Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 589618, 17 pages
doi:10.1155/2010/589618

Research Article

Contiguous Extensions of Dixon's Theorem on the Sum of a $_3F_2$

Junesang Choi

Department of Mathematics, Dongguk University, Gyeongju 780-714, South Korea

Correspondence should be addressed to Junesang Choi, junesang@mail.dongguk.ac.kr

Received 7 October 2009; Accepted 25 February 2010

Academic Editor: Yeol J. E. Cho

In 1994, Lavoie et al. have succeeded in artificially constructing a formula consisting of twenty three interesting results, except for five cases, closely related to the classical Dixon's theorem on the sum of a $_3F_2$ by making a systematic use of some known relations among contiguous functions. We aim at presenting summation formulas for those five exceptional cases that can be derived by using the same technique developed by Bailey with the help of Gauss's summation theorem and generalized Kummer's theorem.

1. Introduction and Preliminaries

The *generalized hypergeometric series* $_pF_q$ is defined by (see [1, page 73])

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z), \end{aligned} \tag{1.1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [2, pages 2 and 6])

$$\begin{aligned} (\lambda)_n &:= \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in \mathbb{N} := \{1,2,3,\ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-), \end{aligned} \tag{1.2}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is,

$$\left(\beta_j \notin \mathbb{Z}_0^-;\ j = 1, \ldots, q\right). \tag{1.3}$$

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates in view of the identity (see [2, page 7])

$$(-n)_k = \begin{cases} \dfrac{(-1)^k\ n!}{(n-k)!}, & (0 \le k \le n;\ n \in \mathbb{N}), \\ 0, & (k > n). \end{cases} \tag{1.4}$$

In fact, ${}_pF_q$ is a natural generalization of the hypergeometric function (or series)

$${}_2F_1\ (a, b;\ c;\ z) = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| z \right] = F(a, b;\ c;\ z). \tag{1.5}$$

Gauss proved his famous summation theorem (see [1, page 49, Theorem 18])

$${}_2F_1(a, b;\ c;\ 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ \left(\Re(c-a-b) > 0;\ c \notin \mathbb{Z}_0^-\right). \tag{1.6}$$

Kummer presented the summation theorem for ${}_2F_1(-1)$ (see [1, page 68, equation (1)])

$${}_2F_1\left[\begin{matrix} a,\ b \\ 1+a-b \end{matrix} \middle| -1 \right] = \frac{\Gamma(1+a-b)\Gamma(1/2)}{2^a\Gamma(1+(1/2)a-b)\Gamma(1/2+(1/2)a)} \\ \left(\Re(b) < 1;\ 1+a-b \notin \mathbb{Z}_0^-\right). \tag{1.7}$$

Dixon gave the following classical summation formula for ${}_3F_2(1)$ (see [1, page 92]):

$${}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix} \middle| 1 \right] = \frac{\Gamma(1+a/2)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a/2-b-c)}{\Gamma(1+a)\Gamma(1+a/2-b)\Gamma(1+a/2-c)\Gamma(1+a-b-c)}, \tag{1.8}$$

where $\Re(a-2b-2c) > -2$.

Lavoie et al. [3] presented a general, artificially constructed, form of the Dixon's theorem (1.8):

$$f_{i,j}(a,b,c) := {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b+i,\ 1+a-c+i+j \end{matrix}\middle| 1\right] \quad (i=-3,-2,-1,0,1,2;\ j=0,1,2,3) \tag{1.9}$$

by making a systematic use of the relations among contiguous functions given by Rainville [1, page 80], except for the cases

$$(i,j) = (3,1),\ (3,2),\ (3,3),\ (2,3),\ \text{and}\ (1,3). \tag{1.10}$$

Very recently, Kim and Rathie [4] derived twenty five transformation formulas in the form of a single identity for the hypergeometric series X_8 introduced by Exton by making use of generalized Watson's theorem [5].

Here, in order to present the five exceptional formulas not given by Lavoie et al. [3, equation (2), page 268], we will first give further extension tables, as in Lemma 1.1, of the generalized formulas of the Kummer's theorem (1.7) obtained by Lavoie et al. [6] and then derive the summation formulas of

$$I_{i,j}(a,b,c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)} f_{i,j}(a,b,c) \tag{1.11}$$

for the cases in (1.10), by using the same technique developed by Bailey [7] with the help of Gauss's theorem (1.6) and some identities in Lemma 1.1.

Lemma 1.1. *One gives further extension tables of the generalized formulas of the Kummer's theorem* (1.7) *obtained by Lavoie et al. [6]:*

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,b \\ 1+a-b+i \end{matrix}\middle| -1\right] &= \frac{\Gamma(1/2)\Gamma(1-b)\Gamma(1+a-b+i)}{2^a\Gamma(1-b+i/2+|i|/2)} \\ &\quad \cdot \left\{ \frac{\mathcal{A}_i}{\Gamma(a/2-b+i/2+1)\Gamma(a/2+i/2+1/2-[(1+i)/2])} \right. \\ &\quad \left. + \frac{\mathcal{B}_i}{\Gamma(a/2-b+i/2+1/2)\Gamma(a/2+i/2+1/2-[i/2])} \right\}. \end{aligned} \tag{1.12}$$

for $i = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5,\ \pm6,\ \pm7,\ \pm8,\ \pm9$. *Here* $[x]$ *denotes the greatest integer less than or equal to* x *and* $|x|$ *its absolute value. The coefficients* $\mathcal{A}_i$ *and* $\mathcal{B}_i$ *are given in Tables 1 and 2.*

Table 1: Table for $\mathscr{A}_i$ and $\mathcal{B}_i$.

i	$\mathscr{A}_i$	$\mathcal{B}_i$
9	$-16a^4+72a^3b-108a^2b^2$ $+60ab^3+23b^4-328a^3$ $+972a^2b-792ab^2+150b^3$ $-2240a^2+3612ab-999b^2$ $-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2$ $-20ab^3+b^4+248a^3$ $-516a^2b+240ab^2-10b^3$ $+1160a^2-1028ab+35b^2$ $+1576a-50b+24$
8	$8a^4-32a^3b+40a^2b^2$ $-16ab^3+b^4+128a^3$ $-312a^2b+176ab^2-10b^3$ $+624a^2-672ab+35b^2$ $+896a-50b+24$	$8b^3-40ab^2+48a^2b$ $-16a^3-192a^2+312ab$ $-88b^2-640a+352b-512$
7	$7b^3-28ab^2+28a^2b-8a^3$ $-100a^2+196ab-70b^2$ $-352a+245b-302$	$8a^3-20a^2b+12ab^2$ $-b^3+68a^2-76ab$ $+6b^2+128a-11b+6$
6	$4a^3-12a^2b+9ab^2-b^3$ $+36a^2-51ab+6b^2$ $+74a-11b+6$	$16ab-8a^2-6b^2$ $-48a+34b-52$
5	$10ab-4a^2-5b^2$ $-26a+25b-32$	$4a^2-6ab+b^2$ $+14a-3b+2$
4	$2a^2-4ab+b^2$ $+8a-3b+2$	$4(b-a-2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0

Proof. It is not difficult, even though a little complicated, to prove the identities given here by making a main use of the following contiguous relation (see [1, equation (15), page 71]):

$$[c-1+(a+b+1-2c)z]F=(c-1)(1-z)\ F(c-)-\frac{1}{c}\ (c-a)(c-b)\ zF(c+). \tag{1.13}$$

□

2. Further Contiguous Extension Formulas of (1.8)

In the sake of a little brevity, summation formulas of $I_{i,j}(a,b,c)$ are given for the cases (1.10).

Theorem 2.1. *Without restrictions for each formula, one just gives the above-mentioned summation formulas:*

$$\begin{aligned} I_{3,1}(a,b,c) &= \alpha_{3,1}\frac{\Gamma(b-3)\Gamma(c-4)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+9/2)}{\Gamma(a-b-c+5)\Gamma((1/2)a-b+5/2)\Gamma((1/2)a-c+7/2)} \\ &\quad+\beta_{3,1}\frac{\Gamma(b-3)\Gamma(c-4)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+5)}{\Gamma(a-b-c+5)\Gamma((1/2)a-b+2)\Gamma((1/2)a-c+3)} \\ &\quad+\gamma_{3,1}\frac{\Gamma(b-3)\Gamma(c-4)\Gamma((1/2)a+1/2)}{\Gamma(a-b-c+5)\Gamma((1/2)a+1)}+\delta_{3,1}\frac{\Gamma(b-3)\Gamma(c-4)}{\Gamma(a-b-c+5)}, \end{aligned} \tag{2.1}$$

Table 2: Table for $\mathcal{A}_i$ and $\mathcal{B}_i$.

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-9	$16a^4-72a^3b+108a^2b^2$ $-60ab^3+9b^4-320a^3$ $+972a^2b-828ab^2+174b^3$ $+2240a^2-3936ab+1323b^2$ $-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2$ $-20ab^3+b^4-256a^3$ $+564a^2b-300ab^2+26b^3$ $+1376a^2-1568ab+251b^2$ $-2816a+1066b+1680$
-8	$8a^4-32a^3b+40a^2b^2$ $-16ab^3+b^4-128a^3$ $+328a^2b-208ab^2+22b^3$ $+688a^2-928ab+179b^2$ $-1408a+638b+840$	$16a^3-48a^2b+40ab^2$ $-8b^3-192a^2+328ab-104b^2$ $+704a-480b-768$
-7	$8a^3-28a^2b+28ab^2-7b^3$ $-96a^2+196ab-77b^2$ $+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3$ $-72a^2+92ab-15b^2$ $+184a-74b-120$
-6	$4a^3-12a^2b+9ab^2-b^3$ $-36a^2+57ab-12b^2$ $+92a-47b-60$	$8a^2-16ab+6b^2$ $-48a+38b+64$
-5	$4a^2-10ab+5b^2$ $-24a+25b+32$	$4a^2-6ab+b^2$ $-16a+7b+12$
-4	$2a^2-4ab+b^2$ $-8a+5b+6$	$4(a-b-2)$
-3	$2a-3b-4$	$2a-b-2$
-2	$a-b-1$	2
-1	1	1

where

$$\begin{aligned}
\alpha_{3,1} &= (a-2)(a-c+1)\left(\frac{a}{2}-b-c+\frac{9}{2}\right)\left(\frac{a}{2}-b-c+\frac{11}{2}\right)\\
&\quad+\frac{1}{2}(b-3)(c-4)(4a-3c+3)\left(\frac{a}{2}-b-c+\frac{9}{2}\right)+\frac{1}{2}(b-2)(b-3)(c-3)(c-4),\\
\beta_{3,1} &= \frac{2}{a}(c-a-1)(a-1)\left(\frac{a}{2}-b-c+5\right),\\
\gamma_{3,1} &= \frac{2}{3}(b-1)(b-2)(b-3)(c-2)(c-3)(c-4)+(a-c+2)(b-2)(b-3)(c-3)(c-4)\\
&\quad+\frac{1}{2}\left\{(c-1)(1-2a)+2a^2\right\}(b-3)(c-4)+\frac{1}{2}(a-1)(a-2)(a-c+1),\\
\delta_{3,1} &= -\frac{2(2a+3)}{3(a+1)(a+3)}(b-1)(b-2)(b-3)(c-2)(c-3)(c-4)\\
&\quad+(c-4a-2)(b-2)(b-3)(c-3)(c-4)+\frac{1}{2}(a-1)(a-2)(c-1)\\
&\quad+\frac{1}{2}(b-3)(c-4)\{(c-1)(2a-1)-4a\},
\end{aligned} \tag{2.2}$$

$$
\begin{aligned}
I_{3,2}(a,b,c) = {} & \alpha_{3,2}\frac{\Gamma(b-3)\Gamma(c-5)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+11/2)}{\Gamma(a-b-c+6)\Gamma((1/2)a-b+5/2)\Gamma((1/2)a-c+9/2)} \\
& + \beta_{3,2}\frac{\Gamma(b-3)\Gamma(c-5)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+5)}{\Gamma(a-b-c+6)\Gamma((1/2)a-b+2)\Gamma((1/2)a-c+4)} \\
& + \gamma_{3,2}\frac{\Gamma(b-3)\Gamma(c-5)\Gamma((1/2)a+1/2)}{\Gamma(a-b-c+6)\Gamma((1/2)a+1)} + \delta_{3,2}\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a-b-c+6)},
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
\alpha_{3,2} = {} & \{(c-1)(2a-c+2)-a(a+1)\}(a-2)\left(\frac{a}{2}-b-c+\frac{13}{2}\right)\left(\frac{a}{2}-b-c+\frac{11}{2}\right) \\
& + \frac{1}{2}\{(c-1)(8a-3c+6)-5a(a+1)\}(b-3)(c-5)\left(\frac{a}{2}-b-c+\frac{11}{2}\right) \\
& + \frac{1}{4}(4c-5a-9)(b-2)(b-3)(c-4)(c-5), \\
\beta_{3,2} = {} & \frac{a+1}{2(a+2)}(b-2)(b-3)(c-4)(c-5) \\
& + \left\{\frac{2}{a}(c-2-2a)(c-1)+\frac{2(a+1)^2}{a+2}\right\}(a-1)\left(\frac{a}{2}-b-c+5\right)\left(\frac{a}{2}-b-c+6\right) \\
& + \left\{\frac{1}{a}(c-1)(c-2)+4(1-c)+\frac{3(a+1)^2}{a+2}\right\}(b-3)(c-5)\left(\frac{a}{2}-b-c+5\right), \\
\gamma_{3,2} = {} & \frac{(a-1)(1-c)(c-2)}{2a}[a(a-2)+2(b-2)(c-5)\{1+(b-2)(c-4)\}] \\
& + \frac{(1-c)(c-2)(b-3)(c-5)}{2a}\{a+2(b-2)(c-4)\} \\
& + \frac{2(b-3)(c-1)(c-5)}{a+2}[a(a+2)+2(b-2)(c-4)\{a+2+(b-1)(c-3)\}] \\
& + \frac{2(a-1)(c-1)}{a+2}\left[\frac{a(a+2)}{2}\{a-2+2(b-3)(c-5)\}\right. \\
& \qquad \left. + (b-2)(b-3)(c-4)(c-5)\left\{a+2+\frac{2}{3}(b-1)(c-3)\right\}\right] \\
& - \frac{(a+1)(b-2)(b-3)(c-4)(c-5)}{2(a+2)^2(a+4)} \\
& \times [(a+2)(a+4)+2(b-1)(c-3)\{a+4+b(c-2)\}] \\
& - \frac{3(a+1)^2(b-3)(c-5)}{2(a+2)^2(a+4)}\left[(a+2)(a+4)\{a+2(b-2)(c-4)\}\right. \\
& \qquad \left. + 2(b-1)(b-2)(c-3)(c-4)\left\{a+4+\frac{2}{3}b(c-2)\right\}\right]
\end{aligned}
$$

$$+\frac{(1-a)(a+1)^2}{2(a+2)^2(a+4)}\Big[a(a+2)(a+4)\{a-2+2(b-3)(c-5)\}$$

$$+2(b-2)(b-3)(c-4)(c-5)$$

$$\times\bigg\{(a+2)(a+4)+\frac{2}{3}(a+4)(b-1)(c-3)$$

$$+\frac{1}{3}b(b-1)(c-2)(c-3)\bigg\}\Big],$$

$$\delta_{3,2}=\frac{(a-2)(c-1)(c-2)}{2(a+1)}[(a-1)(a+1)+2(b-3)(c-5)\{a+1+(b-2)(c-4)\}]$$

$$+\frac{3(b-3)(c-1)(c-2)(c-5)}{2(a+1)}\{a+1+2(b-2)(c-4)\}$$

$$+\frac{2(b-2)(b-3)(1-c)(c-4)(c-5)}{(a+1)(a+3)}\{a+3+2(b-1)(c-3)\}$$

$$+\frac{4a(b-3)(1-c)(c-5)}{(a+1)(a+3)}[(a+1)(a+3)+2(b-2)(c-4)\{a+3+(b-1)(c-3)\}]$$

$$+\frac{2a(a-2)(1-c)}{(a+1)(a+3)}\bigg[(a+1)(a+3)\bigg\{\frac{1}{2}(a-1)+(b-3)(c-5)\bigg\}$$

$$+(b-2)(b-3)(c-4)(c-5)\bigg\{a+3+\frac{2}{3}(b-1)(c-3)\bigg\}\bigg]$$

$$+\frac{5(b-2)(b-3)(c-4)(c-5)}{2(a+3)(a+5)}[(a+3)(a+5)+2(b-1)(c-3)\{a+5+b(c-2)\}]$$

$$+\frac{5a(b-3)(c-5)}{2(a+3)(a+5)}\bigg[(a+5)\{(a+1)(a+3)+2(a+2)(b-2)(c-4)\}$$

$$+2(b-1)(b-2)(c-3)(c-4)\bigg\{a+5+\frac{2}{3}b(c-2)\bigg\}\bigg]$$

$$+\frac{a(a-2)}{(a+3)(a+5)}\bigg[(a+1)(a+3)(a+5)\bigg\{\frac{1}{2}(a-1)+(b-3)(c-5)\bigg\}$$

$$+(a+3)(a+5)(b-2)(b-3)(c-4)(c-5)$$

$$+\frac{1}{3}(b-1)(b-2)(b-3)(c-3)(c-4)$$

$$\times(c-5)\{2(a+5)+b(c-2)\}\bigg],$$

$$I_{3,3}(a,b,c) = \alpha_{3,3}\frac{\Gamma(b-3)\Gamma(c-6)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+11/2)}{\Gamma(a-b-c+7)\Gamma((1/2)a-b+5/2)\Gamma((1/2)a-c+11/2)}$$
$$+\beta_{3,3}\frac{\Gamma(b-3)\Gamma(c-6)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+6)}{\Gamma(a-b-c+7)\Gamma((1/2)a-b+2)\Gamma((1/2)a-c+5)}$$
$$+\gamma_{3,3}\frac{\Gamma(b-3)\Gamma(c-6)\Gamma((1/2)a+1/2)}{\Gamma(a-b-c+7)\Gamma((1/2)a+1)}+\delta_{3,3}\frac{\Gamma(b-3)\Gamma(c-6)}{\Gamma(a-b-c+7)}, \tag{2.4}$$

where

$$\alpha_{3,3} = \left(\frac{a}{2}-b-c+\frac{11}{2}\right)\left(\frac{a}{2}-b-c+\frac{13}{2}\right)\left(\frac{a}{2}-b-c+\frac{15}{2}\right)$$
$$\cdot\left\{(a-2)(1-c)\left(c^2-3ac+2a-5c+2\right)+a(a+1)(a-c+3)\right\}$$
$$+\frac{1}{2}(b-3)(c-6)\left(\frac{a}{2}-b-c+\frac{11}{2}\right)\left(\frac{a}{2}-b-c+\frac{13}{2}\right)$$
$$\times\left[a(a+1)(6a-5c+17)+(1-c)\left\{3\left(c^2-5c+2\right)+4a(2-3c)\right\}\right]$$
$$+\frac{1}{4}(b-2)(b-3)(c-5)(c-6)\left(\frac{a}{2}-b-c+\frac{11}{2}\right)$$
$$\cdot\{2(c-1)(3c-2)+(a+1)(9a-5c+23)\}$$
$$+\frac{1}{4}(a+1)(b-1)(b-2)(b-3)(c-4)(c-5)(c-6),$$
$$\beta_{3,3} = 2(1-a)\left(\frac{a}{2}-b-c+6\right)\left(\frac{a}{2}-b-c+7\right)$$
$$\cdot\left\{(c-1)(3c-2)+\frac{(a-c+3)(a+1)^2}{a+2}\right\}$$
$$+2(1-c)(b-3)(c-6)\left(\frac{a}{2}-b-c+6\right)\left\{3c-2-\frac{3(a+1)}{a+2}\right\}$$
$$+\frac{a+1}{2(a+2)}(b-2)(b-3)(c-5)(c-6)(c-3a-7)-4(a+1)^2(b-3)(c-6),$$
$$\gamma_{3,3} = \frac{1}{2}(a+1)(b-2)(b-3)(c-5)(c-6)$$
$$\cdot\{(a+1)(a-c+4)+(1-c)(a-6c+6)+3\}$$
$$+a(b-3)(c-6)\left\{a(c-1)(3c-2)+(a+1)^3+\frac{(a+1)^2(2a+1)(1-c)}{2(a+2)}\right\}$$
$$+\frac{1}{2}a(a-1)(a-2)\left\{(c-1)(3c-2)+\frac{(a+1)^2(a-c+3)}{a+2}\right\},$$

$$
\begin{aligned}
\delta_{3,3} = {} & (b-2)(b-3)(c-5)(c-6) \\
& \times \left\{ (c-1)\left(c^2-5c+2\right) + (a+1)(1-c)(3c-2) \right. \\
& \quad \left. + \frac{1}{2}(c-1)\left(2a^2+6a+5\right) - \frac{1}{2}(a+2)\left(2a^2+8a+9\right) \right\} \\
& + \frac{1}{2}(b-3)(c-6)\left[(1-c)\left\{2a^2(3c-2)-2a+1\right\} \right. \\
& \qquad \left. + a(a+1)\{(2a+1)(c-1)-2(a+1)(a+2)\} \right] \\
& + \frac{1}{2}(a-1)(a-2)\left[(c-1)\left\{c^2-5c+2+a(a-3c+3)\right\} - a(a+2) \right], \\
I_{2,3}(a,b,c) = {} & \alpha_{2,3}\frac{\Gamma(b-2)\Gamma(c-5)\Gamma((1/2)a)\Gamma((1/2)a-b-c+9/2)}{\Gamma(a-b-c+6)\Gamma((1/2)a-b+3/2)\Gamma((1/2)a-c+9/2)} \\
& + \beta_{2,3}\frac{\Gamma(b-2)\Gamma(c-5)\Gamma((1/2)a)\Gamma((1/2)a-b-c+5)}{\Gamma(a-b-c+6)\Gamma((1/2)a-b+2)\Gamma((1/2)a-c+5)} \\
& + \gamma_{2,3}\frac{\Gamma(b-2)\Gamma(c-5)\Gamma((1/2)a)}{\Gamma(a-b-c+6)\Gamma((1/2)a+1/2)} + \delta_{2,3}\frac{\Gamma(b-2)\Gamma(c-5)}{\Gamma(a-b-c+6)},
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
\alpha_{2,3} = {} & \left\{ \frac{1}{2}\left(c^2-5c+1\right) + \frac{3a^2(1-c)(a-4c+5)}{4(a+1)} \right\} \left(\frac{a}{2}-b-c+\frac{9}{2}\right)\left(\frac{a}{2}-b-c+\frac{11}{2}\right) \\
& + \left[\frac{3a(1-c)(a-2c+3)}{4(a+1)} + \frac{a(a+2)}{4(a+3)}\{2(a+2)(2a+3)+(b-1)(c-4)\} \right] \\
& \cdot (b-2)(c-5)\left(\frac{a}{2}-b-c+\frac{9}{2}\right), \\
\beta_{2,3} = {} & \frac{1}{2}\left[(a-1)\left(c^2-5c+1\right) - 3a(c-1)\left\{(a-1)(c-1)-a^2+2a+4\right\} \right] \\
& \cdot \left(\frac{a}{2}-b-c+5\right)\left(\frac{a}{2}-b-c+6\right) \\
& + \frac{1}{2}\left[c^2-5c+1+\frac{3}{2}a(c-1)(4a-3c+7)-\frac{5}{2}a(a+1)(a+2) \right] \\
& + \frac{1}{8}(6c-5a-16)a(b-1)(b-2)(c-4)(c-5),
\end{aligned}
$$

$$\gamma_{2,3} = (b-2)(c-5)\left[c^2-5c+1-\frac{3a(c-1)^2}{2(a+1)}(2a+1) + \frac{3}{4}a(a+1)(c-1) - \frac{3a(a+2)^2}{2(a+3)}\left(1+\frac{2a}{3}\right)\right] + \frac{a-1}{2}\left\{c^2-5c+1+\frac{3a^2(c-1)(a-4c+5)}{4(a+1)} - \frac{a^2(a+2)^2}{a+3}\right\},$$

$$\delta_{2,3} = (b-2)(c-5)\left\{-\frac{1}{a}\left(c^2-5c+a\right)+\frac{3}{2}(2a+1)(c-1)^2 + 3(1-c)\left(a^2-2\right)+\frac{1}{2}(a+2)\left(2a^2+a-7\right)\right\} + \frac{1}{2}(1-a)\left(c^2-5c+1\right) + \frac{a}{2}\left\{3(a-1)(c-1)^2+3(1-c)\left(a^2-2a+4\right)+(a+2)\left(a^2-2a-6\right)\right\},$$

$$I_{1,3}(a,b,c) = \alpha_{1,3}\frac{\Gamma(b-1)\Gamma(c-4)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+7/2)}{\Gamma(a-b-c+5)\Gamma((1/2)a-b+3/2)\Gamma((1/2)a-c+9/2)} + \beta_{1,3}\frac{\Gamma(b-1)\Gamma(c-4)\Gamma((1/2)a+1/2)\Gamma((1/2)a-b-c+4)}{\Gamma(a-b-c+5)\Gamma((1/2)a-b+1)\Gamma((1/2)a-c+4)} + \gamma_{1,3}\frac{\Gamma(b-1)\Gamma(c-4)\Gamma((1/2)a+1/2)}{\Gamma(a-b-c+5)\Gamma((1/2)a)} + \delta_{1,3}\frac{\Gamma(b-1)\Gamma(c-4)}{\Gamma(a-b-c+5)}, \tag{2.6}$$

where

$$\alpha_{1,3} = \frac{1}{2}\left(\frac{a}{2}-b-c+\frac{7}{2}\right)\left(\frac{a}{2}-b-c+\frac{9}{2}\right) \cdot\left\{a(a+1)(a+2)^2-\left(c^2-5c+1\right)+3a(1-c)(2-c+a)\right\} + \frac{1}{4}(b-1)(c-4)\left(\frac{a}{2}-b-c+\frac{7}{2}\right) \cdot\left\{6(c-1)^2+9(a+1)(1-c)+4(a+1)(a+2)\right\} + \frac{1}{4}(a+1)b(b-1)(c-3)(c-4),$$

$$\beta_{1,3} = \left(\frac{a}{2}-b-c+4\right) \cdot\left\{\frac{c^2-5c+1}{a}-3(c-1)^2+\frac{(a+1)^2}{a+2}(3c-a-5)\right\} + \frac{3(a+1)^2(c-1)}{a+2}+\frac{a+1}{2(a+2)}(b-1)(c-4)(3c-2a-7),$$

$$\gamma_{1,3} = -\frac{c^2-5c+1}{a} + 3(c-1)^2 - \frac{(a+3c-1)(a+1)^2}{a+2},$$
$$\delta_{1,3} = \frac{1}{2}\left(c^2-5c+1\right) + \frac{3}{2}a(c-1)(a-c+2) - \frac{1}{2}a(a+1)(a+2)^2. \tag{2.7}$$

3. Proof of Theorem 2.1

We will prove only $I_{3,2}(a,b,c)$. The other formulas in Theorem 2.1 can be shown as in the proof of $I_{3,2}(a,b,c)$. We begin by writing

$$\begin{aligned} I_{3,2}(a,b,c) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{n!\Gamma(4+a-b+n)\Gamma(6+a-c+n)} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{n!\Gamma(4+a-b+n)\Gamma(6+a-c+n)} \cdot \frac{\boldsymbol{\Gamma(a+2n+4)\Gamma(a-b-c+6)}}{\boldsymbol{\Gamma(a+2n+4)\Gamma(a-b-c+6)}}, \end{aligned} \tag{3.1}$$

by which the bold face factors are multiplied. Rearranging the factors in the last sum to use the Gauss's summation theorem (1.6), we get

$$\begin{aligned} I_{3,2}(a,b,c) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{n!\Gamma(a+2n+4)\Gamma(a-b-c+6)} \cdot \frac{\Gamma(a+2n+4)\Gamma(a-b-c+6)}{\Gamma(4+a-b+n)\Gamma(6+a-c+n)} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{n!\Gamma(a+2n+4)\Gamma(a-b-c+6)} \cdot {}_2F_1\left[\begin{matrix} b+n,\ c-2+n \\ a+2n+4 \end{matrix} \middle| 1\right]. \end{aligned} \tag{3.2}$$

Rewriting the last ${}_2F_1$ and using a manipulation of double series, we obtain

$$\begin{aligned} I_{3,2}(a,b,c) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(c-1+n)(c-2+n)\Gamma(a+n)\Gamma(b+n+m)\Gamma(c-2+n+m)}{n!m!\Gamma(a-b-c+6)\Gamma(a+4+2n+m)} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(c-1+n)(c-2+n)\Gamma(a+n)\Gamma(b+m)\Gamma(c-2+m)}{n!(m-n)!\Gamma(a-b-c+6)\Gamma(a+4+n+m)} \\ &\quad \cdot \frac{\Gamma(a)m!\Gamma(a+4+m)}{\Gamma(a)m!\Gamma(a+4+m)} \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(b+m)\Gamma(c-2+m)\Gamma(a)}{m!\Gamma(a-b-c+6)\Gamma(a+4+m)} \\ &\quad \cdot \sum_{n=0}^{m} \frac{m!(c-1+n)(c-2+n)\Gamma(a+n)\Gamma(a+4+m)}{(m-n)!n!\Gamma(a)\Gamma(a+4+m+n)}. \end{aligned} \tag{3.3}$$

Separating the last summation, we find that

$$I_{3,2}(a,b,c) = \sum_{m=0}^{\infty} \frac{\Gamma(b+m)\Gamma(c-2+m)\Gamma(a)}{m!\Gamma(a-b-c+6)\Gamma(a+4+m)} \cdot \{(c-1)(c-2)\mathcal{P}(a,m) + 2(c-1)\mathcal{Q}(a,m) + \mathcal{R}(a,m)\}, \tag{3.4}$$

where

$$\begin{aligned} \mathcal{P}(a,m) &= \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} \frac{\Gamma(a+n)\Gamma(a+4+m)}{\Gamma(a)\Gamma(a+4+m+n)}, \\ \mathcal{Q}(a,m) &= \sum_{n=0}^{m} \frac{m!n}{n!(m-n)!} \frac{\Gamma(a+n)\Gamma(a+4+m)}{\Gamma(a)\Gamma(a+4+m+n)}, \\ \mathcal{R}(a,m) &= \sum_{n=0}^{m} \frac{m!n(n-1)}{n!(m-n)!} \frac{\Gamma(a+n)\Gamma(a+4+m)}{\Gamma(a)\Gamma(a+4+m+n)}. \end{aligned} \tag{3.5}$$

Using (1.2) and (1.4) to express $\mathcal{P}(a,m)$, $\mathcal{Q}(a,m)$, and $\mathcal{R}(a,m)$ in the forms of ${}_2F_1(-1)$, (3.4) is rewritten as follows:

$$I_{3,2}(a,b,c) = \mathcal{A}(a,b,\mathrm{c}) + \mathcal{B}(a,b,c) + \mathcal{C}(a,b,c), \tag{3.6}$$

where

$$\begin{aligned} \mathcal{A}(a,b,c) &= \frac{(c-1)(c-2)\Gamma(a)}{\Gamma(a-b-c+6)} \sum_{m=0}^{\infty} \frac{\Gamma(b+m)\Gamma(c-2+m)}{m!\Gamma(a+4+m)} \, {}_2F_1\left[\begin{matrix} a, \ -m \\ a+4+m \end{matrix} \,\middle|\, -1\right], \\ \mathcal{B}(a,b,c) &= \frac{2(c-1)\Gamma(a+1)}{\Gamma(a-b-c+6)} \sum_{m=0}^{\infty} \frac{m\Gamma(b+m)\Gamma(c-2+m)}{m!\Gamma(a+5+m)} \\ &\quad \cdot {}_2F_1\left[\begin{matrix} a+1, \ 1-m \\ a+5+m \end{matrix} \,\middle|\, -1\right], \\ \mathcal{C}(a,b,c) &= \frac{\Gamma(a+2)}{\Gamma(a-b-c+6)} \sum_{m=0}^{\infty} \frac{m(m-1)\Gamma(b+m)\Gamma(c-2+m)}{m!\Gamma(a+6+m)} \\ &\quad \cdot {}_2F_1\left[\begin{matrix} a+2, \ 2-m \\ a+6+m \end{matrix} \,\middle|\, -1\right]. \end{aligned} \tag{3.7}$$

Applying some appropriate formulas in Lemma 1.1 to ${}_2F_1(-1)$ in (3.6), we obtain

$$\mathcal{A}(a,b,c) = \sum_{j=1}^{4} \mathcal{A}_j(a,b,c), \tag{3.8}$$

where

$$\mathcal{A}_1(a,b,c) = \frac{(a-2)(1-c)(c-2)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)} \cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \frac{a}{2}-\frac{1}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+3/2)}\right],$$

$$\mathcal{A}_2(a,b,c) = \frac{3(1-c)(c-2)\Gamma(a/2+1/2)}{2\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} \cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ a/2+1/2 \end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)}\right],$$

$$\mathcal{A}_3(a,b,c) = \frac{2(a-1)(c-1)(c-2)\Gamma(a/2+1/2)}{a\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)} \cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \frac{a}{2}-1 \end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+1)}\right],$$

$$\mathcal{A}_4(a,b,c) = \frac{(c-1)(c-2)\Gamma(a/2+1/2)}{a\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma\left(\frac{a}{2}\right)} \cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ \frac{a}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+1)}\right], \tag{3.9}$$

$$\mathcal{B}(a,b,c) = \sum_{j=1}^{5} \mathcal{B}_j(a,b,c), \tag{3.10}$$

where

$$\mathcal{B}_1(a,b,c) = \frac{(c-1)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)} \cdot {}_2F_1\left[\begin{matrix} b-1,\ c-3 \\ \frac{a}{2}+\frac{3}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{\Gamma(a/2+5/2)}\right],$$

$$\mathcal{B}_2(a,b,c) = \frac{4a(c-1)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} \cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ \frac{a}{2}+\frac{1}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{2\Gamma(a/2+5/2)}\right],$$

$$\mathcal{B}_3(a,b,c) = \frac{2a(a-2)(c-1)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)} \cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \frac{a}{2}-\frac{1}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{6\Gamma(a/2+5/2)}\right],$$

$$\mathcal{B}_4(a,b,c) = \frac{4(1-c)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} \cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ \frac{a}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+1)} - \frac{\Gamma(b)\Gamma(c-2)}{2\Gamma(a/2+2)}\right],$$

$$\mathcal{B}_5(a,b,c) = \frac{4(a-1)(1-c)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)}\cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \dfrac{a}{2}-1 \end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+1)} - \frac{\Gamma(b)\Gamma(c-2)}{6\Gamma(a/2+2)}\right]. \tag{3.11}$$

$$\mathcal{C}(a,b,c) = \sum_{j=1}^{6} \mathcal{C}_j(a,b,c), \tag{3.12}$$

where

$$\mathcal{C}_1(a,b,c) = -\frac{5(a+1)}{4}\frac{\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)}\cdot {}_2F_1\left[\begin{matrix} b-1,\ c-3 \\ \dfrac{a}{2}+\dfrac{3}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{\Gamma(a/2+5/2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{2\Gamma(a/2+7/2)}\right],$$

$$\mathcal{C}_2(a,b,c) = -\frac{5a(a+1)}{2}\frac{\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma\left(\dfrac{a}{2}+\dfrac{1}{2}\right)}\cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ \dfrac{a}{2}+\dfrac{1}{2} \end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{2\Gamma(a/2+5/2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{6\Gamma(a/2+7/2)}\right],$$

$$\mathcal{C}_3(a,b,c) = -\frac{a(a-2)(a+1)\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)}\cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \dfrac{a}{2}-\dfrac{1}{2}\end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1/2)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2+1/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+3/2)} - \frac{\Gamma(b)\Gamma(c-2)}{6\Gamma(a/2+5/2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{24\Gamma(a/2+7/2)}\right],$$

$$\mathcal{C}_4(a,b,c) = \frac{a+1}{2(a+2)}\frac{\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+1)}\cdot {}_2F_1\left[\begin{matrix} b-1,\ c-3 \\ \dfrac{a}{2}+1\end{matrix}\middle| 1\right] - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+1)} - \frac{\Gamma(b)\Gamma(c-2)}{\Gamma(a/2+2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{2\Gamma(a/2+3)}\right],$$

$$\mathcal{C}_5(a,b,c) = \frac{3(a+1)^2}{a+2}\frac{\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)}\cdot {}_2F_1\left[\begin{matrix} b-2,\ c-4 \\ \dfrac{a}{2}\end{matrix}\middle| 1\right] - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{\Gamma(a/2+1)} - \frac{\Gamma(b)\Gamma(c-2)}{2\Gamma(a/2+2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{6\Gamma(a/2+3)}\right],$$

$$\mathcal{C}_6(a,b,c) = \frac{2(a-1)(a+1)^2}{a+2}\frac{\Gamma(a/2+1/2)}{\Gamma(a-b-c+6)} \times \left[\frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)}\cdot {}_2F_1\left[\begin{matrix} b-3,\ c-5 \\ \dfrac{a}{2}-1\end{matrix}\middle| 1\right] - \frac{\Gamma(b-3)\Gamma(c-5)}{\Gamma(a/2-1)} - \frac{\Gamma(b-2)\Gamma(c-4)}{\Gamma(a/2)} - \frac{\Gamma(b-1)\Gamma(c-3)}{2\Gamma(a/2+1)} - \frac{\Gamma(b)\Gamma(c-2)}{6\Gamma(a/2+2)} - \frac{\Gamma(b+1)\Gamma(c-1)}{24\Gamma(a/2+3)}\right]. \tag{3.13}$$

Finally applying the Gauss's summation theorem (1.6) to $\mathcal{A}_j$'s in (3.8), $\mathcal{B}_j$'s in (3.10), and $\mathcal{C}_j$'s in (3.12), after a simplification by making a main use of $\Gamma(z+1) = z\Gamma(z)$, we can readily show the summation formula (2.3) in Theorem 2.1.

We conclude this paper by noting that by extending Tables 1 and 2 and using the same technique given here, all other known formulas in [3] (see (1.9) can be proved and further extension summation formulas for $f_{i,j}(a,b,c)$ in (1.9)

$$\left((i,j) \in \mathbb{Z} \setminus \{-3,-2,-1,0,1,2\} \times \mathbb{N}_0 \setminus \{0,1,2,3\}\right) \tag{3.14}$$

can be presented, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ denotes the set of integers.

References

[1] E. D. Rainville, *Special Functions*, The Macmillan, New York, NY, USA, 1960.

[2] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[3] J. L. Lavoie, F. Grondin, A. K. Rathie, and K. Arora, "Generalizations of Dixon's theorem on the sum of a $_3F_2$," *Mathematics of Computation*, vol. 62, no. 205, pp. 267–276, 1994.

[4] Y. S. Kim and A. K. Rathie, "On an extension formulas for the triple hypergeometric series X_8 due to Exton," *Bulletin of the Korean Mathematical Society*, vol. 44, no. 4, pp. 743–751, 2007.

[5] J. L. Lavoie, F. Grondin, and A. K. Rathie, "Generalizations of Watson's theorem on the sum of a $_3F_2$," *Indian Journal of Mathematics*, vol. 34, no. 2, pp. 23–32, 1992.

[6] J. L. Lavoie, F. Grondin, and A. K. Rathie, "Generalizations of Whipple's theorem on the sum of a $_3F_2$," *Journal of Computational and Applied Mathematics*, vol. 72, no. 2, pp. 293–300, 1996.

[7] W. N. Bailey, *Generalized Hypergeometric Series*, Stechert-Hafner, New York, NY, USA, 1964.