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Research Article

Asymptotic Behavior of Solutions of a Periodic Diffusion Equation

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We consider a degenerate parabolic equation with logistic periodic sources. First, we establish the existence of nontrivial nonnegative periodic solutions by monotonicity method. Then by using Moser iterative technique and the method of contradiction, we establish the boundedness estimate of nonnegative periodic solutions, by which we show that the attraction of nontrivial nonnegative periodic solutions, that is, all non-trivial nonnegative solutions of the initial boundary value problem, will lie between a minimal and a maximal nonnegative nontrivial periodic solutions, as time tends to infinity.

1. Introduction

In this paper, we consider the following periodic degenerate parabolic equation:

$$\frac{\partial u}{\partial t} - \Delta u^m = u(a - bu), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $m > 1$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $u_0(x)$ is a nonnegative bounded smooth function, $a = a(x,t)$ and $b = b(x,t)$ are positive continuous functions and of T-periodic ($T > 0$) with respect to t.

The problem (1.1)–(1.3) describes the evolution of the population density of a species living in a habitat Ω and can be proposed for many problems in mathematical biology and fisheries management. The term Δu^m models a tendency to avoid crowding and the reaction term $u(a - bu)$ models the contribution of the population supply due to births and deaths; see [1]. The homogeneous Dirichlet boundary conditions model the inhospitality of the boundary. The time dependence of the coefficients reflects the fact that the time periodic variations of the habitat are taken into account. Reaction diffusion equations with such reaction term can be regarded as generalization of Fisher or Kolomogorv-Petrovsky-Piscunov equations which are used to model the growth of population (see [2, 3]). Especially, when $m = 1$, the (1.1) is the classical Logistic equation and some related problems have attracted much attention of researchers (see [4–6], etc.).

In the recent years, there are a lot of work dedicated to the existence, uniqueness, regularity, and some other qualitative properties, of weak solutions of this kind of degenerate parabolic equations (see [7–9], etc.). But to our knowledge, there is few work that has been accomplished in the literature for periodic degeneracy parabolic equation, and most of the known results so far only concerned with the existence of periodic solutions but not consider the attraction (see [10, 11], etc.). So our work is not a simple extension to the previous work.

The purpose of this paper is to investigate the asymptotic behavior of nontrivial nonnegative solutions of the initial boundary value problem (1.1)–(1.3). Since the equation has periodic sources, it is of no meaning to consider the steady state. So we have to seek some new approaches. Our idea is to consider all nonnegative periodic solutions. We first establish the existence of nontrivial nonnegative periodic solutions by monotone iterative method. Then we establish the a priori upper bound R and a priori lower bound r according to the maximum norm for all nontrivial nonnegative periodic solutions. By which we obtain asymptotic behavior of nontrivial nonnegative solutions of the problem (1.1)–(1.3). That is all nontrivial nonnegative solutions will lie between a minimal and a maximal nonnegative nontrivial periodic solutions, as time tends to infinity.

The paper is organized as follows. In Section 2, we introduce some necessary preliminaries. In Section 3, we establish the existence of nontrivial nonnegative periodic solutions by monotonicity method. In Section 4, we show the asymptotic behavior of nontrivial nonnegative solutions of (1.1)–(1.3).

2. Preliminaries

In this section, we present the definitions of weak solutions and some useful principles.

Since (1.1) is degenerate at points where $u = 0$, problem (1.1)–(1.3) might not have classical solutions in general. Therefore, we focus our main efforts on the discussion of weak solutions in the sense of the following.

Definition 2.1. A nonnegative function u is called to be a weak solution of problem (1.1)–(1.3) in $Q_T = \Omega \times (0,T)$, if $u \in L^2(0,T; H_0^1(\Omega)) \cap C_T(\overline{Q}_T)$ and u satisfies

$$\iint_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi - u(a - bu)\varphi \right) dx\, dt = \int_\Omega u_0 \varphi(x,0) dx - \int_\Omega u(x,T)\varphi(x,T) dx, \tag{2.1}$$

for any test functions $\varphi \in C^1(\overline{Q}_T)$ with $\varphi|_{\partial\Omega\times(0,T)} = 0$, where $C_T(\overline{Q}_T)$ denotes the set of functions which are continuous in $\overline{Q}_T = \overline{\Omega} \times (0,T)$ and of T-periodic with respect to t.

A supersolution $\overline{u}$ (resp., a subsolution $\underline{u}$) is defined in the same way except that the "=" in (2.1) is replaced by "≥" ("≤") and φ is taken to be nonnegative.

Definition 2.2. A function u is called to be a T-periodic solution of problem (1.1)-(1.2) if it is a solution such that $u \in C_T(\overline{Q_T})$. A function $\overline{u}$ is called to be a T-periodic subsolution if it is a subsolution such that $\overline{u}(\cdot,0) \le \overline{u}(\cdot,T)$ in Ω. A function $\underline{u}$ is called to be a T-periodic supersolution if it is a supersolution such that $\underline{u}(\cdot,0) \ge \underline{u}(\cdot,T)$ in Ω. A pair of T-periodic supersolution $\overline{u}$ and subsolution $\underline{u}$ is called to be ordered if $\overline{u} \ge \underline{u}$ in $\overline{Q_T}$.

Several properties of solutions of problem (1.1)–(1.3) are needed in this paper. We first show the comparison principle.

Lemma 2.3 (comparison). *Assume $\underline{u}_0(x) \le \overline{u}_0(x)$, if $\underline{u}(x,t)$ is a subsolution of* (1.1)–(1.3) *corresponding to the initial datum $\underline{u}_0(x)$, and $\overline{u}(x,t)$ is a supersolution of* (1.1)–(1.3) *corresponding to the initial datum $\overline{u}_0(x)$, then $\underline{u}(x,t) \le \overline{u}(x,t)$.*

Proof. Without loss of generality, we might assume that $\overline{u}(x,t), \underline{u}(x,t)$ are bounded. From Definition 2.1, we have

$$\int_0^t\int_\Omega -\overline{u}\frac{\partial\varphi}{\partial t} + \nabla\overline{u}^m\nabla\varphi\,dx\,dt + \int_\Omega \overline{u}(x,t)\varphi(x,t)dx - \int_\Omega \overline{u}_0(x)\varphi(x,0)dx \ge \int_0^t\int_\Omega \overline{u}(a-b\overline{u})\varphi\,dx\,dt,$$

$$\int_0^t\int_\Omega -\underline{u}\frac{\partial\varphi}{\partial t} + \nabla\underline{u}^m\nabla\varphi\,dx\,dt + \int_\Omega \underline{u}(x,t)\varphi(x,t)dx - \int_\Omega \underline{u}_0(x)\varphi(x,0)dx \le \int_0^t\int_\Omega \underline{u}(a-b\underline{u})\varphi\,dx\,dt, \tag{2.2}$$

with nonnegative test function φ and $0 < t \le T$. Subtracting the above inequalities, we get

$$\begin{aligned}\int_\Omega (\underline{u}(x,t) - \overline{u}(x,t))\varphi(x,t)dx \le{} & \int_\Omega (\underline{u}(x,0) - \overline{u}(x,0))\varphi(x,0)dx \\ & + \int_0^t\int_\Omega (\underline{u}(x,s) - \overline{u}(x,s))(\varphi_s + \Phi(x,s)\Delta\varphi)dx\,ds \\ & + \int_0^t\int_\Omega a(\underline{u}(x,s) - \overline{u}(x,s))\varphi\,dx\,ds \\ & - \int_0^t\int_\Omega b(\underline{u}(x,s) - \overline{u}(x,s))\varphi\,dx\,ds,\end{aligned} \tag{2.3}$$

where

$$\Phi(x,s) \equiv \int_0^1 m(\theta\overline{u} + (1-\theta)\underline{u})^{m-1}d\theta. \tag{2.4}$$

Since $\overline{u}$, $\underline{u}$ and $a(x,t)$ are bounded on Q_t, it follows from $m > 1$ that $\Phi(x,s)$ is a bounded nonnegative function. Thus, by choosing appropriate test function φ exactly as [12, Pages 118–123], we obtain

$$\int_{\Omega}\left[\underline{u}(x,t)-\overline{u}(x,t)\right]_{+}dx \le \|\varphi\|_{\infty}\int_{\Omega}\left[\underline{u}(x,0)-\overline{u}(x,0)\right]_{+}dx + C\int_{0}^{t}\int_{\Omega}\left[\underline{u}(x,t)-\overline{u}(x,t)\right]_{+}dx\,ds, \tag{2.5}$$

where $s_{+} = \max\{s,0\}$ and $C > 0$ is a bounded constant. Since $\underline{u}(x,0) \le \overline{u}(x,0)$, combining with the Gronwall's lemma, we see that $\underline{u}(x,t) \le \overline{u}(x,t)$ a.e. in Ω for any $0 < t \le T$. The proof is completed. □

Lemma 2.4 (global existence). *For any nonnegative bounded initial value $u_0(x)$, problem* (1.1)–(1.3) *admits a global nonnegative solution.*

Proof. Local existence can be proved as [13]. Global existence and nonnegativity follow from Lemma 2.3 by standard arguments. □

Lemma 2.5 (regularity [7]). *Let $u(x,t)$ be a weak solution of problem* (1.1)–(1.3), *then there exist positive constants K and $\beta \in (0,1)$, such that*

$$|u(x_1,t_1)-u(x_2,t_2)| \le K\left(|x_1-x_2|^{\beta}+|t_1-t_2|^{\beta/2}\right), \tag{2.6}$$

for every pair of points $(x_1,t_1),(x_2,t_2) \in \overline{Q}_T$.

3. Existence of Periodic Solutions

In this section, we show the existence of nontrivial nonnegative periodic solutions of the problem (1.1)-(1.2) by monotonicity method. First, we introduce the following remark.

Remark 3.1 (see [14]). According to Lemmas 2.3–2.5, the semiflow associated with the solution $u \equiv u(x,t;u_0)$ of problem (1.1)–(1.3), namely, the map

$$\begin{gathered} P_t : L^{\infty}(\Omega) \longrightarrow L^{\infty}(\Omega), \\ P_t(u_0) := u(\cdot,t;u_0) \quad (t>0) \end{gathered} \tag{3.1}$$

has the following properties:

(i) P_t is well defined for any $t > 0$ (Lemma 2.3);

(ii) P_t is order preserving (Lemma 2.4);

(iii) P_t is compact. In fact, the family $\{P_r u_0 \mid \|u_0\|_{\infty} \le M\}_{r\in[0,t]}$ $(M > 0)$ is uniformly bounded in $L^{\infty}(\Omega)$ by Lemma 2.3. Then by Lemma 2.5, the set $\{P_r u_0 \mid \|u_0\|_{\infty} \le M\}$ consists of equicontinuous functions, thus the conclusion follows from Ascoli-Arzelà's theorem.

Theorem 3.2. *If problem* (1.1)-(1.2) *admits a pair of ordered nontrivial nonnegative T-periodic subsolution $\underline{u}(x,t)$ and T-periodic supersolution $\overline{u}(x,t)$, then problem* (1.1)-(1.2) *admits a nontrivial nonnegative periodic solutions.*

Proof. From Remark 3.1, we just need to construct a pair of ordered T-periodic subsolution and T-periodic supersolution. The existence of nontrivial nonnegative periodic solutions of problem (1.1)-(1.2) will come from the similar iteration procedure as that in [15].

Let $\lambda_1,\ \varphi_1$ be the first eigenvalue and its corresponding eigenfunction to the Laplacian operator $-\Delta$ on the domain Ω, μ_1, ϕ_1 the first eigenvalue and its corresponding eigenfunction to the Laplacian operator $-\Delta$ on some domain $\Omega' \supset \Omega$, with respect to homogeneous Dirichlet data, respectively. It is clear that $\phi_1(x) > 0$ for all $x \in \overline{\Omega}$. Denote

$$a_m = \min_{\overline{Q}_T} a(x,t), \qquad a_M = \max_{\overline{Q}_T} a(x,t), \qquad b_M = \max_{\overline{Q}_T} b(x,t), \tag{3.2}$$

and define

$$\underline{u} = (r\varphi_1)^{1/m}, \qquad \overline{u} = (R\phi_1)^{1/m}, \tag{3.3}$$

where

$$r = \min\left\{ \frac{1}{\max_{\overline{\Omega}}\varphi_1}\left(\frac{a_m}{2\lambda_1}\right)^{m/(m-1)}, \frac{(a_m/2b_M)^m}{\max_{\overline{\Omega}}\varphi_1} \right\}, \qquad R = \frac{(a_M/\mu_1)^{m/(m-1)}}{\min_{\overline{\Omega}}\phi_1}. \tag{3.4}$$

Clearly, $\overline{u}$ and $\underline{u}$ are the T-periodic subsolution and supersolution of (1.1) subject to the condition (1.2), respectively. Further, we may assume $\underline{u} \le \overline{u}$, else we may change Ω' and then r, R appropriately. Thus we complete the proof. □

4. Asymptotic Behavior

In this section, we show the asymptotic behavior of nontrivial nonnegative solutions of the initial boundary value problem. First, we employ Moser's technique to obtain the upper bound of L^∞ norm for a nonnegative periodic solution u.

Lemma 4.1. *Let u be a nontrivial nonnegative periodic solution of* (1.1)-(1.2)*, then there exists a positive constant R which is independent of u, such that*

$$\|u(t)\|_{L^\infty(Q_T)} < R, \tag{4.1}$$

where $u(t) = u(\cdot,t)$.

Proof. Let u be a nontrivial nonnegative periodic solution of (1.1)-(1.2), multiply the (1.1) by u^{p+1} $(p \ge 0)$ and integrate the resulting relation over Ω, we have

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2} + \frac{4m(p+1)}{(p+m+1)^2}\left\|\nabla u^{(m+p+1)/2}(t)\right\|_2^2 \le M\|u(t)\|_{p+2}^{p+2}, \tag{4.2}$$

where $M = \sup_{(x,t)\in\overline{Q}_T} a(x,t)$. Hence

$$\frac{d}{dt}\|u(t)\|_{p+2}^{p+2} + C\left\|\nabla u^{(m+p+1)/2}(t)\right\|_2^2 \le C(p+1)\|u(t)\|_{p+2}^{p+2}, \tag{4.3}$$

where C denotes various positive constants independent of p and u. Set

$$p_k = 2^k + m - 3, \qquad \alpha_k = \frac{2(p_k+2)}{p_k+m+1}, \qquad u_k = u^{(p_k+m+1)/2}(t), \quad k = 1,2,\ldots, \tag{4.4}$$

from (4.3) we have

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + C\|\nabla u_k(t)\|_2^2 \le C(p_k+1)\|u_k(t)\|_{\alpha_k}^{\alpha_k}. \tag{4.5}$$

Here we appeal to the Gagliardo-Nirenberg inequality

$$\|u_k(t)\|_{\alpha_k} \le C\|\nabla u_k(t)\|_2^{\theta_k}\|u_k(t)\|_1^{1-\theta_k}, \tag{4.6}$$

with

$$\theta_k = \left(1 - \frac{1}{\alpha_k}\right)\left(\frac{1}{N} - \frac{1}{2} + 1\right)^{-1} = \frac{(p_k - m + 3)N}{(p_k+2)(N+2)} \in (0,1), \tag{4.7}$$

where C denotes a positive constant independent of k and p. From (4.5), (4.6), and the fact that $\|u_k(t)\|_1 = \|u_{k-1}(t)\|_{\alpha_{k-1}}^{\alpha_{k-1}}$, we have

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} \le \|u_k(t)\|_{\alpha_k}^{\alpha_k}\left\{-C\|u_k(t)\|_{\alpha_k}^{2/\theta_k-\alpha_k}\chi_{k-1}^{2\alpha_{k-1}(1-1/\theta_k)} + C(p_k+1)\right\}, \tag{4.8}$$

with $\chi_k = \max\{1, \sup_t\|u_k(t)\|_{\alpha_k}\}$. Taking the periodicity of $\|u_k(t)\|_{\alpha_k}$ into account, we can obtain

$$\|u_k(t)\|_{\alpha_k} \le \left\{C(p_k+1)\chi_{k-1}^{2\alpha_{k-1}(1/\theta_k-1)}\right\}^{1/c_k}, \tag{4.9}$$

where $c_k = 2/\theta_k - \alpha_k$. From the boundedness of α_k and c_k, we can see that

$$\|u_k(t)\|_{\alpha_k} \le C2^{k\beta}\chi_{k-1}^{2\alpha_{k-1}(1-\theta_k)/(2-\theta_k\alpha_k)}, \tag{4.10}$$

where β is a positive constant independent of k. Noticing that $\alpha_k < 2$ implies

$$\frac{2(1-\theta_k)\alpha_{k-1}}{2-\theta_k\alpha_k} < \frac{2(1-\theta_k)\alpha_{k-1}}{2-2\theta_k} < 2, \tag{4.11}$$

with $\chi_{k-1} \geq 1$, we get

$$\|u_k(t)\|_{\alpha_k} \leq C\lambda^k \chi_{k-1}^2, \tag{4.12}$$

where $\lambda = 2^\beta > 1$. That is

$$\ln\|u_k(t)\|_{\alpha_k} \leq \ln\chi_k \leq \ln C + k\ln\lambda + 2\ln\chi_{k-1}, \tag{4.13}$$

and thus

$$\begin{aligned}\ln\|u_k(t)\|_{\alpha_k} &\leq \ln C\sum_{i=0}^{k-2}2^i + 2^{k-1}\ln\chi_1 + \sum_{j=0}^{k-2}\left((k-j)2^j\right)\ln\lambda \\ &\leq \left(2^{k-1}-1\right)\ln C + 2^{k-1}\ln\chi_1 + f(k)\ln\lambda,\end{aligned} \tag{4.14}$$

or

$$\|u(t)\|_{p_k+2} \leq \left\{C^{2^{k-1}}\chi_1^{2^{k-1}}\lambda^{f(k)}\right\}^{2/(p_k+2)}, \tag{4.15}$$

where

$$f(k) = 2^{k+1} - 2^{k-1} - k - 2. \tag{4.16}$$

Letting $k \to \infty$, we get

$$\|u(t)\|_\infty \leq C\chi_1 \leq C\left(\max\left\{1, \sup_t\|u(t)\|_{m+1}^m\right\}\right). \tag{4.17}$$

Now we estimate $\|u(t)\|_{m+1}$. Set $p = m-1$ in (4.3), we get

$$\frac{d}{dt}\|u(t)\|_{m+1}^{m+1} + C\|\nabla u^m(t)\|_2^2 \leq C\|u(t)\|_{m+1}^{m+1}. \tag{4.18}$$

By Hölder's inequality and Sobolev's theorem, we have

$$\|u(t)\|_{m+1}^{2m} \leq |\Omega|^{(m-1)/(m+1)}\|u(t)\|_{2m}^{2m} \leq C\|\nabla u^m(t)\|_2^2. \tag{4.19}$$

From (4.18), (4.19) we can obtain

$$\frac{d}{dt}\|u(t)\|_{m+1}^{m+1} + C\|u(t)\|_{m+1}^{2m} \le C\|u(t)\|_{m+1}^{m+1}. \tag{4.20}$$

By Young's inequality, we get

$$\frac{d}{dt}\|u(t)\|_{m+1}^{m+1} + C\|u(t)\|_{m+1}^{2m} \le C, \tag{4.21}$$

where C denotes different positive constants independent of u. By (4.21) and the periodicity of $u(t)$, we have

$$\|u(t)\|_{m+1}^{m} \le C. \tag{4.22}$$

Together with (4.17), we complete the proof of this lemma. □

Lemma 4.2. *There exists a constant r with $0 < r < R$, such that no nontrivial nonnegative periodic solutions u of problem* (1.1)-(1.2) *satisfies*

$$0 < \|u\|_{L^\infty(Q_T)} \le r. \tag{4.23}$$

Proof. To arrive at a contradiction, we assume that problem (1.1)-(1.2) admits a nontrivial nonnegative periodic solution u satisfying $0 < \|u\|_{L^\infty(Q_T)} \le r$. For any given $\phi(x) \in C_0^\infty(\Omega)$, we can choose ϕ^2/u as a test function. Multiplying (1.1) by ϕ^2/u and integrating over Q_T, we obtain

$$\iint_{Q_T} \frac{\phi^2}{u}\frac{\partial u}{\partial t}dt\,dx + \iint_{Q_T} mu^{m-1}\nabla u\nabla\left(\frac{\phi^2}{u}\right)dt\,dx = \iint_{Q_T} \phi^2(a-bu)dt\,dx. \tag{4.24}$$

By the periodicity of u, the first term of the left-hand side in (4.24) satisfies

$$\iint_{Q_T} \frac{\phi^2}{u}\frac{\partial u}{\partial t}dt\,dx = \int_\Omega \phi^2\int_0^T \frac{\partial(\ln u)}{\partial t}dt\,dx = 0. \tag{4.25}$$

The second term of the left-hand side in (4.24) can be rewritten as

$$\begin{aligned}
&\iint_{Q_T} mu^{m-1}\nabla u\nabla\left(\frac{\phi^2}{u}\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}\nabla u\nabla\left(\phi\cdot\frac{\phi}{\mathrm{u}}\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}\nabla u\left(\frac{\phi}{u}\nabla\phi+\phi\nabla\left(\frac{\phi}{u}\right)\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}\left(\frac{\nabla\phi}{u}\right)\phi\nabla u\,dt\,dx+\iint_{Q_T} mu^{m-1}\phi\nabla u\nabla\left(\frac{\phi}{u}\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}\left(\frac{\nabla\phi}{u}\right)\left(u\nabla\phi-u^2\nabla\left(\frac{\phi}{u}\right)\right)dt\,dx+\iint_{Q_T} mu^{m-1}\phi\nabla u\nabla\left(\frac{\phi}{u}\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}|\nabla\phi|^2dt\,dx-\iint_{Q_T} mu^{m-1}(u\nabla\phi-\phi\nabla u)\nabla\left(\frac{\phi}{u}\right)dt\,dx\\
&\quad=\iint_{Q_T} mu^{m-1}|\nabla\phi|^2dt\,dx-\iint_{Q_T} mu^{m-1}u^2\left|\nabla\left(\frac{\phi}{u}\right)\right|^2dt\,dx.
\end{aligned}\tag{4.26}$$

Thus

$$\iint_{Q_T} mu^{m-1}\nabla u\nabla\left(\frac{\phi^2}{u}\right)dt\,dx\le\iint_{Q_T} mu^{m-1}|\nabla\phi|^2dt\,dx.\tag{4.27}$$

Combining (4.24) with (4.25) and (4.27), we obtain

$$\iint_{Q_T}\phi^2(a-bu)dt\,dx\le\iint_{Q_T} mu^{m-1}|\nabla\phi|^2dt\,dx.\tag{4.28}$$

By an approximating process, we can choose $\phi=\phi_1$ with μ_1 is the first eigenvalue and ϕ_1 is its corresponding eigenfunction to the eigenvalue problem

$$\begin{aligned}
-\Delta u&=\mu u,\quad x\in\Omega'\supset\Omega,\\
u&=0,\quad x\in\partial\Omega'.
\end{aligned}\tag{4.29}$$

Then $\mu_1 > 0$ and $\phi_1(x)$ is strictly positive in Ω. From (4.28) we have

$$\begin{aligned}
0 &\leq \iint_{Q_T} \left(m u^{m-1} |\nabla \phi_1|^2 - \phi_1^2 (a - bu) \right) dt\, dx \\
&\leq \iint_{Q_T} \left(-m r^{m-1} \phi_1 \Delta \phi_1 - \phi_1^2 (a - bu) \right) dt\, dx \\
&= \iint_{Q_T} \phi_1^2 \left(\mu_1 m r^{m-1} - (a - bu) \right) dt\, dx \\
&= \int_\Omega \phi_1^2 \int_0^T \left(\mu_1 m r^{m-1} - a + bu \right) dt\, dx.
\end{aligned} \tag{4.30}$$

Thus there exists $y_0 \in \Omega$ such that $g(y_0) = \int_0^T (\mu_1 m r^{m-1} - a(y_0, t) + b(y_0, t) u(y_0, t)) dt \geq 0$, then

$$\frac{1}{T} \int_0^T a(y_0, t) dt \leq \mu_1 m r^{m-1} + b(y_0, t) r. \tag{4.31}$$

Obviously, we can choose suitable small $r_0 < R$, such that for any $r < r_0$, the above inequality does not hold. It is a contradiction. The proof is completed. □

In the following, we will make use of the a priori boundedness of all nontrivial nonnegative periodic solutions to show the asymptotic behavior of nontrivial nonnegative solutions of the initial boundary value problem (1.1)–(1.3).

Theorem 4.3. *Problem* (1.1)-(1.2) *admits a minimal and a maximal nonnegative nontrivial periodic solutions $u_*(x,t)$ and $u^*(x,t)$. Moreover, if $u(x,t)$ is the solution of the initial boundary value problem* (1.1)–(1.3) *with initial value $u_0(x) > 0$, then for any $\varepsilon > 0$, there exists t_0 depending on $u_0(x)$ and ε, such that*

$$u_*(x,t) - \varepsilon \leq u(x,t) \leq u^*(x,t) + \varepsilon, \quad \text{for } x \in \Omega,\ t \geq t_0. \tag{4.32}$$

Proof. First, we show the existence of the maximal periodic solution $u^*(x,t)$. Define a Poincaré map

$$P_T : C\left(\overline{\Omega}\right) \longrightarrow C\left(\overline{\Omega}\right), \quad \text{with } P_T(u_0(x)) = u(x,T), \tag{4.33}$$

where $u(x,t)$ is the solution of (1.1)–(1.3) with initial value $u_0(x)$. By Remark 3.1, the map P_T is well defined. Let $u_n(x,t)$ be the solution of (1.1)–(1.3) with initial value $u_0(x) = P_T^{n-1}(\overline{u}(x))$, where $\overline{u}(x) = K\phi_1^{1/m}(x)$ and K is a positive constant satisfying

$$K^{m-1} \mu_1 \min_{x \in \overline{\Omega}} \phi_1^{(m-1)/m}(x) \geq \|a(x,t)\|_{C(Q_T)}, \tag{4.34}$$

where μ_1, ϕ_1 are chosen as those in Theorem 3.2. It is observed that $u_n(x,T) = P_T^n(\overline{u}(x))$, $n = 1, 2, \ldots$, and $u_{n+1}(x,t) \leq u_n(x,t) \leq \overline{u}(x)$ by the comparison principle. By a rather standard argument, we conclude that there exist a function $u^*(x) \in C(\overline{\Omega})$ and a subsequence of $\{P_T^n(\overline{u}(x))\}$, denoted by itself for simplicity, such that

$$u^*(x) = \lim_{n\to\infty} P_T^n(\overline{u}(x)). \tag{4.35}$$

Similar to the proof of Theorem 4.1 in [4], we can prove that $u^*(x,t)$, which is the even extension of the solution of the initial boundary value problem (1.1)–(1.3) with initial value $u^*(x)$, is a periodic solution of the problem (1.1)-(1.2). Moreover, Lemma 4.1 shows that any nonnegative periodic solution $u(x,t)$ of (1.1)-(1.2) must satisfy $u(x,t) \leq R$ for $(x,t) \in Q_T$. Therefore, if we take K is larger than $R/\min_{x\in\overline{\Omega}}\phi_1^{1/m}(x)$, by the comparison principle we have $u^*(x) \geq u(x,0)$ and thus $u^*(x,t) \geq u(x,t)$, which means that $u^*(x,t)$ is the maximal periodic solution of problem (1.1)-(1.2). The existence of the minimal periodic solution can be obtained with the same method.

Let $u(x,t)$ be the solution of the initial boundary value problem (1.1)–(1.3) with any given nonnegative initial value $u_0(x)$, and let $v(x,t)$ be the solution of (1.1)–(1.3) with initial value $v(x,0) = K\phi_1^{1/m}(x)$, where K is a positive constant satisfying

$$K \geq \max\left\{ \frac{\|u_0\|_{L^\infty(\Omega)}}{\min_{x\in\overline{\Omega}} \phi_1^{1/m}(x)}, \frac{1}{\min_{x\in\overline{\Omega}} \phi_1^{1/m}(x)} \left(\frac{\|a(x,t)\|_{C(Q_T)}}{\mu_1}\right)^{1/(m-1)} \right\}, \tag{4.36}$$

then for any $(x,t) \in \overline{Q}_T$, we have

$$u(x,t+kT) \leq v(x,t+kT), \quad k = 0,1,2,\dots. \tag{4.37}$$

A similar argument as [4] shows that $v^*(x,t) = \lim_{k\to\infty} v(x,t+kT)$, and $v^*(x,t)$ is a nontrivial nonnegative periodic solution of (1.1)-(1.2). Therefore, there exists k_0 such that

$$u(x,t+kT) \leq v^*(x,t) \leq u^*(x,t), \tag{4.38}$$

for any $k \geq k_0$ and $(x,t) \in \overline{Q}_T$. Provided that the periods of $v^*(x,t)$ and $u^*(x,t)$ are taken into account, for any $\varepsilon > 0$, there exists t_0 depending on $u_0(x)$ and ε such that

$$u(x,t) \leq u^*(x,t) + \varepsilon, \quad \text{for } x \in \Omega, t \geq t_0. \tag{4.39}$$

By the similar way and Lemma 4.2, we can obtain

$$u_*(x,t) - \varepsilon \leq u(x,t), \quad \text{for } x \in \Omega, t \geq t_0. \tag{4.40}$$

Thus we complete the proof. □

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