j) + d(w_j, x_0) \leq C_{\lambda,N}\rho(x_0), \tag{2.17}$$

for some positive constant $C_{\lambda,N}$. This implies that $\bigcup_{j\in J_N} \widetilde{B}_j \subset B(x_0, C_{\lambda,N}\rho(x_0))$. Likewise, there exists a positive constant $\widetilde{C_{\lambda,N}}$ such that for all $j \in J_N$, $B(x_0, C_{\lambda,N}\rho(x_0)) \subset \widetilde{C_{\lambda,N}}\widetilde{B}_j$. By this and (2.2), we obtain

$$\#(J_N)\mu(B(x_0, C_{\lambda,N}\rho(x_0))) \lesssim \sum_{j\in J_N} \mu\left(\widetilde{B}_j\right) \sim \mu\left(\bigcup_{j\in J_N} \widetilde{B}_j\right) \lesssim \mu(B(x_0, C_{\lambda,N}\rho(x_0))), \tag{2.18}$$

and hence $\#(J_N) \lesssim 1$. This combined with the fact that $\mathcal{X} = \bigcup_{N=1}^{\infty} \mathcal{X}_N$ implies the claim.

For any $z \in \mathcal{X}$, by the maximal property of $\mathcal{I}$, there exists some j such that

$$B\left(z, \left([\Theta_\lambda]^2 + 1\right)^{-1} \lambda\rho(z)\right) \cap B\left(x_j, \left([\Theta_\lambda]^2 + 1\right)^{-1} \lambda\rho(x_j)\right) \neq \emptyset, \tag{2.19}$$

which combined with (2.1) implies that $\rho(z) \le [\Theta_\lambda]^2\rho(x_j)$ and $d(z, x_j) < \lambda\rho(x_j)$. This proves (i).

For any $z \in \mathcal{X}$, set $J(z) \equiv \{j : z \in \tau B_j\}$. By (2.1), $\rho(x_j) \sim \rho(z)$ for all $j \in J(z)$. Then by an argument similar to the proof for the above claim, we obtain (iii), which completes the proof of Lemma 2.7. □

For any $a > 0$, we consider the *noncentered local Hardy-Littlewood maximal operator* $\mathcal{M}_a$ on $(\mathcal{X}, d, \mu)_\rho$, which is defined by setting, for all locally integrable functions f and $x \in \mathcal{X}$,

$$\mathcal{M}_a f(x) \equiv \sup_{B\in\mathcal{B}_a(x)} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y), \tag{2.20}$$

where $\mathcal{B}_a(x)$ is the collection of balls $B \in \mathcal{B}_a$ containing x. Observe that if $(\mathcal{X}, d, \mu)_\rho$ is the Gauss measure metric space and $\rho(x) \equiv \min\{1, 1/|x|\}$, then (2.20) is exactly the noncentered local Hardy-Littlewood maximal function introduced in [12, (3.1)]; see also [18, (7.1)].

Theorem 2.8. (i) *For any $a > 0$, the operator $\mathcal{M}_a$ in* (2.20) *is of weak type* (1,1) *and bounded on $L^p(\mathcal{X})$ for $p \in (1, \infty]$.*

(ii) *For any locally integrable function f and almost all $x \in \mathcal{X}$,*

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| d\mu(y) = 0. \tag{2.21}$$

Proof. A similar argument as in [26, Theorem 2.2] together with (2.2) shows (i). Following the procedure in [26, Theorem 1.8], we obtain that for almost all $x \in \mathcal{X}$,

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y) = f(x), \tag{2.22}$$

which together with an argument similar to that of the Euclidean case (see [28]) yields (ii). This finishes the proof of Theorem 2.8. □

3. Approximations of the Identity

Motivated by [6, 20], we introduce the following inhomogeneous approximation of the identity on the locally doubling measure metric space $(\mathcal{X}, d, \mu)_\rho$.

Definition 3.1. Let $\ell_0 \in \mathbb{Z}$. A sequence of bounded linear operators, $\{S_k\}_{k=\ell_0}^\infty$, on $L^2(\mathcal{X})$ is called an ℓ_0-approximation of the identity on $(\mathcal{X}, d, \mu)_\rho$ (for short, ℓ_0-AOTI) if there exist positive constants C_1 and C_2 (may depend on ℓ_0) such that for all $k \geq \ell_0$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of S_k, is a measurable function from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{C}$ satisfying that

(i) $S_k(x, y) = 0$ if $d(x, y) \geq C_1 2^{-k}[\rho(x) \wedge \rho(y)]$ and $|S_k(x, y)| \leq C_2(1/(V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)))$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq C_2(d(x, x')/(2^{-k}\rho(x)))(1/(V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)))$ if $d(x, x') \leq (C_1 \vee 1)2^{-k+1}\rho(x)$;

(iii) $|S_k(x, y) - S_k(x, y')| \leq C_2(d(y, y')/(2^{-k}\rho(y)))(1/(V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)))$ if $d(y, y') \leq (C_1 \vee 1)(2^{-k+1}\rho(y))$;

(iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C_2(d(x, x')/(2^{-k}\rho(x)))(d(y, y')/(2^{-k}\rho(y))(1/(V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)))$ if $d(x, x') \leq (C_1 \vee 1)2^{-k+1}\rho(x)$ and $d(y, y') \leq (C_1 \vee 1)2^{-k+1}\rho(y)$;

(v) $\int_{\mathcal{X}} S_k(x, w)\,d\mu(w) = 1 = \int_{\mathcal{X}} S_k(w, y)\,d\mu(w)$ for all $k \geq \ell_0$.

The existence of the approximation of the identity on $(\mathcal{X}, d, \mu)_\rho$ follows from a subtle modification on the construction of Coifman in [20, Lemma 2.2] (see also [6]). Different from [20], here we define $S_k = M_k T_k W_k T_k^* M_k$, where T_k is an integral operator whose kernel is defined via the admissible function ρ, and M_k and W_k are the operators of multiplication by $(1/T_k 1)$ and $[T_k^*(1/T_k 1)]^{-1}$, respectively; see (3.2), (3.3), and (3.4) below. We remark that the idea of using the dual operator T_k^* here was used before by Tolsa [9].

Proposition 3.2. *For any given $\ell_0 \in \mathbb{Z}$, there exists a nonnegative symmetric ℓ_0-AOTI $\{S_k\}_{k=\ell_0}^\infty$, where the symmetric means that $S_k(x, y) = S_k(y, x)$ for all $k \geq \ell_0$ and $x, y \in \mathcal{X}$. Moreover, there exists a positive constant C_3 (may depend on ℓ_0) such that for all $k \geq \ell_0$ and $x, y \in \mathcal{X}$ satisfying $d(x, y) \leq 2^{-k}\rho(x)$,*

$$C_3 V_{2^{-k}\rho(x)}(x) S_k(x, y) \geq 1. \tag{3.1}$$

Proof. Let h be a differentiable radial function on $\mathbb{R}$ satisfying $\chi_{[0,a_0]} \leq h \leq \chi_{[0,2a_0)}$ with $a_0 \equiv 2\Theta_{2^{-\ell_0}}$. For any $k \geq \ell_0$, $f \in L^1_{\text{loc}}(\mathcal{X})$, and $u \in \mathcal{X}$, define

$$T_k(f)(u) \equiv \int_{\mathcal{X}} h\left(\frac{d(u, w)}{2^{-k}\rho(w)}\right) f(w)\,d\mu(w), \tag{3.2}$$

and its dual operator

$$T_k^*(f)(u) \equiv \int_{\mathcal{X}} h\left(\frac{d(u, w)}{2^{-k}\rho(u)}\right) f(w)\,d\mu(w). \tag{3.3}$$

Then, for all $x,y \in \mathcal{X}$, set

$$S_k(x,y) \equiv \frac{1}{T_k 1(x)} \left\{ \int_{\mathcal{X}} h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) \frac{1}{T_k^*(1/T_k 1)(z)} h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) d\mu(z) \right\} \frac{1}{T_k 1(y)}. \tag{3.4}$$

It is easy to see that S_k is nonnegative, $S_k(x,y) = S_k(y,x)$, and $\int_{\mathcal{X}} S_k(x,y)d\mu(y) = 1$. The support condition of h together with (2.1) and (2.2) implies that for any $u \in \mathcal{X}$,

$$T_k 1(u) \sim V_{2^{-k}\rho(u)}(u) \sim T_k^* 1(u), \tag{3.5}$$

with constants depending on ℓ_0.

If $S_k(x,y) \neq 0$, then by (3.4), there exists $z \in \mathcal{X}$ such that $d(x,z) \le a_0 2^{-k+1}\rho(z)$ and $d(z,y) \le a_0 2^{-k+1}\rho(z)$, which together with (2.1) implies that

$$d(x,y) \le a_0 [\Theta_{a_0 2^{-\ell_0+1}}] 2^{-k+2} [\rho(x) \wedge \rho(y)], \tag{3.6}$$

and that the integral domain in (3.4) is $B(x, a_0[\Theta_{a_0 2^{-\ell_0+1}}]2^{-k+1}\rho(x))$.

For any $z \in B(x, a_0[\Theta_{a_0 2^{-\ell_0+1}}]2^{-k+1}\rho(x))$, by (3.5), (2.2), the support condition of h, and Proposition 2.6(a), we obtain

$$\begin{gathered} T_k^*\left(\frac{1}{T_k 1}\right)(z) = \int_{\mathcal{X}} h\left(\frac{d(z,w)}{2^{-k}\rho(z)}\right) \frac{1}{T_k 1(w)}\, d\mu(w) \gtrsim \int_{B(z,2^{-k}\rho(z))} \frac{1}{V_{2^{-k}\rho(w)}(w)}\, d\mu(w) \gtrsim 1, \\ T_k^*\left(\frac{1}{T_k 1}\right)(z) \lesssim \int_{B(z,a_0 2^{-k+1}\rho(z))} \frac{1}{T_k 1(w)}\, d\mu(w) \lesssim 1, \end{gathered} \tag{3.7}$$

which further implies that for all $z \in B(x, a_0[\Theta_{a_0 2^{-\ell_0+1}}]2^{-k+1}\rho(x))$,

$$T_k^*\left(\frac{1}{T_k 1}\right)(z) \sim 1. \tag{3.8}$$

By (3.5), (3.8), Proposition 2.6(a), and the fact that the integral domain in (3.4) is $B(x, a_0[\Theta_{a_0 2^{-\ell_0+1}}]2^{-k+1}\rho(x))$, we obtain

$$0 \le S_k(x,y) \lesssim \frac{1}{V_{2^{-k}\rho(x)}(x)} \lesssim \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \tag{3.9}$$

Thus, (i) of Definition 3.1 holds with positive constants C_1 and C_2 depending only on ℓ_0.

To show (3.1), by the fact $h \ge \chi_{[0,a_0]}$ and (3.8), we obtain that when $d(x,y) \le 2^{-k}\rho(x)$,

$$S_k(x,y) \gtrsim \frac{1}{T_k 1(x)} \left\{ \int_{\mathcal{X}} \left[\chi_{\{d(x,z)\le a_0^{-1}2^{-k}\rho(z)\}}(z)\right] h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) d\mu(z) \right\} \frac{1}{T_k 1(y)}. \tag{3.10}$$

When $d(x,y) \le 2^{-k}\rho(x)$ and $d(x,z) \le a_0^{-1}2^{-k}\rho(z)$, by (2.1) and the fact that $a_0 > 1$, we have $[\Theta_{2^{-\ell_0}}]^{-1}\rho(z) \le \rho(x) \le \Theta_{2^{-\ell_0}}\rho(z)$ and

$$d(y,z) \le d(y,x) + d(x,z) \le 2^{-k}\rho(x) + a_0^{-1}2^{-k}\Theta_{2^{-\ell_0}}\rho(x) \le 2^{-k+1}\rho(x) \le a_0 2^{-k}\rho(z), \tag{3.11}$$

which implies that $h(d(z,y)/2^{-k}\rho(z)) = 1$. Inserting this into (3.10) and then using (3.5), we obtain (3.1).

Now we show that S_k satisfies the desired regularity in the first variable when $d(x,x') \le (C_1 \vee 1)2^{-k+1}\rho(x)$. Notice that in this case, $S_k(x,y) - S_k(x',y) \neq 0$ implies that $d(x,y) \lesssim 2^{-k}\rho(x)$, and hence $\rho(y) \sim \rho(x) \sim \rho(x')$ by (2.1). Write

$$\begin{aligned}
&S_k(x,y) - S_k(x',y)\\
&\quad= \left[\frac{1}{T_k1(x)} - \frac{1}{T_k1(x')}\right]\left\{\int_{\mathcal{X}} h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right)\frac{1}{T_k^*(1/T_k1)(z)}h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right)d\mu(z)\right\}\frac{1}{T_k1(y)}\\
&\qquad+ \frac{1}{T_k1(x')}\left\{\int_{\mathcal{X}}\left[h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(x',z)}{2^{-k}\rho(z)}\right)\right]\frac{1}{T_k^*(1/T_k1)(z)}h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right)d\mu(z)\right\}\\
&\qquad\times\frac{1}{T_k1(y)} \equiv Z_1 + Z_2.
\end{aligned} \tag{3.12}$$

If $d(x,x') \le (C_1 \vee 1)2^{-k+1}\rho(x)$, then by the mean value theorem, (2.1), (2.2), (3.5), and Proposition 2.6(a),

$$\begin{aligned}
\left|\frac{1}{T_k1(x)} - \frac{1}{T_k1(x')}\right| &\le \frac{1}{T_k1(x)T_k1(x')}\int_{\mathcal{X}}\left|h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(x',z)}{2^{-k}\rho(z)}\right)\right|d\mu(z)\\
&\lesssim \frac{1}{V_{2^{-k}\rho(x)}(x)V_{2^{-k}\rho(x')}(x')}\int_{\substack{d(x,z)\le a_0 2^{-k+1}\rho(z)\\ \text{or } d(x',z)\le a_0 2^{-k+1}\rho(z)}}\frac{d(x,x')}{2^{-k}\rho(z)}d\mu(z)\\
&\lesssim \frac{d(x,x')}{2^{-k}\rho(x)}\frac{1}{V_{2^{-k}\rho(x)}(x)}.
\end{aligned} \tag{3.13}$$

By this, (3.5), (3.8), $\rho(x') \sim \rho(x)$, $d(x,y) \lesssim 2^{-k}\rho(x)$, and Proposition 2.6(a), we obtain

$$Z_1 \lesssim \frac{d(x,x')}{2^{-k}\rho(x)}\frac{1}{V_{2^{-k}\rho(x)}(x)} \sim \frac{d(x,x')}{2^{-k}\rho(x)}\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \tag{3.14}$$

Now we estimate Z_2. If $Z_2 \neq 0$, from the support condition of h and Proposition 2.6(a), we deduce that $d(x,z) \le C2^{-k}\rho(x)$ for some positive constant C that depends on ℓ_0. Therefore, by the mean value theorem and (3.8),

$$Z_2 \lesssim \frac{1}{T_k1(x')}\left\{\int_{d(x,z)\le C2^{-k}\rho(x)}\frac{d(x,x')}{2^{-k}\rho(z)}h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right)d\mu(z)\right\}\frac{1}{T_k1(y)}, \tag{3.15}$$

which combined with (2.1), (3.5), $d(x,y) \lesssim 2^{-k}\rho(x)$, $d(x,x') \le C2^{-k}\rho(x)$, and Proposition 2.6(a) further implies that

$$Z_2 \lesssim \frac{d(x,x')}{2^{-k}\rho(x)} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \tag{3.16}$$

Combining the estimates of Z_1 and Z_2 yields that S_k satisfies (ii) of Definition 3.1.

We finally prove that S_k satisfies (iv) of Definition 3.1 if $d(x,x') \le (C_1 \vee 1)2^{-k+1}\rho(x)$ and $d(y,y') \le (C_1 \vee 1)2^{-k+1}\rho(y)$. In this case, $[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')] \neq 0$ implies that $d(x,y) \lesssim 2^{-k}\rho(x)$ and hence $\rho(x') \sim \rho(x) \sim \rho(y) \sim \rho(y')$ by (2.1). Write

$$\begin{aligned}
&[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')] \\
&\quad = \left[\frac{1}{T_k1(x)} - \frac{1}{T_k1(x')}\right] \left\{ \int_{\mathcal{X}} h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) \frac{1}{T_k^*(1/T_k1)(z)} h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) d\mu(z) \right\} \\
&\qquad \times \left[\frac{1}{T_k1(y)} - \frac{1}{T_k1(y')}\right] + \left[\frac{1}{T_k1(x)} - \frac{1}{T_k1(x')}\right] \frac{1}{T_k1(y')} \\
&\qquad \times \left\{ \int_{\mathcal{X}} h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) \frac{1}{T_k^*(1/T_k1)(z)} \left[h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(z,y')}{2^{-k}\rho(z)}\right) \right] d\mu(z) \right\} \\
&\qquad + \frac{1}{T_k1(x')} \left\{ \int_{\mathcal{X}} \left[h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(x',z)}{2^{-k}\rho(z)}\right) \right] \frac{1}{T_k^*(1/T_k1)(z)} h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) d\mu(z) \right\} \\
&\qquad \times \left[\frac{1}{T_k1(y)} - \frac{1}{T_k1(y')}\right] + \frac{1}{T_k1(x')} \frac{1}{T_k1(y')} \\
&\qquad \times \left\{ \int_{\mathcal{X}} \left[h\left(\frac{d(x,z)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(x',z)}{2^{-k}\rho(z)}\right) \right] \frac{1}{T_k^*(1/T_k1)(z)} \right. \\
&\qquad\qquad \left. \times \left[h\left(\frac{d(z,y)}{2^{-k}\rho(z)}\right) - h\left(\frac{d(z,y')}{2^{-k}\rho(z)}\right) \right] d\mu(z) \right\} \\
&\quad \equiv Z_3 + Z_4 + Z_5 + Z_6.
\end{aligned} \tag{3.17}$$

By (3.13), (3.5), (3.8), (3.6), the fact $\rho(x') \sim \rho(x) \sim \rho(y) \sim \rho(y')$, and Proposition 2.6(a), we obtain

$$\begin{aligned}
Z_3 &\lesssim \frac{d(x,x')}{2^{-k}\rho(x)} \frac{1}{V_{2^{-k}\rho(x)}(x)} \frac{d(y,y')}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(y)}(y)} \mu\left(B\left(x, a_0\Theta_{a_0 2^{-\ell_0+1}} 2^{-k}\rho(x)\right)\right) \\
&\lesssim \frac{d(x,x')}{2^{-k}\rho(x)} \frac{d(y,y')}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}.
\end{aligned} \tag{3.18}$$

The estimates for Z_4 through Z_5 are similar to those of Z_3 or Z_2 and hence omitted. Therefore, S_k satisfies (iv) of Definition 3.1. This finishes the proof of Proposition 3.2. □

Remark 3.3. (a) It should be mentioned that (3.1) is crucial in establishing the vector-valued Fefferman-Stein maximal function inequality; see Theorem 4.4 below.

(b) Let $\ell_0 \in \mathbb{Z}$. Given any $\tau > 0$, if $\{S_k\}_{k=\ell_0}^{\infty}$ satisfy (i) and (ii) of Definition 3.1, then by (2.1) and (2.2), we have that there exists a positive constant C (depending on τ) such that for all $k \geq \ell_0$ and all $d(x,x') \leq \tau 2^{-k}\rho(x)$,

$$\left|S_k(x,y) - S_k(x',y)\right| \leq C\frac{d(x,x')}{2^{-k}\rho(x)}\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \tag{3.19}$$

If $\{S_k\}_{k=\ell_0}^{\infty}$ satisfy (i) and (iii) of Definition 3.1, then a symmetric estimate as in (3.19) holds for the second variable. Analogously, if $\{S_k\}_{k=\ell_0}^{\infty}$ satisfy (i) through (iv) of Definition 3.1, then for all $d(x,x') \leq \tau 2^{-k}\rho(x)$ and $d(y,y') \leq \tau 2^{-k}\rho(y)$,

$$\begin{aligned} &\left|\left[S_k(x,y) - S_k(x,y')\right] - \left[S_k(x',y) - S_k(x',y')\right]\right| \\ &\quad \leq C\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(y,y')}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \end{aligned} \tag{3.20}$$

The following technical lemma in some sense illustrates that the composition of two ℓ_0-AOTI's is still an ℓ_0-AOTI (except Definition 3.1(v)).

Lemma 3.4. *Let $\ell_0 \in \mathbb{Z}$ and let $\{S_k\}_{k=\ell_0}^{\infty}$ and $\{E_k\}_{k=\ell_0}^{\infty}$ be two ℓ_0-AOTI's. Set $D_{\ell_0} \equiv S_{\ell_0}$, $Q_{\ell_0} \equiv E_{\ell_0}$, $D_k \equiv S_k - S_{k-1}$, and $Q_k \equiv E_k - E_{k-1}$ for $k > \ell_0$. Then for any $\eta, \sigma, \delta \in (0,1)$ and $\sigma + \delta \in (0,1]$, there exists a positive constant C, depending on η, σ, δ, C_1, and C_2, such that the kernel of D_kQ_j, which is still denoted by D_kQ_j, satisfies that for all k, $j \geq \ell_0$,*

(i) *if $D_kQ_j(x,y) \neq 0$, then $d(x,y) \leq C_4 2^{-(k\wedge j)}[\rho(x) \wedge \rho(y)]$ with $C_4 \equiv 4C_1\Theta_{C_1 2^{-\ell_0+1}}$;*

(ii) *for all x, $y \in \mathcal{X}$,*

$$\left|D_kQ_j(x,y)\right| \leq C2^{-|k-j|}\frac{1}{V_{2^{-(k\wedge j)}\rho(x)}(x) + V_{2^{-(k\wedge j)}\rho(y)}(y)}; \tag{3.21}$$

(iii) *for all x, y, $y' \in \mathcal{X}$ satisfying $d(y,y') \leq (C_4 \vee 1)2^{-(k\wedge j)+1}\rho(y)$,*

$$\begin{aligned} &\left|D_kQ_j(x,y) - D_kQ_j(x,y')\right| \\ &\quad \leq C2^{-|k-j|(1-\eta)}\left(\frac{d(y,y')}{2^{-(k\wedge j)}\rho(y)}\right)^{\eta}\frac{1}{V_{2^{-(k\wedge j)}\rho(x)}(x) + V_{2^{-(k\wedge j)}\rho(y)}(y)}; \end{aligned} \tag{3.22}$$

(iv) *for all* $x,y,x' \in \mathcal{X}$ *satisfying* $d(x,x') \le (C_4 \vee 1)2^{-(k\wedge j)+1}\rho(x)$,

$$\begin{aligned}&\left|D_kQ_j(x,y) - D_kQ_j(x',y)\right| \\ &\quad\le C2^{-|k-j|(1-\eta)}\left(\frac{d(x,x')}{2^{-(k\wedge j)}\rho(x)}\right)^{\eta}\frac{1}{V_{2^{-(k\wedge j)}\rho(x)}(x) + V_{2^{-(k\wedge j)}\rho(y)}(y)};\end{aligned} \tag{3.23}$$

(v) *for all* $x,\ y,\ x',\ y' \in \mathcal{X}$ *satisfying* $d(x,x') \le (C_4 \vee 1)2^{-(k\wedge j)+1}\rho(x)$ *and* $d(y,y') \le (C_4 \vee 1)2^{-(k\wedge j)+1}\rho(y)$,

$$\begin{aligned}&\left|\left[D_kQ_j(x,y) - D_kQ_j(x',y)\right] - \left[D_kQ_j(x,y') - D_kQ_j(x',y')\right]\right| \\ &\quad\le C2^{-|k-j|(1-\eta)(\sigma+\delta)}\left(\frac{d(x,x')}{2^{-(k\wedge j)}\rho(x)}\right)^{\eta(1-\sigma)}\left(\frac{d(y,y')}{2^{-(k\wedge j)}\rho(y)}\right)^{\eta(1-\delta)} \\ &\qquad\times\frac{1}{V_{2^{-(k\wedge j)}\rho(x)}(x) + V_{2^{-(k\wedge j)}\rho(y)}(y)};\end{aligned} \tag{3.24}$$

(vi) *for all* $x,y \in \mathcal{X}$, $\int_{\mathcal{X}} D_kQ_j(x,y)d\mu(x) = \int_{\mathcal{X}} D_kQ_j(x,y)d\mu(y) = 0$ *when* $(k \vee j) > \ell_0$; $= 1$ *when* $k = j = \ell_0$.

Proof. Without loss of generality, we may assume that $j \ge k \ge \ell_0$. By Definition 3.1(i), for all $j \ge \ell_0$, $Q_j(x,y) \ne 0$ implies that

$$d(x,y) \le C_1 2^{-(j-1)}\left[\rho(x) \wedge \rho(y)\right], \tag{3.25}$$

likewise for D_k. Therefore, if $D_kQ_j(x,y) = \int_{\mathcal{X}} D_k(x,z)Q_j(z,y)d\mu(z) \ne 0$, then there exists $z \in \mathcal{X}$ such that $d(x,z) \le C_1 2^{-(k-1)}[\rho(x)\wedge\rho(z)]$ and $d(z,y) \le C_1 2^{-(j-1)}[\rho(z)\wedge\rho(y)]$, which together with (2.1) yields (i).

The support and size conditions of S_{ℓ_0} and E_{ℓ_0} together with (2.1), (2.2), and Proposition 2.6(a) imply that (ii) holds when $j = k = \ell_0$. To show that (ii) holds when $j > \ell_0$, by the fact $\int_{\mathcal{X}} Q_j(z,y)d\mu(z) = 0$, (3.25), the size condition of Q_j, and the regularity of D_k, we obtain that for all $x, y \in \mathcal{X}$,

$$\begin{aligned}\left|D_kQ_j(x,y)\right| &= \left|\int_{\mathcal{X}}\left[D_k(x,z) - D_k(x,y)\right]Q_j(z,y)d\mu(z)\right| \\ &\le \int_{d(z,y)\le C_1 2^{-j+1}\rho(y)}\left|D_k(x,z) - D_k(x,y)\right|\left|Q_j(z,y)\right|d\mu(z) \\ &\lesssim \frac{1}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)}\int_{d(z,y)\le C_1 2^{-j+1}\rho(y)}\frac{d(y,z)}{V_{2^{-j}\rho(y)}(y) + V(y,z)}d\mu(z),\end{aligned} \tag{3.26}$$

which combined with Proposition 2.6(c) and $j \geq k$ further implies that

$$\left|D_k Q_j(x,y)\right| \lesssim \frac{1}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(x)}(x)} \int_{d(z,y)\leq C_1 2^{-j+1}\rho(y)} \frac{d(z,y)}{V(z,y)} d\mu(z) \lesssim 2^{k-j} \frac{1}{V_{2^{-k}\rho(x)}(x)}. \quad (3.27)$$

This together with (i) of this lemma and Proposition 2.6(a) yields (ii).

The proofs for (iii) and (iv) are similar and we only show (iii). To this end, it suffices to prove that when $d(y,y') \leq (C_4 \vee 1)2^{-k+1}\rho(y)$,

$$\left|D_k Q_j(x,y) - D_k Q_j(x,y')\right| \lesssim \frac{d(y,y')}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \quad (3.28)$$

To see this, notice that if $D_k Q_j(x,y) - D_k Q_j(x,y') \neq 0$, then the assumption of (iii) combined with (i) and (2.1) yields that

$$\rho(x) \sim \rho(y) \sim \rho(y'), \quad d(x,y) \lesssim 2^{-k}\left[\rho(x) \wedge \rho(y)\right]. \quad (3.29)$$

This together with (ii) and Proposition 2.6(a) further implies that

$$\begin{aligned} \left|D_k Q_j(x,y) - D_k Q_j(x,y')\right| &\lesssim 2^{-|k-j|} \chi_{\{d(x,y)\leq C_4 2^{-k}\rho(x)\}}(x,y) \left\{ \frac{1}{V_{2^{-k}\rho(x)}(x)} + \frac{1}{V_{2^{-k}\rho(x)}(x)} \right\} \\ &\lesssim 2^{-|k-j|} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \end{aligned} \quad (3.30)$$

Taking the geometric mean between (3.28) and (3.30) gives the desired estimate of (iii).

Now we verify (3.28). Indeed, by the observation (3.19) on the regularity of the second variable, it suffices to show (3.28) for $d(y,y') \leq C_1 2^{-k}\rho(y)/4$. In fact, we show that (3.28) holds for $d(y,y') \leq [C_1 2^{-k}\rho(y) + d(x,y)]/4$. To this end, by Definition 3.1(v), we write

$$\begin{aligned} \left|D_k Q_j(x,y) - D_k Q_j(x,y')\right| &= \left| \int_{\mathcal{X}} \left[D_k(x,z) - D_k(x,y)\right]\left[Q_j(z,y) - Q_j(z,y')\right] d\mu(z) \right| \\ &\leq \sum_{i=1}^{2} \int_{W_i} \left|D_k(x,z) - D_k(x,y)\right| \left|Q_j(z,y) - Q_j(z,y')\right| d\mu(z) \\ &\equiv \sum_{i=1}^{2} Z_i, \end{aligned} \quad (3.31)$$

where $W_1 \equiv \{z \in \mathcal{X} : \ d(y,y') \leq [C_1 2^{-j}\rho(y) + d(z,y)]/2\}$ and $W_2 \equiv \{z \in \mathcal{X} : \ d(y,y') > [C_1 2^{-j}\rho(y) + d(z,y)]/2\}$.

We first estimate Z_1. If $z \in W_1$ and $Q_j(z,y) - Q_j(z,y') \neq 0$, then either $d(z,y) \leq C_1 2^{-j}[\rho(z) \wedge \rho(y)]$ or $d(z,y') \leq C_1 2^{-j}[\rho(z) \wedge \rho(y')]$, which together with (2.1) yields that

$d(y,y') \lesssim 2^{-j}\rho(y)$ and $d(z,y) \lesssim 2^{-j}[\rho(z) \wedge \rho(y)] \lesssim 2^{-k}\rho(y)$. These facts and (3.29) together with Proposition 2.6(a) and the regularities of $\{D_k\}_{k=\ell_0}^{\infty}$ and $\{Q_j\}_{j=\ell_0}^{\infty}$ yield that

$$\begin{aligned} Z_1 &\lesssim \frac{d(y,y')}{2^{-j}\rho(y)} \int_{W_2} \frac{d(z,y)}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(y)}(y)} \frac{\chi_j(z,y)}{V_{2^{-k}\rho(z)}(z) + V_{2^{-k}\rho(y)}(y)} d\mu(z) \chi_k(x,y) \\ &\lesssim \frac{d(y,y')}{2^{-k}\rho(x)} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}, \end{aligned} \tag{3.32}$$

where and in what follows, $\chi_j(z,y) \equiv \chi_{\{d(z,y) \lesssim 2^{-j}[\rho(z)\wedge\rho(y)]\}}(z,y)$ for all $j \geq \ell_0$ and $z, y \in \mathcal{X}$.

To estimate Z_2, notice that for any $z \in W_2$, by (3.29) and (2.1), we have

$$d(z,y) \leq 2d(y,y') \leq \frac{[C_1 2^{-k}\rho(y) + d(x,y)]}{2} \lesssim 2^{-k}\rho(y) \sim 2^{-k}\rho(x). \tag{3.33}$$

This combined with (3.19) and Proposition 2.6(b) gives that

$$\begin{aligned} Z_2 &\lesssim \chi_k(x,y) \int_{W_2} \frac{d(z,y)}{2^{-k}\rho(x)} \frac{1}{V_{2^{-k}\rho(y)}(y)} [|Q_j(z,y)| + |Q_j(z,y')|] d\mu(z) \\ &\lesssim \chi_k(x,y) \int_{W_2} \frac{d(z,y)}{2^{-k}\rho(x)} \frac{d(y,y')}{C_1 2^{-j}\rho(y) + d(z,y)} \frac{1}{V_{2^{-k}\rho(y)}(y)} [|Q_j(z,y)| + |Q_j(z,y')|] d\mu(z) \\ &\lesssim \chi_k(x,y) \frac{d(y,y')}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(y)}(y)} \int_{\mathcal{X}} [|Q_j(z,y)| + |Q_j(z,y')|] d\mu(z) \\ &\lesssim \frac{d(y,y')}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \end{aligned} \tag{3.34}$$

Combining the estimates of Z_1 and Z_2 yields (3.28) and hence (iii) holds.

When $j \geq k$, to prove (v), it suffices to verify that for any $\eta \in (0,1)$, $d(x,x') \leq (C_4 \vee 1)2^{-k+1}\rho(x)$ and $d(y,y') \leq (C_4 \vee 1)2^{-k+1}\rho(y)$,

$$\begin{aligned} &|[D_kQ_j(x,y) - D_kQ_j(x',y)] - [D_kQ_j(x,y') - D_kQ_j(x',y')]| \\ &\quad \lesssim \left(\frac{d(x,x')}{2^{-k}\rho(x)}\right)^{\eta} \left(\frac{d(y,y')}{2^{-k}\rho(y)}\right)^{\eta} \frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}. \end{aligned} \tag{3.35}$$

To see this, notice that if $D_kQ_j(x,y') - D_kQ_j(x',y') \neq 0$, then by (i) and the assumption $d(x,x') \leq (C_4 \vee 1)2^{-k+1}\rho(x)$ together with (2.1), we have $d(x,y') \lesssim 2^{-k}\rho(x)$, which combined with $d(y,y') \leq (C_4 \vee 1)2^{-k}\rho(x)$ further implies that $d(x,y) \lesssim 2^{-k}\rho(x)$. By this, (iv) of this lemma, (3.19), and Proposition 2.6(a), we obtain

$$\begin{aligned}
&\left|\left[D_kQ_j(x,y) - D_kQ_j(x',y)\right] - \left[D_kQ_j(x,y') - D_kQ_j(x',y')\right]\right| \\
&\quad\lesssim 2^{-|k-j|(1-\eta)}\left(\frac{d(x,x')}{2^{-k}\rho(x)}\right)^{\eta}\left\{\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)} + \frac{\chi_k(x,y)}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y')}(y')}\right\} \\
&\quad\lesssim 2^{-|k-j|(1-\eta)}\left(\frac{d(x,x')}{2^{-k}\rho(x)}\right)^{\eta}\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}.
\end{aligned} \tag{3.36}$$

Using (iii) of this lemma and a symmetric argument, we obtain that

$$\begin{aligned}
&\left|\left[D_kQ_j(x,y) - D_kQ_j(x',y)\right] - \left[D_kQ_j(x,y') - D_kQ_j(x',y')\right]\right| \\
&\quad\lesssim 2^{-|k-j|(1-\eta)}\left(\frac{d(y,y')}{2^{-k}\rho(y)}\right)^{\eta}\frac{1}{V_{2^{-k}\rho(x)}(x) + V_{2^{-k}\rho(y)}(y)}.
\end{aligned} \tag{3.37}$$

Then the geometric mean among (3.35), (3.36), and (3.37) gives the desired estimate of (v).

By the observation (3.20), we only need to show (3.35) for $d(y,y') \leq C_1 2^{-k}\rho(y)/8$ and $d(x,x') \leq C_1 2^{-k}\rho(y)/8$. Actually, we now establish (3.35) for $d(y,y') \leq [C_1 2^{-k}\rho(y) + d(x,y)]/8$ and $d(x,x') \leq [C_1 2^{-k}\rho(y) + d(x,y)]/8$. To this end, notice that if $|[D_kQ_j(x,y) - D_kQ_j(x',y)] - [D_kQ_j(x,y') - D_kQ_j(x',y')] \neq 0$, then (i) of this lemma implies that at least one of the following four inequalities holds: $d(x,y) \leq C_4 2^{-k}[\rho(x) \wedge \rho(y)]$, $d(x',y) \leq C_4 2^{-k}[\rho(x') \wedge \rho(y)]$, $d(x,y') \leq C_4 2^{-k}[\rho(x) \wedge \rho(y')]$, and $d(x',y') \leq C_4 2^{-k}[\rho(x') \wedge \rho(y')]$. This and (2.1) together with the assumptions $d(y,y') \leq [C_1 2^{-k}\rho(y) + d(x,y)]/8$ and $d(x,x') \leq [C_1 2^{-k}\rho(y) + d(x,y)]/8$ imply that

$$d(x,y) \lesssim 2^{-k}\rho(x), \quad \rho(x) \sim \rho(y) \sim \rho(x') \sim \rho(y'), \tag{3.38}$$

and hence

$$d(x,x') \lesssim 2^{-k}\rho(x), \quad d(y,y') \lesssim 2^{-k}\rho(y). \tag{3.39}$$

Then we write

$$
\begin{aligned}
&\left|[D_kQ_j(x,y)-D_kQ_j(x',y)]-[D_kQ_j(x,y')-D_kQ_j(x',y')]\right| \\
&\quad=\left|\int_{\mathcal{X}}\{[D_k(x,z)-D_k(x',z)]-[D_k(x,y)-D_k(x',y)]\}\,\{Q_j(z,y)-Q_j(z,y')\}d\mu(z)\right| \\
&\quad\le\sum_{i=1}^{2}\int_{U_i}\left|[D_k(x,z)-D_k(x',z)]-[D_k(x,y)-D_k(x',y)]\right|\left|Q_j(z,y)-Q_j(z,y')\right|d\mu(z) \\
&\quad\equiv\sum_{i=1}^{2}J_i,
\end{aligned}
\tag{3.40}
$$

where $U_1\equiv\{z\in\mathcal{X}:d(y,y')\le[C_12^{-j}\rho(y)+d(z,y)]/2\}$ and $U_2\equiv\{z\in\mathcal{X}:d(y,y')>[C_12^{-j}\rho(y)+d(z,y)]/2\}$.

If $z\in U_1$ and $Q_j(z,y)-Q_j(z,y')\neq 0$, then by the support condition of Q_j and the fact $d(y,y')\le[C_12^{-j}\rho(y)+d(z,y)]/2$ together with (3.38), we have

$$
d(z,y)\lesssim 2^{-j}\rho(y)\lesssim 2^{-k}\rho(y),
\tag{3.41}
$$

and hence $d(y,y')\lesssim 2^{-j}\rho(y)$. By this, (3.41), (3.39), the second-order difference condition of D_k, and Remark 3.3(b), we then obtain

$$
\begin{aligned}
J_1&\lesssim\int_{U_1}\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(z,y)}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)}\frac{d(y,y')}{2^{-j}\rho(y)}\frac{\chi_{\{d(z,y)\lesssim 2^{-j}\rho(y)\}}(z)}{V_{2^{-j}\rho(z)}(z)+V_{2^{-j}\rho(y)}(y)}d\mu(z) \\
&\lesssim\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(y,y')}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)}.
\end{aligned}
\tag{3.42}
$$

If $z\in U_2$, then by (3.39), we have $d(z,y)\le 2d(y,y')\lesssim 2^{-k}\rho(y)$. This and (3.39) together with the second-order difference condition of D_k and (3.19) yield that

$$
\begin{aligned}
J_2&\lesssim\int_{U_2}\frac{d(y,y')}{C_12^{-j}\rho(y)+d(z,y)}\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(z,y)}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)} \\
&\quad\times\left|Q_j(z,y)-Q_j(z,y')\right|d\mu(z) \\
&\lesssim\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(y,y')}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)}\int_{\mathcal{X}}\left|Q_j(z,y)-Q_j(z,y')\right|d\mu(z) \\
&\lesssim\frac{d(x,x')}{2^{-k}\rho(x)}\frac{d(y,y')}{2^{-k}\rho(y)}\frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)}.
\end{aligned}
\tag{3.43}
$$

Combining the estimates of J_1 and J_2 yields (3.35). Hence, (v) holds.

Property (vi) can be obtained simply by using Definition 3.1(v). This finishes the proof of Lemma 3.4. □

We conclude this section with some basic properties of ℓ_0-AOTI, which are used in Section 5. For all $f \in L^p(\mathcal{X})$ with $p \in [1,\infty]$ and $x \in \mathcal{X}$, set $S_k(f)(x) \equiv \int_{\mathcal{X}} S_k(x,y)f(y)d\mu(y)$. Denote by $L_b^\infty(\mathcal{X})$ the collection of all $f \in L^\infty(\mathcal{X})$ with bounded support.

Proposition 3.5. *Let $\ell_0 \in \mathbb{Z}$ and $\{S_k\}_{k=\ell_0}^\infty$ be an ℓ_0-AOTI as in Definition 3.1.*

(i) *There exists a positive constant C depending only ℓ_0 such that for all $x, y \in \mathcal{X}$ and $k \geq \ell_0$, $\int_{\mathcal{X}} |S_k(x,y)|d\mu(y) \leq C$ and $\int_{\mathcal{X}} |S_k(x,y)|d\mu(x) \leq C$.*

(ii) *There exists a positive constant C depending only on ℓ_0 such that for all $k \geq \ell_0$, locally integrable functions f, and $x \in \mathcal{X}$, $|S_k(f)(x)| \leq C\mathcal{M}_{C_1 2^{-\ell_0}} f(x)$, where C_1 is the constant appearing in Definition 3.1(i).*

(iii) *For $p \in [1,\infty]$, there exists a positive constant C_p, depending on p and ℓ_0, such that for all $k \geq \ell_0$ and $f \in L^p(\mathcal{X})$, $\|S_k(f)\|_{L^p(\mathcal{X})} \leq C_p\|f\|_{L^p(\mathcal{X})}$.*

(iv) *Set $D_{\ell_0} \equiv S_{\ell_0}$ and $D_k \equiv S_k - S_{k-1}$ for $k > \ell_0$. Then $I = \sum_{k=\ell_0}^\infty D_k$ in $L^p(\mathcal{X})$, where $p \in [1,\infty)$ and I is the identity operator on $L^p(\mathcal{X})$.*

Proof. (i) can be easily deduced from the support and size conditions of S_k together with Proposition 2.6(c). We can easily show (ii) by using (2.20) and Definition 3.1(i). Property (iii) is a simple corollary of (i) and Hölder's inequality.

To prove (iv), it suffices to show that $\lim_{N\to\infty}\|f - \sum_{k=\ell_0}^N D_k(f)\|_{L^p(\mathcal{X})} = 0$ for all $f \in L^p(\mathcal{X})$ with $p \in [1,\infty)$. Since $\|f - \sum_{k=\ell_0}^N D_k(f)\|_{L^p(\mathcal{X})} = \|f - S_N(f)\|_{L^p(\mathcal{X})}$, it is enough to show

$$\lim_{N\to\infty} \int_{\mathcal{X}} |f(x) - S_N(f)(x)|^p d\mu(x) = 0. \tag{3.44}$$

Now we prove (3.44) for $p \in (1,\infty)$. Let $x \in \mathcal{X}$ be a point such that Theorem 2.8(ii) holds for f. Then using (v) and (i) of Definition 3.1, we obtain

$$\begin{aligned} |f(x) - S_N(f)(x)| &\leq \int_{\mathcal{X}} |S_N(x,y)||f(x) - f(y)|d\mu(y) \\ &\lesssim \frac{1}{\mu(B(x, C_1 2^{-N}\rho(x)))} \int_{B(x,C_1 2^{-N}\rho(x))} |f(x) - f(y)|d\mu(y), \end{aligned} \tag{3.45}$$

which tends to 0 as $N \to \infty$, by Theorem 2.8(ii). This and $|S_N(f)(x)| \lesssim \mathcal{M}_{C_1 2^{-\ell_0}} f(x)$ together with the dominated convergence theorem and Theorem 2.8(i) imply that (3.44) holds for $p \in (1,\infty)$.

To prove (3.44) for the case $p = 1$, we first consider $f \in L_b^\infty(\mathcal{X})$. Assume that $\operatorname{supp} f \subset B(x_0, r_0\rho(x_0))$ for some $x_0 \in \mathcal{X}$ and $r_0 > 0$. Combining this with (2.1) gives $\operatorname{supp} S_k(f) \subset B(x_0, (C_1\Theta_{r_0} + r_0)\rho(x_0))$. By Hölder's inequality and $L_b^\infty(\mathcal{X}) \subset L^2(\mathcal{X})$ together with the fact that (3.44) holds for $p = 2$, we obtain that for all $f \in L_b^\infty(\mathcal{X})$,

$$\lim_{N\to\infty} \|S_N(f) - f\|_{L^1(\mathcal{X})} \leq \lim_{N\to\infty} [\mu(B(x_0, (C_1\Theta_{r_0} + r_0)\rho(x_0)))]^{1/2} \|S_N(f) - f\|_{L^2(\mathcal{X})} = 0, \tag{3.46}$$

which combined with the density of $L_b^\infty(\mathcal{X})$ in $L^1(\mathcal{X})$ and Proposition 3.5(iii) yields that (3.44) holds for $p = 1$. Thus, we obtain (iv), which completes the proof of Proposition 3.5. □

4. Local Vector-Valued Singular Integral Operators

In this section, let $(\mathcal{X}, d)$ be a metric space and μ a regular Borel measure satisfying (2.2). Denote by $\mathfrak{B}$ a complex Banach space with norm $\|\cdot\|_{\mathfrak{B}}$, and by $\mathfrak{B}^*$ its dual space with norm $\|\cdot\|_{\mathfrak{B}^*}$. A function F defined on a σ-finite measure space $(\mathcal{X}, \mu)$ and taking values in $\mathfrak{B}$ is called $\mathfrak{B}$-measurable if there exists a measurable subset $\mathcal{X}_0$ of $\mathcal{X}$ such that $\mu(\mathcal{X} \setminus \mathcal{X}_0) = 0$ and $F(\mathcal{X}_0)$ is contained in some separable subspace $\mathfrak{B}_0$ of $\mathfrak{B}$, and for every $u^* \in \mathfrak{B}^*$, the complex-valued map $x \to \langle u^*, F(x)\rangle$ is measurable. From this definition and the theorem in [29, page 131], it follows that the function $x \to \|F(x)\|_{\mathfrak{B}}$ on $\mathcal{X}$ is measurable.

For any $p \in (0,\infty]$, we define $L^p(\mathcal{X}, \mathfrak{B})$ to be the space of all $\mathfrak{B}$-measurable functions F on $\mathcal{X}$ satisfying $\|F\|_{L^p(\mathcal{X},\mathfrak{B})} < \infty$, where $\|F\|_{L^p(\mathcal{X},\mathfrak{B})} = \{\int_{\mathcal{X}} \|F(x)\|_{\mathfrak{B}}^p \, d\mu(x)\}^{1/p}$ with a usual modification made when $p = \infty$. Define $L^{p,\infty}(\mathcal{X}, \mathfrak{B})$ to be the space of all $\mathfrak{B}$-measurable functions F on $\mathcal{X}$ satisfying $\|F\|_{L^{p,\infty}(\mathcal{X},\mathfrak{B})} < \infty$, where

$$\|F\|_{L^{p,\infty}(\mathcal{X},\mathfrak{B})} = \sup_{\alpha>0}\left\{\alpha\left[\mu(\{x \in \mathcal{X} : \|F(x)\|_{\mathfrak{B}} > \alpha\})\right]^{1/p}\right\}. \tag{4.1}$$

Denote by $L_b^\infty(\mathcal{X}, \mathfrak{B})$ the set of all functions in $L^\infty(\mathcal{X}, \mathfrak{B})$ with bounded support. For $p \in (0,\infty)$, let $L^p(\mathcal{X}) \otimes \mathfrak{B}$ be the set of all finite linear combinations of elements of $\mathfrak{B}$ with coefficients in $L^p(\mathcal{X})$, that is, elements of the form,

$$F = f_1 u_1 + \cdots + f_m u_m, \tag{4.2}$$

where $m \in \mathbb{N}$, $f_j \in L^p(\mathcal{X})$, and $u_j \in \mathfrak{B}$ for $j \in \{1, \ldots, m\}$. Both $L_b^\infty(\mathcal{X}, \mathfrak{B})$ and $L^p(\mathcal{X}) \otimes \mathfrak{B}$ are dense in $L^p(\mathcal{X}, \mathfrak{B})$; see, for example, [3] or [30, Lemma 2.1]. Given $F \in L^1(\mathcal{X}) \otimes \mathfrak{B}$ as in (4.2), we define its integral to be the following element of $\mathfrak{B}$:

$$\int_{\mathcal{X}} F(x) d\mu(x) \equiv \sum_{j=1}^{m}\left\{\int_{\mathcal{X}} f_j(x) d\mu(x)\right\} u_j. \tag{4.3}$$

Therefore, for any $F \in L^1(\mathcal{X}, \mathfrak{B})$, the integral $\int_{\mathcal{X}} F(x) d\mu(x)$, as a unique extension of the integral of functions in $L^1(\mathcal{X}) \otimes \mathfrak{B}$, is well defined; it is not difficult to show that

$$\left\|\int_{\mathcal{X}} F(x) d\mu(x)\right\|_{\mathfrak{B}} \le \int_{\mathcal{X}} \|F(x)\|_{\mathfrak{B}} d\mu(x); \tag{4.4}$$

see, for instance, [3] or [29]. Here we refer the reader to [3, 31, 32] for more detailed knowledge on Banach space-valued functions.

In what follows, we consider a kernel $\overrightarrow{K}$ defined on $(\mathcal{X} \times \mathcal{X}) \setminus \Delta$ with $\Delta = \{(x, x) : x \in \mathcal{X}\}$ that takes values in the space $\mathcal{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ of all bounded linear operators from Banach space $\mathfrak{B}_1$ to Banach space $\mathfrak{B}_2$. Then $\overrightarrow{K}(x, y)$ is a bounded linear operator from $\mathfrak{B}_1$ to $\mathfrak{B}_2$

whose norm is denoted by $\|\vec{K}(x,y)\|_{\mathfrak{B}_1\to\mathfrak{B}_2}$. Assume that $\vec{K}(x,y)$ is $\mathcal{L}(\mathfrak{B}_1,\mathfrak{B}_2)$-measurable and locally integrable on $(\mathcal{X}\times\mathcal{X})\setminus\Delta$ such that the integral

$$\vec{T}(F)(x)=\int_{\mathcal{X}}\vec{K}(x,y)F(y)\,d\mu(y) \tag{4.5}$$

is well defined as an element of $\mathfrak{B}_2$ for all $F\in L_b^\infty(\mathcal{X},\mathfrak{B}_1)$ and $x\notin\operatorname{supp}F$. Set $A\equiv\liminf_{\tau\to0}\Theta_\tau$. Suppose that there exist constants $C_5>2A^2\Theta_{A^2}\Theta_A+A$ and $C_6>0$ such that for all $x,y\in\mathcal{X}$ satisfying $d(x,y)\le C_5\rho(x)$,

$$\left\|\vec{K}(x,y)\right\|_{\mathfrak{B}_1\to\mathfrak{B}_2}\le C_6\frac{1}{V(x,y)}, \tag{4.6}$$

$$\int_{d(x,z)\ge2d(x,y)}\left\|\vec{K}(z,x)-\vec{K}(z,y)\right\|_{\mathfrak{B}_1\to\mathfrak{B}_2}d\mu(z)\le C_6, \tag{4.7}$$

$$\int_{d(x,z)\ge2d(x,y)}\left\|\vec{K}(x,z)-\vec{K}(y,z)\right\|_{\mathfrak{B}_1\to\mathfrak{B}_2}d\mu(z)\le C_6. \tag{4.8}$$

Let $\mathcal{N}\equiv\{(x,y)\in\mathcal{X}\times\mathcal{X}:d(x,y)\le[\rho(x)\wedge\rho(y)]\}$ and $\mathcal{N}_x\equiv\{y\in\mathcal{X}:(x,y)\in\mathcal{N}\}$. Then for all $x\in\mathcal{X}$, set

$$\vec{T}_{\mathrm{local}}(F)(x)\equiv\vec{T}(\chi_{\mathcal{N}_x}F)(x), \tag{4.9}$$

where $\chi_{\mathcal{N}_x}$ represents the characteristic function of the set $\mathcal{N}_x$. A conclusion concerned such locally vector-valued singular integrals is as follows.

Theorem 4.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Banach spaces. Suppose that $\vec{T}$ given by* (4.5) *is a bounded linear operator from $L^r(\mathcal{X},\mathfrak{B}_1)$ to $L^r(\mathcal{X},\mathfrak{B}_2)$ for some $r\in(1,\infty]$ with norm $A_r>0$. Assume that $\vec{K}$ satisfies* (4.6) *through* (4.8)*. Then $\vec{T}_{local}$ as in* (4.9) *has well-defined extensions on $L^p(\mathcal{X},\mathfrak{B}_1)$ for all $p\in[1,\infty)$. Moreover, there exists a positive constant C depending on $\mathcal{X}$, p, and C_5 such that*

(i) *whenever $p\in[1,r)$, for all $F\in L^p(\mathcal{X},\mathfrak{B}_1)$,*

$$\left\|\vec{T}_{local}(F)\right\|_{L^{p,\infty}(\mathcal{X},\mathfrak{B}_2)}\le C(C_6+A_r)\|F\|_{L^p(\mathcal{X},\mathfrak{B}_1)}; \tag{4.10}$$

(ii) *whenever $p\in(1,\infty)$, for all $F\in L^p(\mathcal{X},\mathfrak{B}_1)$,*

$$\left\|\vec{T}_{local}(F)\right\|_{L^p(\mathcal{X},\mathfrak{B}_2)}\le C(C_6+A_r)\|F\|_{L^p(\mathcal{X},\mathfrak{B}_1)}. \tag{4.11}$$

Proof. It suffices to show the theorem for $F\in L_b^\infty(\mathcal{X},\mathfrak{B}_1)$, since $L_b^\infty(\mathcal{X},\mathfrak{B}_1)$ is dense in $L^p(\mathcal{X},\mathfrak{B}_1)$ for $p\in[1,\infty)$. We further assume that $\mu(\mathcal{X})<\infty$, since the proof for the case $\mu(\mathcal{X})=\infty$ is similar and simpler.

Suppose that $r < \infty$ and $p \in [1, r)$. If $0 < A_r^{-1}\lambda \le \|F\|_{L^p(\mathcal{X},\mathfrak{B}_1)}/\mu(\mathcal{X})^{1/p}$ (this happens only when $\mu(\mathcal{X}) < \infty$), then

$$\mu\left(\left\{x \in \mathcal{X} : \left\|\overrightarrow{T}_{\text{local}}(F)(x)\right\|_{\mathfrak{B}_2} > \lambda\right\}\right) \le \mu(\mathcal{X}) \le \left(\frac{A_r}{\lambda}\right)^p \|F\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}. \tag{4.12}$$

Assume now that $A_r^{-1}\lambda > \|F\|_{L^p(\mathcal{X},\mathfrak{B}_1)}/\mu(\mathcal{X})^{1/p}$. By Lemma 2.7, for any given sufficiently small positive number t, which will be determined later, there exists a sequence of balls, $B_j \equiv B(x_j, t\rho(x_j))$, such that $\mathcal{X} = \bigcup_j B_j$ and $\{\eta \mathrm{B}_j\}_j$ has finite overlapping property whenever $\eta > 0$. Set $B_j^* \equiv B(x_j, [\Theta_t + t]\rho(x_j))$. It follows easily from (2.1) that for any given j, $\bigcup_{x\in B_j} \mathcal{N}_x \subset B_j^*$. Therefore, by (4.9),

$$\begin{aligned}
\mu&\left(\left\{x \in \mathcal{X} : \left\|\overrightarrow{T}_{\text{local}}\,(F)(x)\right\|_{\mathfrak{B}_2} > \lambda\right\}\right) \\
&\le \sum_j \mu\left(\left\{x \in B_j : \left\|\overrightarrow{T}(\chi_{\mathcal{N}_x}F)(x)\right\|_{\mathfrak{B}_2} > \lambda\right\}\right) \\
&\le \sum_j \mu\left(\left\{x \in B_j : \left\|\overrightarrow{T}\left(\chi_{B_j^*}F\right)(x)\right\|_{\mathfrak{B}_2} > \frac{\lambda}{2}\right\}\right) \\
&\quad + \sum_j \mu\left(\left\{x \in B_j : \left\|\overrightarrow{T}\left(\chi_{B_j^*\setminus\mathcal{N}_x}F\right)(x)\right\|_{\mathfrak{B}_2} > \frac{\lambda}{2}\right\}\right) \equiv \sum_j Y_j + Y.
\end{aligned} \tag{4.13}$$

Observe that if $y \in B_j^*$ and $x \in B_j$, then $d(x,y) \le (\Theta_t + 2t)\rho(x_j) \le \Theta_t(\Theta_t + 2t)\rho(x)$, so $d(x,y) < C_5\rho(x)$ if we choose $t > 0$ sufficiently small. Thus, for any $x \in B_j$, by (4.4) and (4.6), we have

$$\begin{aligned}
\left\|\overrightarrow{T}\left(\chi_{B_j^*\setminus\mathcal{N}_x}F\right)(x)\right\|_{\mathfrak{B}_2} &= \left\|\int_{B_j^*\setminus\mathcal{N}_x} \overrightarrow{K}(x,y)F(y)\,d\mu(y)\right\|_{\mathfrak{B}_2} \\
&\le C_6 \int_{B_j^*\setminus\mathcal{N}_x} \frac{\|F(y)\|_{\mathfrak{B}_1}}{V(x,y)}\,d\mu(y).
\end{aligned} \tag{4.14}$$

If $x \in B_j$ and $y \in B_j^* \setminus \mathcal{N}_x$, then (2.1) implies that $B_j^* \subset B(x, \Theta_t(\Theta_t + 2t)\rho(x))$ and $d(x,y) > [\Theta_t]^{-1}[\Theta_{\Theta_t+t}]^{-1}\rho(x)$. This combined with (4.14) and (2.2) yields that there exists a positive constant $\widetilde{C}$, depending only on $\mathcal{X}$ and t, such that

$$\left\|\overrightarrow{T}\left(\chi_{B_j^*\setminus\mathcal{N}_x}F\right)(x)\right\|_{\mathfrak{B}_2} \le \widetilde{C}C_6\left[\mathcal{M}_a\left(\|F(\cdot)\|^p_{\mathfrak{B}_1}\chi_{B_j^*}\right)(x)\right]^{1/p}, \tag{4.15}$$

where $a \equiv \Theta_t(\Theta_t + 2t)$. By (4.15), Theorem 2.8(i), Lemma 2.7(ii), the finite overlapping property of $\{B_j^*\}_j$, and the fact that for all $\kappa > 1$ and $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$,

$$\sum_{j\in\mathbb{N}} |a_j|^{\kappa} \le \left(\sum_{j\in\mathbb{N}} |a_j|\right)^{\kappa}, \tag{4.16}$$

we then obtain

$$\begin{aligned} Y &\le \sum_j \mu\left(\left\{x \in B_j^* : \mathcal{M}_a\left(\|F(\cdot)\|_{\mathfrak{B}_1}^p \chi_{B_j^*}\right)(x) > \frac{\lambda^p}{\widetilde{C}C_6}\right\}\right) \\ &\lesssim C_6 \sum_j \frac{\left\|F\chi_{B_j^*}\right\|_{L^p(\mathcal{X},\mathfrak{B}_1)}^p}{\lambda^p} \lesssim C_6 \frac{\|F\|_{L^p(\mathcal{X},\mathfrak{B}_1)}^p}{\lambda^p}. \end{aligned} \tag{4.17}$$

Now we estimate $\sum_j Y_j$. Set $f_j \equiv \|F(\cdot)\|_{\mathfrak{B}_1} \chi_{B_j^*}$. Then $\operatorname{supp} f_j \subset B_j^*$. We claim that there exists a positive number t sufficiently small such that

$$\operatorname{supp}\left[\mathcal{M}_a\left(f_j^p\right)\right]^{1/p} \subset B(x_j, C_5\rho(x_j)). \tag{4.18}$$

In fact, for any $x \in \mathcal{X}$, from (2.20), we deduce that if $[\mathcal{M}_a(f_j^p)(x)]^{1/p} \neq 0$, then there exists a ball $B \ni x$ satisfying that $B \cap B_j^* \neq \emptyset$ and $r_B \le a\rho(c_B)$. From this and (2.1) together with the triangular inequality of d, it follows that $d(x, x_j) \le [2a\Theta_a\Theta_{\Theta_t+t} + \Theta_t + t]\rho(x_j)$. Combined this with the facts $C_5 > 2A^2\Theta_{A^2}\Theta_A + A$, $A \equiv \lim\inf_{\tau\to 0}\Theta_\tau$, and $a = \Theta_t(\Theta_t + 2t)$, we obtain that $x \in B(x_j, C_5\rho(x_j))$ if t is sufficiently small. Thus, (4.18) holds.

For any $\lambda > 0$, the set $\Omega_\lambda \equiv \{x \in \mathcal{X} : [\mathcal{M}_a(f_j^p)(x)]^{1/p} > \lambda\}$ is open and, by (4.18), Ω_λ is contained in the ball $B(x_j, C_5\rho(x_j))$. Following the procedure of the proof for the Whitney covering lemma (see [33, page 277] and [7, pages 70–71]), we obtain that for any fixed j, there exists a sequence $\{B_j^i\}_{i\in I_j}$ of balls, where I_j is an index set depending on j and a positive number M (depending on D_{C_5} in (2.2) with $a = C_5$, but not on j) such that

(i) $\Omega_{A_r^{-1}\lambda} = \bigcup_{i\in I_j} B_j^i \subset B(x_j, C_5\rho(x_j))$;

(ii) $r_{B_j^i} \le C_5\rho(x_j)/(2\Theta_{C_5} + 2)$;

(iii) every point of $\mathcal{X}$ belongs to no more than M balls of $\{3B_j^i\}_{i\in I_j}$;

(iv) the balls $\{(1/4)B_j^i\}_{i\in I_j}$ are mutually disjoint and $(3(\Theta_{C_5} + 1)B_j^i) \cap (\Omega_{A_r^{-1}\lambda})^{\complement} \neq \emptyset$.

For any given j, $i \in I_j$, and $x \in \mathcal{X}$, we set $\zeta_j^i(x) \equiv \chi_{B_j^i}(x)/\sum_{i\in I_j}\chi_{B_j^i}(x)$, and define

$$g_j(x) \equiv F(x)\chi_{B_j^*\setminus\Omega_{A_r^{-1}\lambda}}(x) + \sum_{i\in I_j}\left\{\frac{1}{\mu\left(B_j^i\right)}\int_{B_j^i}F(y)\chi_{B_j^*}(y)\zeta_j^i(y)d\mu(y)\right\}\chi_{B_j^i}(x), \tag{4.19}$$

$$h_j^i(x) \equiv F(x)\chi_{B_j^*}(x)\zeta_j^i(x) - \left\{\frac{1}{\mu\left(B_j^i\right)}\int_{B_j^i}F(y)\chi_{B_j^*}(y)\zeta_j^i(y)d\mu(y)\right\}\chi_{B_j^i}(x). \tag{4.20}$$

By Properties (i), (iii), and (iv) above together with Lemma 2.7, it is not difficult to show that there exists a positive constant C, independent of j, such that

(v) for all $x \in \mathcal{X}$, $F(x)\chi_{B_j^*}(x) = g_j(x) + h_j(x)$, where $h_j(x) = \sum_{i\in I_j} h_j^i(x)$;

(vi) for almost every $x \in \mathcal{X}$, $\|g_j(x)\|_{\mathfrak{B}_1} \le CA_r^{-1}\lambda$;

(vii) $\|g_j\|_{L^p(\mathcal{X},\mathfrak{B}_1)} \le C\|F\chi_{B_j^*}\|_{L^p(\mathcal{X},\mathfrak{B}_1)}$;

(viii) for any $i \in I_j$, supp $h_j^i \subset B_j^i$ and $\sum_{i\in I_j}\mu(B_j^i) \le C\|F\chi_{B_j^*}\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}/(A_r^{-1}\lambda)^p$;

(ix) for any $i \in I_j$, $\int_{\mathcal{X}} h_j^i(x)d\mu(x) = \theta_{\mathfrak{B}_1}$, where $\theta_{\mathfrak{B}_1}$ denotes the zero element of $\mathfrak{B}_1$;

(x) $\sum_{i\in I_j}\|h_j^i\|_{L^p(\mathcal{X},\mathfrak{B}_1)} \le C\|F\chi_{B_j^*}\|_{L^p(\mathcal{X},\mathfrak{B}_1)}$;

(xi) $\sum_{i\in I_j}\|h_j^i\|_{L^1(\mathcal{X},\mathfrak{B}_1)} \le C[A_r^{-1}\lambda]^{1-p}\|F\chi_{B_j^*}\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}$ by (viii).

Then, by Property (v), we obtain

$$\begin{aligned}
Y_j &\le \mu\left(\left\{x \in B_j : \left\|\vec{T}(g_j)(x)\right\|_{\mathfrak{B}_2} > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x \in B_j : \left\|\vec{T}(h_j)(x)\right\|_{\mathfrak{B}_2} > \frac{\lambda}{2}\right\}\right) \\
&\le \frac{2^r}{\lambda^r}\left\|\left\|\vec{T}(g_j)(\cdot)\right\|_{\mathfrak{B}_2}\right\|^r_{L^r(\mathcal{X})} + \mu\left(\bigcup_{i\in I_j}\left(3B_j^i\right)\right) \\
&\quad + \mu\left(\left\{x \in B_j \setminus \bigcup_{i\in I_j}\left(3B_j^i\right) : \|T(h)(x)\|_{\mathfrak{B}_2} > \frac{\lambda}{2}\right\}\right) \equiv L_j + H_j + N_j.
\end{aligned} \tag{4.21}$$

Recall that $p \in [1, r)$. The boundedness of $\vec{T}$ from $L^r(\mathcal{X},\mathfrak{B}_1)$ to $L^r(\mathcal{X},\mathfrak{B}_2)$ together with Properties (vi) and (vii) implies that

$$L_j \lesssim \frac{(A_r)^r}{\lambda^r}\|g_j\|^r_{L^r(\mathcal{X},\mathfrak{B}_1)} \lesssim \left(\frac{A_r}{\lambda}\right)^p\|g_j\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)} \lesssim \left(\frac{A_r}{\lambda}\right)^p\left\|F\chi_{B_j^*}\right\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}. \tag{4.22}$$

By (2.2) and (viii) together with Theorem 2.8(i), we have

$$H_j \lesssim \sum_{i\in I_j}\mu\left(3B_j^i\right) \lesssim \sum_{i\in I_j}\mu\left(B_j^i\right) \lesssim \left(\frac{A_r}{\lambda}\right)^p\left\|F\chi_{B_j^*}\right\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}. \tag{4.23}$$

To estimate N_j, for any $x \in B_j \setminus (\bigcup_{i\in I_j}(3B_j^i))$, by (ix), (4.4), and (4.5),

$$\begin{aligned}\left\|\overrightarrow{T}(h_j)(x)\right\|_{\mathfrak{B}_2} &= \left\|\sum_{i\in I_j}\int_{B_j^i}\left[\overrightarrow{K}(x,y)-\overrightarrow{K}\left(x,c_{B_j^i}\right)\right]h_j^i(y)d\mu(y)\right\|_{\mathfrak{B}_2} \\ &\le \sum_{i\in I_j}\int_{B_j^i}\left\|\left[\overrightarrow{K}(x,y)-\overrightarrow{K}\left(x,c_{B_j^i}\right)\right]h_j^i(y)\right\|_{\mathfrak{B}_2}d\mu(y) \\ &\le \sum_{i\in I_j}\int_{B_j^i}\left\|\overrightarrow{K}(x,y)-\overrightarrow{K}\left(x,c_{B_j^i}\right)\right\|_{\mathfrak{B}_1\to\mathfrak{B}_2}\left\|h_j^i(y)\right\|_{\mathfrak{B}_1}d\mu(y).\end{aligned} \tag{4.24}$$

For any j and i, if $x \in B_j \setminus (\bigcup_{i\in I_j}(3B_j^i))$, then $x \in B_j \setminus (3B_j^i)$, which implies that for any $y \in B_j^i$, $d(x,y) > 2d(y,c_{B_j^i})$. By (ii) and (2.1), we have

$$d\left(y,c_{B_j^i}\right) < r_{B_j^i} < (2\Theta_{C_5}+2)^{-1}C_5\rho(x_j) \le (2\Theta_{C_5}+2)^{-1}C_5\Theta_{C_5}\rho(y) \le C_5\rho(y). \tag{4.25}$$

Therefore, applying (4.7) and (xi) yields that

$$\begin{aligned}N_j &\le \frac{2}{\lambda}\int_{B_j\setminus\bigcup_{i\in I_j}(3B_j^i)}\left\|\overrightarrow{T}(h_j)(x)\right\|_{\mathfrak{B}_2}d\mu(x) \\ &\le \frac{2}{\lambda}\sum_{i\in I_j}\int_{B_j^i}\int_{B_j\setminus(3B_j^i)}\left\|\overrightarrow{K}(x,y)-\overrightarrow{K}\left(x,c_{B_j^i}\right)\right\|_{\mathfrak{B}_1\to\mathfrak{B}_2}\left\|h_j^i(y)\right\|_{\mathfrak{B}_1}d\mu(x)\,d\mu(y) \\ &\lesssim \frac{C_6(A_r)^{p-1}}{\lambda^p}\left\|F\chi_{B_j^*}\right\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}.\end{aligned} \tag{4.26}$$

The estimates of L_j, H_j, and N_j together with the finite overlapping property of $\{B_j^*\}_j$ and (4.16) imply

$$\sum_j \Upsilon_j \lesssim \left(\frac{A_r+C_6}{\lambda}\right)^p\sum_j\left\|F\chi_{B_j^*}\right\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)} \lesssim \left(\frac{A_r+C_6}{\lambda}\right)^p\|F\|^p_{L^p(\mathcal{X},\mathfrak{B}_1)}, \tag{4.27}$$

which combined with the estimate of Υ and (4.13) yields that (4.10) holds when $r < \infty$.

Following the proof of [30, Theorem 1.1], we interpolate between the estimates $\overrightarrow{T}_{\mathrm{local}}$: $L^{(1+p)/2}\ (\mathcal{X},\mathcal{B}_1) \to L^{(1+p)/2,\infty}\ (\mathcal{X},\mathcal{B}_2)$ and $\overrightarrow{T}_{\mathrm{local}}$: $L^{(p+r)/2}\ (\mathcal{X},\mathcal{B}_1) \to L^{(p+r)/2,\infty}\ (\mathcal{X},\mathcal{B}_2)$, and then obtain that (4.11) holds whenever $p \in (1,r)$. A standard duality argument shows that (4.11) holds for $p \in (r,\infty)$; see, for example, [30].

The case $r = \infty$ of the theorem can be proved by a slight modification of the above argument (see [3, 30]) and we omit the details. This finishes the proof of Theorem 4.1. □

As an application of Theorem 4.1, by an argument similar to that used in [3], we obtain the following conclusion. The details are omitted.

Proposition 4.2. *Let $p,q \in (1,\infty)$ and $\mathfrak{B}_1$, $\mathfrak{B}_2$ be Banach spaces. Suppose that $\overrightarrow{T}$ in* (4.5) *is a bounded linear operator from $L^q(\mathcal{X},\mathcal{B}_1)$ to $L^q(\mathcal{X},\mathcal{B}_2)$ with norm $A_q > 0$. Assume that $\overrightarrow{K}$ satisfies* (4.6) *through* (4.8) *with positive constants C_5 and C_6. Let $\overrightarrow{T}_{local}$ be as in* (4.9). *Then there exists a positive constant C, depending on $\mathcal{X}$ and C_5, such that for all $\mathfrak{B}_1$-valued functions $\{F_j\}_{j\in\mathbb{N}}$,*

$$\left\| \left(\sum_{j\in\mathbb{N}} \left\| \overrightarrow{T}_{local}(F_j) \right\|_{\mathfrak{B}_2}^q \right)^{1/q} \right\|_{L^{1,\infty}(\mathcal{X},\mathfrak{B}_2)} \le C(C_6 + A_q) \left\| \left(\sum_{j\in\mathbb{N}} \|F_j\|_{\mathfrak{B}_1}^q \right)^{1/q} \right\|_{L^1(\mathcal{X},\mathfrak{B}_1)},$$
$$\left\| \left(\sum_{j\in\mathbb{N}} \left\| \overrightarrow{T}_{local}(F_j) \right\|_{\mathfrak{B}_2}^q \right)^{1/q} \right\|_{L^p(\mathcal{X},\mathfrak{B}_2)} \le C(C_6 + A_q) \left\| \left(\sum_{j\in\mathbb{N}} \|F_j\|_{\mathfrak{B}_1}^q \right)^{1/q} \right\|_{L^p(\mathcal{X},\mathfrak{B}_1)}. \tag{4.28}$$

Remark 4.3. It is not difficult to see that Theorem 4.1 and Proposition 4.2 still hold if $\mathcal{N}$ in (4.9) is replaced by any set $\{(x,y) \in \mathcal{X}\times\mathcal{X} : d(x,y) \le \widetilde{C}\rho(x)\}$ with $\widetilde{C} > 0$ and $C_5 > 2\widetilde{C}A^2\Theta_{\widetilde{C}A^2}\Theta_{\widetilde{C}A} + \widetilde{C}A$, where $A \equiv \liminf_{\tau\to\infty}\Theta_\tau$. In fact, in this case, we only need to replace B_j^* in the proof of Theorem 4.1 by $\widetilde{B_j^*} \equiv B(x_j, (\widetilde{C}\Theta_t + t)\rho(x_j))$ and make some corresponding modifications for the succedent proof.

Using the approximation of the identity constructed in Proposition 3.2 together with Theorem 4.1 and Proposition 4.2, we obtain the following Fefferman-Stein vector-valued inequality, which was first established by Fefferman and Stein in [34] for the setting of Euclidean spaces; see also [30] for the setting of RD-spaces.

Theorem 4.4. *Let $a > 0$ and $\mathcal{M}_a$ be as in* (2.20). *For $p \in (1,\infty)$ and $q \in (1,\infty]$, there exists a positive constant C, depending on a, p, and q, such that for all measurable functions $\{f_j\}_{j\in\mathbb{N}}$,*

$$\left\| \left(\sum_{j\in\mathbb{N}} [\mathcal{M}_a(f_j)]^q \right)^{1/q} \right\|_{L^{1,\infty}(\mathcal{X})} \le C \left\| \left(\sum_{j\in\mathbb{N}} |f_j|^q \right)^{1/q} \right\|_{L^1(\mathcal{X})},$$
$$\left\| \left(\sum_{j\in\mathbb{N}} [\mathcal{M}_a(f_j)]^q \right)^{1/q} \right\|_{L^p(\mathcal{X})} \le C \left\| \left(\sum_{j\in\mathbb{N}} |f_j|^q \right)^{1/q} \right\|_{L^p(\mathcal{X})}. \tag{4.29}$$

Proof. If $q = \infty$, then (4.29) can be deduced from the boundedness of $\mathcal{M}_a$ in Theorem 2.8 and the fact that $\sup_{j\in\mathbb{N}}\mathcal{M}_a(f_j)(x) \le \mathcal{M}_a(\sup_{j\in\mathbb{N}}|f_j|)(x)$ for all $x \in \mathcal{X}$.

Now we assume that $q < \infty$. In this case, by (2.1) and (2.2), we have

$$\mathcal{M}_a(f)(x) \lesssim \sup_{r\le 2a\Theta_a} \frac{1}{\mu(B(x,r\rho(x)))} \int_{B(x,r\rho(x))} |f(y)|\,d\mu(y), \tag{4.30}$$

uniformly in $x \in \mathcal{X}$. Choose $\ell_0 \in \mathbb{Z}$ such that $2^{-\ell_0-1} < 2a\Theta_a \leq 2^{-\ell_0}$. Let $\{S_k\}_{k=\ell_0}^{\infty}$ be the ℓ_0-AOTI constructed in Proposition 3.2. For any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, set $\mathcal{M}(f)(x) \equiv \sup_{k\geq\ell_0}|S_k(f)(x)|$ (the operator $\mathcal{M}$ here in fact depends on a). This combined with (4.30) and (3.1) yields that

$$\begin{aligned}\mathcal{M}_a(f)(x) &\lesssim \sup_{k\geq\ell_0}\frac{1}{\mu(B(x,2^{-k}\rho(x)))}\int_{B(x,2^{-k}\rho(x))}|f(y)|d\mu(y)\\ &\lesssim \sup_{k\geq\ell_0}\int_{B(x,2^{-k}\rho(x))}S_k(x,y)|f(y)|d\mu(y)\sim\mathcal{M}(|f|)(x).\end{aligned}\tag{4.31}$$

Now the proof of Theorem 4.4 falls into proving the inequalities (4.29) and for the operator $\mathcal{M}$. To this end, we set $\mathfrak{B}_1 \equiv \mathbb{C}$, $\mathfrak{B}_2 \equiv \ell^\infty$, and view $\mathcal{M}$ as the linear operator that maps $\mathfrak{B}_1$-valued functions f to $\mathfrak{B}_2$-valued functions $\{S_k(f)\}_{k=\ell_0}^{\infty}$. Precisely, define $\overrightarrow{\mathcal{M}}(f) \equiv \{S_k(f)\}_{k=\ell_0}^{\infty}$. The corresponding kernel of $\overrightarrow{\mathcal{M}}$, say $\overrightarrow{K}(x,y) = \{S_k(x,y)\}_{k=\ell_0}^{\infty}$, is defined by that for any $t \in \mathbb{C}$, $\overrightarrow{K}(x,y)(t) \equiv \{S_k(x,y)t\}_{k=\ell_0}^{\infty}$. If we appropriately choose $\mathcal{N} \equiv \{(x,y) \in \mathcal{X}\times\mathcal{X} : d(x,y) \leq C_1 2^{-\ell_0}\rho(x)\}$, then $\overrightarrow{\mathcal{M}}_{\mathrm{local}} = \overrightarrow{\mathcal{M}}$. Recall that we are using the ℓ_0-AOTI $\{S_k\}_{k=\ell_0}$ constructed in Proposition 3.2, which is nonnegative and $\int_{\mathcal{X}} S_k(x,y)d\mu(y) = 1$ for all $x \in \mathcal{X}$. Therefore, for all $f \in L^\infty(\mathcal{X})$ and $x \in \mathcal{X}$,

$$|S_k(f)(x)| \leq \|S_k(x,\cdot)\|_{L^1(\mathcal{X})}\|f\|_{L^\infty(\mathcal{X})} \leq \|f\|_{L^\infty(\mathcal{X})},\tag{4.32}$$

which implies that $\overrightarrow{\mathcal{M}}$ is bounded from $L^\infty(\mathcal{X},\mathfrak{B}_1)$ to $L^\infty(\mathcal{X},\mathfrak{B}_2)$.

Now we show that $\overrightarrow{K}$ satisfies (4.6) through (4.8). In fact, for all $x,y \in \mathcal{X}$, since S_k is nonnegative and satisfies Definition 3.1(i),

$$\left\|\overrightarrow{K}(x,y)\right\|_{\mathbb{C}\to\ell^\infty} = \sup_{k\geq\ell_0} S_k(x,y) \lesssim \sup_{k\geq\ell_0}\frac{\chi_{\{d(x,y)\leq C_1 2^{-k}\rho(x)\}}(x,y)}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)} \lesssim \frac{1}{V(x,y)}.\tag{4.33}$$

Observe that

$$\begin{aligned}&\int_{d(x,y)\geq 2d(z,y)}\left\|\overrightarrow{K}(x,y)-\overrightarrow{K}(x,z)\right\|_{\mathbb{C}\to\ell_\infty}d\mu(x)\\ &\quad\leq\sum_{k=\ell_0}^{\infty}\int_{d(x,y)\geq 2d(z,y)}|S_k(x,y)-S_k(x,z)|d\mu(x)\equiv\sum_{k=\ell_0}^{\infty}I_k.\end{aligned}\tag{4.34}$$

When $d(z,y) > C_1 2^{-k}[\rho(z)\vee\rho(y)]$ and $d(x,y) \geq 2d(z,y)$, by the support condition of S_k, we have $S_k(x,y) = S_k(x,z) = 0$. So the summation $\sum_{k=\ell_0}^{\infty}$ is valid only for k satisfying $d(z,y) \leq C_1 2^{-k}\Theta_{C_1 2^{-\ell_0}}\rho(y)$. Moreover, for these k's, if $S_k(x,y) - S_k(x,z) \neq 0$, then $d(x,y) \leq \widetilde{C}2^{-k}\rho(y)$, where $\tilde{C}$ is a positive constant depending only on ℓ_0. Since $\{S_k\}_{k=\ell_0}^{\infty}$ satisfy (i) and (iii) of Definition 3.1, by Remark 3.3 and (2.2),

$$\begin{aligned} \mathrm{I}_k &\lesssim \int_{d(x,y)\le\tilde{C}2^{-k}\rho(y)} \frac{d(y,z)}{2^{-k}\rho(y)} \frac{1}{V_{2^{-k}\rho(x)}(x)+V_{2^{-k}\rho(y)}(y)}\, d\mu(x) \\ &\lesssim \frac{d(y,z)}{2^{-k}\rho(y)} \frac{\mu\left(B\left(y,\tilde{C}2^{-k}\rho(y)\right)\right)}{V_{2^{-k}\rho(y)}(y)} \lesssim \frac{d(y,z)}{2^{-k}\rho(y)}. \end{aligned} \tag{4.35}$$

Taking summation over all k satisfying $d(y,z) \le C_1 2^{-k}\Theta_{C_1 2^{-\ell_0}}\rho(y)$ yields that $\sum_{k=\ell_0}^{\infty} I_k \lesssim 1$. Thus, (4.7) holds. Likewise, (4.8) holds by symmetry.

Applying Theorem 4.1 yields that $\overrightarrow{\mathcal{M}}$ is bounded from $L^q(\mathcal{X},\mathfrak{B}_1)$ to $L^q(\mathcal{X},\mathfrak{B}_2)$ when $q\in(1,\infty)$. Then, by Proposition 4.2 and $\|\overrightarrow{\mathcal{M}}(f_j)(x)\|_{\mathfrak{B}_2} = \mathcal{M}(f_j)(x)$, we obtain that (4.29) hold for the operator $\mathcal{M}$. This finishes the proof of Theorem 4.4 □

5. Littlewood-Paley Operators

Fix $\ell_0\in\mathbb{Z}$. Let $\{S_k\}_{k=\ell_0}^{\infty}$ be an ℓ_0-AOTI as in Definition 3.1. Set $D_{\ell_0}\equiv S_{\ell_0}$, and $D_k = S_k - S_{k-1}$ for $k>\ell_0$. Without loss of generality, we may assume that $D_k\equiv 0$ for $k<\ell_0$.

For any given measurable function f on $\mathcal{X}$, we define the Littlewood-Paley g-function $g(f)$ by setting, for all $x\in\mathcal{X}$,

$$g(f)(x) \equiv \left\{\sum_{j=\ell_0}^{\infty} |D_j(f)(x)|^2\right\}^{1/2}. \tag{5.1}$$

Recall that $I = \sum_{k=\ell_0}^{\infty} D_k$ in $L^p(\mathcal{X})$ with $p\in[1,\infty)$; see Proposition 3.5(iv). Following Coifman's idea in [20], we know that for any $N\in\mathbb{N}$,

$$I = \left(\sum_{k=\ell_0}^{\infty} D_k\right)\left(\sum_{j=\ell_0}^{\infty} D_j\right) = \sum_{|\ell|>N}\sum_{k=\ell_0}^{\infty} D_{k+\ell}D_k + \sum_{k=\ell_0}^{\infty} D_k^N D_k \equiv R_N + T_N, \tag{5.2}$$

in $L^p(\mathcal{X})$, where $D_k^N \equiv \sum_{|\ell|\le N} D_{k+\ell}$.

Applying an argument similar to the proof of Lemma 3.4 in [6] yields the following conclusion.

Lemma 5.1. *Let $N\in\mathbb{N}$ and R_N be as in* (5.2). *Then there exists a positive constant C, depending on ℓ_0, C_1, and C_2, such that for all $N\in\mathbb{N}$,*

$$\|R_N\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \le C2^{-N}. \tag{5.3}$$

Proof. For any $|\ell| \le N$, $k \ge \ell_0$, and $j \ge \ell_0$, by Lemma 3.4 and Proposition 3.5(iii), we obtain that for all $f \in L^2(\mathcal{X})$,

$$\begin{aligned}
\left\|D_{k+\ell}D_k\left(D_{j+\ell}D_j\right)^*f\right\|_{L^2(\mathcal{X})} &\lesssim 2^{-|\ell|}\left\|\left(D_{j+\ell}D_j\right)^*f\right\|_{L^2(\mathcal{X})} \lesssim 2^{-2|\ell|}\|f\|_{L^2(\mathcal{X})},\\
\left\|D_{k+\ell}D_k\left(D_{j+\ell}D_j\right)^*f\right\|_{L^2(\mathcal{X})} &= \left\|D_{k+\ell}D_kD_j^*D_{j+\ell}^*f\right\|_{L^2(\mathcal{X})} \lesssim \left\|D_kD_j^*D_{j+\ell}^*f\right\|_{L^2(\mathcal{X})}\\
&\lesssim 2^{-|k-j|}\left\|D_{j+\ell}^*f\right\|_{L^2(\mathcal{X})} \lesssim 2^{-|k-j|}\|f\|_{L^2(\mathcal{X})}.
\end{aligned} \tag{5.4}$$

The geometric mean between (5.4) gives

$$\left\|D_{k+\ell}D_k\left(D_{j+\ell}D_j\right)^*\right\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \lesssim 2^{-|\ell|}2^{-|k-j|/2}. \tag{5.5}$$

Similarly,

$$\left\|D_{j+\ell}D_j\left(D_{k+\ell}D_k\right)^*\right\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \lesssim 2^{-|\ell|}2^{-|k-j|/2}. \tag{5.6}$$

Applying the Cotlar-Knapp-Stein lemma (see [32, page 280]) together with (5.5) and (5.6) yields that $\|\sum_{k=\ell_0}^{\infty} D_{k+\ell}D_k\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \lesssim 2^{-|\ell|}$. Thus, we obtain

$$\|R_N\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \le \sum_{|\ell|>N}\left\|\sum_{k=\ell_0}^{\infty} D_{k+\ell}D_k\right\|_{L^2(\mathcal{X})\to L^2(\mathcal{X})} \lesssim 2^{-N}, \tag{5.7}$$

which completes the proof of Lemma 5.1. □

The locally reverse doubling condition (2.3) is used in the proof of following lemma.

Lemma 5.2. *For any $\eta > 0$ and $\ell_0 \in \mathbb{Z}$, there exists a positive constant C depending on η and ℓ_0 such that*

$$\sum_{k=\ell_0}^{\infty} \frac{\chi_{\{d(x,y)\le\eta 2^{-k}\rho(x)\}}(x,y)}{\mu\left(B\left(x,2^{-k}\rho(x)\right)\right)} \le C\frac{1}{V(x,y)}. \tag{5.8}$$

Proof. Notice that

$$\sum_{k=\ell_0}^{\infty} \frac{\chi_{\{d(x,y)\le\eta 2^{-k}\rho(x)\}}(x,y)}{\mu\left(B\left(x,2^{-k}\rho(x)\right)\right)} \lesssim \sum_{k=\ell_0}^{\infty} \frac{\chi_{\{d(x,y)\le\eta 2^{-k}\rho(x)\}}(x,y)}{\mu\left(B\left(x,2^{-k}\rho(x)\right)\right)}\frac{2^{-k}\rho(x)}{2^{-k}\rho(x)+d(x,y)}. \tag{5.9}$$

Observe that the summation in the above inequality equals to 0 if $d(x,y) > \eta 2^{-\ell_0}\rho(x)$. Choose $k_0 \in \mathbb{N}$ such that $2^{-k_0-\ell_0}\eta\rho(x) \le d(x,y) < 2^{-k_0-\ell_0+1}\eta\rho(x)$. If $\ell_0 \le k \le k_0 + \ell_0$, by Proposition 2.3(iv), there exists a positive constant $\kappa_{\tilde{a}}$ with $\tilde{a} \equiv 2^{-\ell_0+1}\eta$ such that

$$\begin{aligned} V(x,y) &\le \mu\Big(B\Big(x, 2^{-k_0-\ell_0+1}\eta\rho(x)\Big)\Big) \\ &\lesssim 2^{(-k_0-\ell_0+k)\kappa_{\tilde{a}}}\mu\Big(B\Big(x, 2^{-k+1}\eta\rho(x)\Big)\Big) \\ &\lesssim 2^{(-k_0-\ell_0+k)\kappa_{\tilde{a}}}\mu\Big(B\Big(x, 2^{-k}\rho(x)\Big)\Big), \end{aligned} \tag{5.10}$$

where in the last inequality we used (2.2). This further implies that

$$\sum_{k=\ell_0}^{k_0+\ell_0} \frac{\chi_{\{d(x,y)\le\eta 2^{-k}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))}\frac{2^{-k}\rho(x)}{2^{-k}\rho(x)+d(x,y)} \lesssim \frac{1}{V(x,y)}\sum_{k=\ell_0}^{k_0+\ell_0} 2^{(-k_0-\ell_0+k)\kappa_{\tilde{a}}} \lesssim \frac{1}{V(x,y)}. \tag{5.11}$$

If $k > k_0 + \ell_0$ and $d(x,y) \le \eta 2^{-k}\rho(x)$, then by (2.2),

$$V(x,y) \le \mu\Big(B\Big(x, 2^{-k}\eta d(x,y)\Big)\Big) \le D_\eta \mu\Big(B\Big(x, 2^{-k}d(x,y)\Big)\Big), \tag{5.12}$$

and hence

$$\begin{aligned} \sum_{k=k_0+\ell_0+1}^{\infty} \frac{\chi_{\{d(x,y)\le\eta 2^{-k}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))}\frac{2^{-k}\rho(x)}{2^{-k}\rho(x)+d(x,y)} &\lesssim \frac{1}{V(x,y)}\sum_{k=k_0+\ell_0+1}^{\infty}\frac{2^{-k}\rho(x)}{d(x,y)} \\ &\lesssim \frac{1}{V(x,y)}. \end{aligned} \tag{5.13}$$

Combining the last two formulae yields the desired result of Lemma 5.2. □

Lemmas 5.1 and 5.2 together with the scalar version of Theorem 4.1 yield the following conclusion.

Corollary 5.3. *Let $N \in \mathbb{N}$ and R_N be as in* (5.2)*. Then, for any $p \in (1,\infty)$, there exists a positive constant C, depending on p, ℓ_0, C_1, and C_2, such that for all $N \in \mathbb{N}$,*

$$\|R_N\|_{L^p(\mathcal{X})\to L^p(\mathcal{X})} \le C2^{-N/2}; \tag{5.14}$$

while $p = 1$, $\|R_N\|_{L^1(\mathcal{X})\to L^{1,\infty}(\mathcal{X})} \le C2^{-N/2}$.

Proof. For all $x, y \in \mathcal{X}$, we write

$$R_N(x,y) = \sum_{\ell=N}^{\infty}\sum_{k=\ell_0}^{\infty} D_{k+\ell}D_k(x,y) + \sum_{\ell=-\infty}^{-N}\sum_{k=\ell_0}^{\infty}\cdots \equiv R_N^1(x,y) + R_N^2(x,y). \tag{5.15}$$

By (i) and (ii) of Lemma 3.4 together with Lemma 5.2, we obtain

$$\left|R_N^1(x,y)\right| \lesssim \sum_{\ell=N}^{\infty}\sum_{k=\ell_0}^{\infty} 2^{-\ell}\frac{\chi_{\{d(x,y)\le C_4 2^{-k}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))} \lesssim 2^{-N}\frac{\chi_{\{d(x,y)\le C_4\rho(x)\}}(x,y)}{V(x,y)}, \tag{5.16}$$

where C_4 is as in Lemma 3.4(i). If $d(y,y') \le d(x,y)/2$ and $D_{k+\ell}D_k(x,y) - D_{k+\ell}D_k(x,y') \ne 0$, then by Lemma 3.4(i),

$$d(x,y) \le 2C_4 2^{-k}\rho(x). \tag{5.17}$$

Thus, when $d(y,y') \le d(x,y)/2$, using (5.17), Lemma 3.4(iii), and (3.20), we obtain

$$\begin{aligned}
\left|R_N^1(x,y) - R_N^1(x,y')\right| &\le \sum_{\ell=N}^{\infty}\sum_{k=\ell_0}^{\infty}\left|D_{k+\ell}D_k(x,y) - D_{k+\ell}D_k(x,y')\right| \\
&\lesssim \sum_{\ell=N}^{\infty}\sum_{k=\ell_0}^{\infty} 2^{-\ell/2}\left(\frac{d(y,y')}{2^{-k}\rho(x)}\right)^{1/2}\frac{\chi_{\{d(x,y)\le 2C_4 2^{-k}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))} \\
&\lesssim 2^{-N/2}\left(\frac{d(y,y')}{d(x,y)}\right)^{1/2}\sum_{k=\ell_0}^{\infty}\frac{\chi_{\{d(x,y)\le 2C_4 2^{-k}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))},
\end{aligned} \tag{5.18}$$

which together with Lemma 5.2 and (2.2) implies that for the constant $\widetilde{C} \equiv 2^{-\ell_0+1}C_4$,

$$\begin{aligned}
&\int_{d(x,y)\ge 2d(y,y')}\left|R_N^1(x,y) - R_N^1(x',y)\right| d\mu(x) \\
&\quad\lesssim 2^{-N/2}\sum_{k=1}^{\infty}\int_{2^k d(y,y')\le d(x,y)<2^{k+1}d(y,y')}\left[\frac{d(y,y')}{d(x,y)}\right]^{1/2}\frac{\chi_{\{d(x,y)\le \widetilde{C}\rho(x)\}}(x,y)}{V(x,y)}d\mu(x) \\
&\quad\lesssim 2^{-N/2}.
\end{aligned} \tag{5.19}$$

Similarly, R_N^2 satisfies the same estimates as in (5.16) and (5.19). From this and a symmetric argument, we deduce that R_N satisfies (4.6) through (4.8). Notice that $(R_N)_{\text{local}} = R_N$ if we choose $\mathcal{N} \equiv \{(x,y) \in \mathcal{X} \times \mathcal{X} : d(x,y) \le 2^{-\ell_0+1}C_4\rho(x)\}$. Applying Lemma 5.1, Theorem 4.1, and Remark 4.3, we obtain Corollary 5.3. □

From Corollary 5.3, we easily deduce the following Calderón reproducing formulae, which are basic tools connecting considered operators with corresponding function spaces.

Corollary 5.4. *For any given $p \in (1, \infty)$, there exists $N \in \mathbb{N}$ large enough, depending on p, ℓ_0, and $\mathcal{X}$, such that for all $f \in L^p(\mathcal{X})$,*

$$f = \sum_{k=\ell_0}^{\infty} T_N^{-1} D_k^N D_k(f) = \sum_{k=\ell_0}^{\infty} D_k^N D_k T_N^{-1}(f), \tag{5.20}$$

in $L^p(\mathcal{X})$.

Proof. For any given $p \in (1, \infty)$, by Corollary 5.3, there exists $N \in \mathbb{N}$ large enough such that $\|R_N\|_{L^p(\mathcal{X}) \to L^p(\mathcal{X})} < 1/2$. This combined with (5.2) implies that T_N^{-1} exists and is bounded on $L^p(\mathcal{X})$; see [29, page 69,Theorem 2]. In view of (5.2), $T_N = \sum_{k=\ell_0}^{\infty} D_k^N D_k$ in $L^p(\mathcal{X})$. Therefore, the desired conclusions of Corollary 5.4 hold. □

Proposition 5.5. *There exists a positive constant C, depending on ℓ_0, C_1, and C_2, such that for all $f \in L^2(\mathcal{X})$,*

$$C^{-1} \|f\|_{L^2(\mathcal{X})}^2 \le \sum_{k=\ell_0}^{\infty} \|D_k(f)\|_{L^2(\mathcal{X})}^2 \le C \|f\|_{L^2(\mathcal{X})}^2. \tag{5.21}$$

Proof. By Proposition 3.5(iii), $\|S_{\ell_0}(f)\|_{L^2(\mathcal{X})} \lesssim \|f\|_{L^2(\mathcal{X})}$. Therefore, to show the second inequality of (5.21), it suffices to show that

$$\sum_{k=\ell_0+1}^{\infty} \|D_k(f)\|_{L^2(\mathcal{X})}^2 \lesssim \|f\|_{L^2(\mathcal{X})}^2. \tag{5.22}$$

In fact, by Lemma 3.4 and the proof of Proposition 3.5(iii), there exists a positive constant C, which depends on ℓ_0, C_1, and C_2, such that for all $k, j \ge \ell_0 + 1$,

$$\left\| D_k D_j^* \right\|_{L^2(\mathcal{X}) \to L^2(\mathcal{X})} + \left\| D_j^* D_k \right\|_{L^2(\mathcal{X}) \to L^2(\mathcal{X})} \le C 2^{-|k-j|}. \tag{5.23}$$

This combined with the Cotlar-Knapp-Stein lemma yields (5.22). Thus, the second inequality of (5.21) holds.

Now we show the first inequality of (5.21). By Lemma 5.1, for large $N \in \mathbb{N}$, we have

$$\|I - T_N\|_{L^2(\mathcal{X}) \to L^2(\mathcal{X})} = \|R_N\|_{L^2(\mathcal{X}) \to L^2(\mathcal{X})} < \frac{1}{2}, \tag{5.24}$$

and hence T_N is invertible on $L^2(\mathcal{X})$. Thus, for large N, there exists a positive constant C, depending on N, such that $\|f\|_{L^2(\mathcal{X})} \le C \|T_N(f)\|_{L^2(\mathcal{X})}$ for all $f \in L^2(\mathcal{X})$.

For any $f \in L^2(\mathcal{X})$ and $h \in L^2(\mathcal{X})$, by Hölder's inequality,

$$\begin{aligned}\left|\langle T_N(f), h\rangle\right| &= \left|\left\langle \sum_{k=\ell_0}^{\infty} D_k^N D_k(f), h\right\rangle\right| = \left|\sum_{k=\ell_0}^{\infty}\left\langle D_k(f), \left(D_k^N\right)^* h\right\rangle\right| \\ &\leq \left\{\sum_{k=\ell_0}^{\infty}\|D_k(f)\|_{L^2(\mathcal{X})}^2\right\}^{1/2}\left\{\sum_{k=\ell_0}^{\infty}\left\|D_k^N(h)\right\|_{L^2(\mathcal{X})}^2\right\}^{1/2}.\end{aligned} \tag{5.25}$$

Then, using $D_k^N \equiv \sum_{|\ell|\leq N} D_{k+\ell}$, Minkowski's inequality, and the second inequality of (5.21), we obtain

$$\begin{aligned}\sum_{k=\ell_0}^{\infty}\left\|D_k^N(h)\right\|_{L^2(\mathcal{X})}^2 &\lesssim \sum_{k=0}^{\infty}\sum_{|\ell|\leq N}\|D_{k+\ell}(h)\|_{L^2(\mathcal{X})}^2 \\ &\lesssim \sum_{k=\ell_0}^{\infty}\|D_k(h)\|_{L^2(\mathcal{X})}^2 \lesssim \|h\|_{L^2(\mathcal{X})}^2.\end{aligned} \tag{5.26}$$

This combined with (5.25) together with a dual argument yields that

$$\|f\|_{L^2(\mathcal{X})}^2 \lesssim \|T_N(f)\|_{L^2(\mathcal{X})}^2 \lesssim \sum_{k=\ell_0}^{\infty}\|D_k(f)\|_{L^2(\mathcal{X})}^2, \tag{5.27}$$

which completes the proof of Proposition 5.5. □

The main result of this section is the following characterization of $L^p(\mathcal{X})$ for $p \in (1, \infty)$ by using the Littlewood-Paley g-function.

Theorem 5.6. *Let $p \in (1, \infty)$. Then there exists a positive constant C, depending on p, ℓ_0, C_1, and C_2, such that for all $f \in L^p(\mathcal{X})$,*

$$C^{-1}\|f\|_{L^p(\mathcal{X})} \leq \|g(f)\|_{L^p(\mathcal{X})} \leq C\|f\|_{L^p(\mathcal{X})}, \tag{5.28}$$

while $p = 1$, for all $f \in L^1(\mathcal{X})$,

$$\|g(f)\|_{L^{1,\infty}(\mathcal{X})} \leq C\|f\|_{L^1(\mathcal{X})}. \tag{5.29}$$

Proof. We first show (5.29) and $\|g(f)\|_{L^p(\mathcal{X})} \lesssim \|f\|_{L^p(\mathcal{X})}$ by using Theorem 4.1. To this end, set $\mathfrak{B}_1 \equiv \mathbb{C}$, $\mathfrak{B}_2 \equiv \ell_2$, and define $\overrightarrow{T}(f) \equiv \{D_k(f)\}_{k=0}^{\infty}$. The corresponding kernel of $\overrightarrow{T}$, say $\overrightarrow{K}(x, y)$, is given by that for any $t \in \mathbb{C}$, $\overrightarrow{K}(x, y)(t) = \{D_k(x, y)t\}_{k=\ell_0}^{\infty}$. By Proposition 5.5, $\overrightarrow{T}$ is bounded from $L^2(\mathcal{X}, \mathfrak{B}_1)$ to $L^2(\mathcal{X}, \mathfrak{B}_2)$. Observe that $D_k(x, y) = 0$ when $d(x, y) > C_1 2^{-k+1}[\rho(x) \wedge \rho(y)]$. Then $\overrightarrow{T}_{\rm local}$ defined as in (4.9) equals to $\overrightarrow{T}$ if we choose $\mathcal{N} \equiv \{(x, y) \in \mathcal{X} \times \mathcal{X} : d(x, y) \leq 2^{-\ell_0+1}C_1\rho(x)\}$. It remains to show that $\overrightarrow{K}(x, y)$ satisfies the hypotheses (4.6) through (4.8).

For any $x, y \in \mathcal{X}$, since D_k satisfies Definition 3.1(i), we apply Lemma 5.2 to obtain

$$\left\|\vec{K}(x,y)\right\|_{\mathbb{C}\to\ell^2} = \left\{\sum_{k=\ell_0}^{\infty}|D_k(x,y)|^2\right\}^{1/2} \lesssim \sum_{k=\ell_0}^{\infty}\frac{\chi_{\{d(x,y)\le C_1 2^{-k+1}\rho(x)\}}(x,y)}{\mu(B(x,2^{-k}\rho(x)))} \lesssim \frac{1}{V(x,y)}, \tag{5.30}$$

and hence $\vec{K}$ satisfies (4.6). An argument similar to the proof of Theorem 4.4 shows that $\vec{K}(x,y)$ satisfies the hypotheses (4.7) and (4.8). Applying Theorem 4.1 and Remark 4.3 yields (5.29) and $\|g(f)\|_{L^p(\mathcal{X})} \lesssim \|f\|_{L^p(\mathcal{X})}$.

The proof for $\|f\|_{L^p(\mathcal{X})} \lesssim \|g(f)\|_{L^p(\mathcal{X})}$ follows by duality. In fact, Corollary 5.3 implies that $\|R_N\|_{L^p(\mathcal{X})\to L^p(\mathcal{X})} < 1/2$ for large enough N. Furthermore, for large N, the operator $T_N \equiv I - R_N$ is invertible on $L^p(\mathcal{X})$ and $\|f\|_{L^p(\mathcal{X})} \lesssim \|T_N(f)\|_{L^p(\mathcal{X})}$.

By the second inequality of (5.28) together with an argument similar to (5.25) and (5.26), we obtain that for all $h \in L^{p'}(\mathcal{X})$ with $\|h\|_{L^{p'}(\mathcal{X})} \le 1$,

$$\begin{aligned}
|\langle T_N(f),h\rangle| &\le \int_{\mathcal{X}}\sum_{k=\ell_0}^{\infty}\left|D_k(f)(x)\left(D_k^N\right)^*(h)(x)\right|d\mu(x)\\
&\le \int_{\mathcal{X}}\left(\sum_{k=\ell_0}^{\infty}|D_k(f)(x)|^2\right)^{1/2}\left(\sum_{k=\ell_0}^{\infty}\left|\left(D_k^N\right)^*(h)(x)\right|^2\right)^{1/2}d\mu(x)\\
&\le \left\|\left(\sum_{k=\ell_0}^{\infty}|D_k(f)|^2\right)^{1/2}\right\|_{L^p(\mathcal{X})}\left\|\left(\sum_{k=\ell_0}^{\infty}\left|\left(D_k^N\right)^*(h)\right|^2\right)^{1/2}\right\|_{L^{p'}(\mathcal{X})} \lesssim \|g(f)\|_{L^p(\mathcal{X})}.
\end{aligned} \tag{5.31}$$

Thus, $\|f\|_{L^p(\mathcal{X})} \lesssim \|T_N(f)\|_{L^p(\mathcal{X})} \lesssim \|g(f)\|_{L^p(\mathcal{X})}$, which completes the proof of Theorem 5.6. □

As an application of Proposition 4.2, Theorem 5.6 has the following vector-valued extension; see, for example, [3] for the Euclidean case. The details are omitted.

Corollary 5.7. *Let $p \in (1,\infty)$. Then there exist positive constants C_p and C, which depend on ℓ_0, such that for all $\{f_j\}_{j\in\mathbb{Z}} \subset L^p(\mathcal{X})$,*

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{k=\ell_0}^{\infty}|D_k(f_j)|^2\right\}^{1/2}\right\|_{L^p(\mathcal{X})} \le C_p\left\|\left\{\sum_{j\in\mathbb{Z}}|f_j|^2\right\}^{1/2}\right\|_{L^p(\mathcal{X})}, \tag{5.32}$$

and for all $\{f_j\}_{j\in\mathbb{Z}} \subset L^1(\mathcal{X})$,

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{k=\ell_0}^{\infty}|D_k(f_j)|^2\right\}^{1/2}\right\|_{L^{1,\infty}(\mathcal{X})} \le C\left\|\left\{\sum_{j\in\mathbb{Z}}|f_j|^2\right\}^{1/2}\right\|_{L^1(\mathcal{X})}. \tag{5.33}$$

In particular, the inequalities (5.32) *and* (5.33) *still hold if their left-hand sides are replaced by* $\|\{\sum_{j=\ell_0}^{\infty}|D_j(f_j)|^2\}^{1/2}\|_{L^p(\mathcal{X})}$ *and* $\|\{\sum_{j=\ell_0}^{\infty}|D_j(f_j)|^2\}^{1/2}\|_{L^{1,\infty}(\mathcal{X})}$*, respectively.*

6. Some Examples

In this section, we present some typical examples of the locally doubling measure metric spaces satisfying Definition 2.1. All results in previous sections, especially, Theorems 4.4 and 5.6 apply to various settings appearing in the following examples and, to the best of our knowledge, are new even for these typical settings.

Example 6.1 (Gauss measure metric spaces). Let $(\mathbb{R}^n, |\cdot|, d\gamma)$ be the Gauss measure metric space and for all $x \in \mathbb{R}^n$, $\rho(x) \equiv \min\{1, 1/|x|\}$; see Mauceri and Meda [12].

The space $(\mathbb{R}^n, |\cdot|, \gamma)_\rho$ is a locally doubling measure metric space in the sense of Definition 2.1. To see this, for $a \in (0,\infty)$, by the definition of $\mathcal{B}_a$, any Euclidean ball $B \subset \mathbb{R}^n$ is in the class $\mathcal{B}_a$ if and only if $r_B \le a\rho(c_B)$, where c_B is the center of B and r_B is its radius. For all $B \in \mathcal{B}_a$ and $y \in B$,

$$(a+1)^{-1}\rho(y) \le \rho(c_B) \le (a+1)\rho(y); \tag{6.1}$$

see [12, (3.4)] when $a = 1$ and see [35] for general $a \in (0,\infty)$. Thus, ρ satisfies (2.1) with $\Theta_a \equiv a+1$. By Proposition 2.5(a) and its proof, the Gauss measure γ satisfies the locally doubling Condition (2.2) and the reverse locally doubling Condition (2.3).

For the admissible function $\widetilde{\rho}(x) \equiv 1/(1+|x|)$ for all $x \in \mathbb{R}^n$, a similar argument shows that $(\mathbb{R}^n, |\cdot|, \gamma)_{\widetilde{\rho}}$ is also a locally doubling measure metric space in the sense of Definition 2.1.

It should be mentioned that the admissible function ρ or $\widetilde{\rho}$ plays important roles in analysis on Gauss measure metric spaces, especially in the study of operators related to Ornstein-Uhlenbeck semigroup; see, for example, [11–15, 17]. These operators can be represented as integral operators associated with kernels K: for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$,

$$T(f)(x) \equiv \int_{\mathbb{R}^n} K(x,y)f(y)\,d\gamma(y); \tag{6.2}$$

see, for example, [12]. Fix $\delta > 0$. Some common properties shared by these operators are that their kernels $K(x,y)$ satisfy that

$$K(x,y) \le C_\delta \frac{e^{|y|^2}}{|x-y|^n}, \quad |\partial_x K(x,y)| + |\partial_y K(x,y)| \le C_\delta \frac{e^{|y|^2}}{|x-y|^{n+1}}, \tag{6.3}$$

only in the regions $\mathcal{N}_\delta \equiv \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y, |x-y| \le \delta\rho(x)\}$ with constants C_δ growing exponentially as $\delta \to \infty$; see, for example, [11, 12, 15, 16]. A usual treatment in considering the boundedness of these operators on $L^p(\mathbb{R}^n, d\gamma)$ ($p \in [1,\infty)$) is by splitting the operator into a local part and a global part, namely, $T_{\mathrm{local}}(f)(x) \equiv T(\chi_{\mathcal{N}_\delta}(x,\cdot)f)(x)$ for all $x \in \mathbb{R}^n$ and $T_{\mathrm{global}} \equiv T - T_{\mathrm{local}}$; see, for instance, [11, 14, 17] and the references therein.

We remark that Theorem 4.1 is an extension of [11, Theorem 2.1] in a more abstract setting, which can be applied to local versions of operators such as Riesz transforms of any order (see, e.g., [11, 14]), imaginary powers of the Ornstein-Uhlenbeck operator (see, e.g., [16]), and some square functions (see [11, 17]).

Example 6.2 (Nondoubling measure metric spaces in [18, 19]). Carbonaro, et al. [18, 19] developed an H^1-BMO theory on the metric space $(M, d, \mu)_\rho$ withholding the following properties:

(i) $\rho \equiv 1$ on M, and the *locally doubling property* (2.2) holds;

(ii) *isoperimetric property when* $\mu(M) = \infty$: there exist positive constants κ_0 and C such that for every bounded open set A,

$$\mu\left(\left\{x \in A : d\left(x, A^{\complement}\right) \leq \kappa\right\}\right) \geq C\kappa\mu(A), \quad \forall \kappa \in (0, \kappa_0]; \tag{6.4}$$

complementary isoperimetric property when $\mu(M) < \infty$: there exists a ball $B_0 \subset M$, $\kappa_0 > 0$, and $C > 0$ such that (6.4) holds for every open set A contained in $M \setminus \overline{B}_0$;

(iii) *approximation midpoint property*: there exist $R_0 \in [0, \infty)$ and $\beta \in (1/2, 1)$ such that for every pair of points $x, y \in M$ with $d(x, y) > R_0$, there exists a point $z \in M$ such that $d(x, z) < \beta d(x, y)$ and $d(z, y) < \beta d(x, y)$.

Applying Proposition 2.3(vi), we can easily show that $(M, d, \mu)_\rho$ satisfying (i) and (ii) as above falls into the scope of Definition 2.1; however, (iii) is not necessary.

Example 6.3 (Complete Riemannian manifold with Ricci curvature bounded from below). Let $(M, d, \mu)_\rho$ be an n-dimensional complete Riemannian manifold M endowed with Riemannian distance d and Riemannian density μ, and also $\rho(x) \equiv 1$ for all $x \in M$. Let g denote the Riemannian metric tensor. When M has the Ricci curvature bounded from below, namely, $\operatorname{Ric} M \geq -Kg$ for some $K \geq 0$, the Bishop-Gromov comparison theorem implies that for all $x \in M$ and $r > 0$,

$$\mu(B(x, 2r)) \leq 2^n \exp\left(2r\sqrt{(n-1)K}\right)\mu(B(x, r)); \tag{6.5}$$

see [36] or [37]. This implies that $(M, d, \mu)_\rho$ satisfies the locally doubling condition (2.2). It is well known that any Riemannian manifold has a differentiable structure of C^∞, and hence the annulus $(2B) \setminus B \neq \emptyset$ for all balls B contained in M. Then Proposition 2.3(vi) implies that the locally reverse doubling condition (2.3) holds for $(M, d, \mu)_\rho$. Therefore, any complete Riemannian manifold with Ricci curvature bounded from below is a locally doubling measure metric space in the sense of Definition 2.1. Some curvature-dimension conditions also guarantee the locally doubling condition (2.2); see, for example, [38].

Example 6.4 ($\mathbb{R}^n$ with certain admissible function associated to Schrödinger operators). Let V be a nonnegative function on $\mathbb{R}^n$ ($n \geq 3$) satisfying some reverse Hölder inequality $RH_q(\mathbb{R}^n)$ with $q \geq n/2$, namely, there exists a positive constant C, depending on q and V, such that $\{(1/|B|)\int_B V(x)^q dx\}^{1/q} \leq (C/|B|)\int_B V(x)dx$ for all balls $B \subset \mathbb{R}^n$. Function spaces and operators related to Schrödinger operators $\mathcal{L} \equiv -\Delta + V$ have been the subject of many literatures; see, for example, [21–25, 39–41]. The admissible function associated to $\mathcal{L}$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x, V) \equiv \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) d(y) \le 1 \right\}; \tag{6.6}$$

see Shen [39] for more details. It was proved in [39, Lemma 1.4] that for every positive constant C, there exists a constant $\widetilde{C} \ge 1$ such that for all $|x - y| \le C\rho(x, V)$,

$$\widetilde{C}^{-1} \rho(y, V) \le \rho(x, V) \le \widetilde{C} \rho(y, V), \tag{6.7}$$

which implies that $\rho(\cdot, V)$ satisfies (2.1). Thus, $(\mathbb{R}^n, |\cdot|, dx)_{\rho(\cdot,V)}$ is a locally doubling measure metric space in the sense of Definition 2.1.

Dziubański [21] also developed a theory of Hardy spaces associated to a class of degenerated Schrödinger operators

$$\mathcal{L} f(x) \equiv -\frac{1}{w(x)} \sum_{i,j} \partial_i \big(a_{ij}(\cdot) \partial_j f \big)(x) + V(x) f(x), \tag{6.8}$$

where $a_{ij}(x)$ is a real symmetric matrix satisfying that there exists a positive constant C such that for all $\xi \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$,

$$C^{-1} w(x) |\xi|^2 \le \sum_{i,j} a_{ij}(x) \xi_i \overline{\xi}_j \le C w(x) |\xi|^2, \tag{6.9}$$

with w being a nonnegative weight from the Muckenhoupt class A_2, and $V \ge 0$ belonging to a reverse Hölder class $RH_q(\mathbb{R}^n, w(x)dx)$ with certain large q. The corresponding admissible function is defined by setting, for all $x \in \mathbb{R}^n$,

$$\widetilde{\rho}(x, V) \equiv \sup\left\{ r > 0 : \frac{r^2}{\int_{B(x,r)} w(z) dz} \int_{B(x,r)} V(y) w(y) d(y) \le 1 \right\}. \tag{6.10}$$

By [21, Lemma 4.3], the underlying space $(\mathbb{R}^n, |\cdot|, w(x)dx)_{\widetilde{\rho}(\cdot,V)}$ is also a locally doubling measure metric space in the sense of Definition 2.1.

We remark that $\rho(\cdot, V)$ or $\widetilde{\rho}(\cdot, V)$ as above plays a crucial role in dealing with function spaces such as Hardy spaces or BMO spaces associated to Schrödinger operators; see [22, 41] and the references therein. In a general setting of the RD-space $(\mathcal{X}, d, \mu)$ (see [6, 10]), a theory of Hardy spaces H^1 related to a certain class of admissible functions ρ satisfying (2.4) was developed in [25], which applies to Schrödinger operators on $(\mathbb{R}^n, |\cdot|, dx)_{\rho(\cdot,V)}$, degenerated Schrödinger operators on $(\mathbb{R}^n, |\cdot|, w(x)dx)_{\widetilde{\rho}(\cdot,V)}$, sub-Laplace Schrödinger operators on Heisenberg groups (see [23, 25]), and sub-Laplace Schrödinger operators on nilpotent Lie groups of polynomial growth (see [24, 25]).

Obviously, when ρ satisfies (2.4) and $(\mathcal{X}, d, \mu)$ is an RD-space, $(\mathcal{X}, d, \mu)_\rho$ in [25] is another typical example of locally doubling measure metric space in Definition 2.1.

Example 6.5. Dziubański [40] studied Hardy spaces on $(\mathbb{R}^+, |\cdot|, dx)$ associated to admissible functions $\rho_1(y) \equiv \min\{1, y\}$ and $\rho_2(y) \equiv \min\{y, y^{-1}\}$ for $y \in \mathbb{R}^+$. It is easy to show that (2.4) fails for ρ_1 and ρ_2, while both ρ_1 and ρ_2 satisfy (2.1). Thus, $(\mathbb{R}^+, |\cdot|, dx)_{\rho_i}$ with $i = 1, 2$ are all locally doubling measure metric spaces in the sense of Definition 2.1, and the notion of admissible function here is more general than the one in [25].

Remark 6.6. When ρ is the constant function 1, as in Examples 6.2 and 6.3, the approximation of the identity constructed in Proposition 3.2 can be easily obtained from [6, 20].

The construction of the approximation of the identity for Example 6.1 can be deduced from the case $\rho = 1$ by changing the metric d. In fact, by [19, Proposition 8.2], for every $a > 0$, there exists a positive constant C depending on a such that for all $x, y \in \mathbb{R}^n$ with $d(x, y) < a$,

$$C^{-1}|x-y|[\rho(x)]^{-1} \le d_0(x,y) \le C|x-y|[\rho(x)]^{-1}, \tag{6.11}$$

where $\rho(x) = \min\{1, |x|^{-1}\}$ for all $x \in \mathbb{R}^n$, and d_0 is the distance function defined by the length element $ds^2 \equiv (1+|\nabla\varphi|)^2(dx_1^2 + \cdots + dx_n^2)$ with $\varphi(x) = |x|^2$. Then, the approximation of the identity on $(\mathbb{R}^n, |\cdot|, d\gamma)_\rho$ follows from (6.11) and the approximation of the identity on $(\mathbb{R}^n, d_0, d\gamma)_1$.

Remark 6.7. By [42], with the modified Agmon metric $ds^2 \equiv \rho(x, V)^{-1}(dx_1^2 + \cdots + dx_n^2)$, we define the corresponding distance function

$$\widetilde{d}(x,y) = \inf_\gamma \int_0^1 \rho(\Gamma(t), V)^{-1}|\Gamma'(t)|dt, \tag{6.12}$$

where $\Gamma : [0,1] \to \mathbb{R}^n$ is absolutely continuous and $\Gamma(0) = x$ and $\Gamma(1) = y$. For any given $a \in (0, \infty)$, using [42, Proposition 1.8(c)] and an argument similar to [42, (3.19)], there exist positive constants C and $\widetilde{C}$ such that for all $x \in \mathbb{R}^n$ and $r \in (0, a]$,

$$\begin{aligned} &|x-y| < r\rho(x,V) \text{ implies that } \widetilde{d}(x,y) \le Cr, \\ &\widetilde{d}(x,y) \le r \quad \text{implies that} \quad |x-y| \le \widetilde{C}r\rho(x,V). \end{aligned} \tag{6.13}$$

Obviously, (6.13) is weaker than (6.11). However, (6.13) is not sufficient to deduce an approximation of the identity on $(\mathbb{R}^n, |\cdot|, dx)_{\rho(\cdot,V)}$ from the constructed approximation of the identity on $(\mathbb{R}^n, \widetilde{d}, dx)_1$; the problem lies in how to derive (ii) and (iii) of Definition 3.1. So far, it is not clear whether there is an argument as in Remark 6.6 also works for Example 6.4.

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