Research Article

A Note on \((C_p, \alpha)\)-Hyponormal Operators

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We study \((C_p, \alpha)\)-normal operators and \((C_p, \alpha)\)-hyponormal operators. We show the inclusion relation between these classes under various hypotheses for \(p\) and \(\alpha\). We also obtain some sufficient conditions for Aluthge transform \(\tilde{T}_{s,t} = |T|^s|U|T|^t\) and \(T^2\) of \((C_p, \alpha)\)-hyponormal operators still to be \((C_p, \alpha)\)-hyponormal.

1. Introduction

Let \(\mathcal{H}\) be a separable, infinite dimensional, complex Hilbert space, and denote by \(\mathcal{L}(\mathcal{H})\) the algebra of all bounded linear operators on \(\mathcal{H}\). Recently, Lauric in [1] introduced \((C_p, \alpha)\)-hyponormal operators. For \(\alpha > 0\) and \(T \in \mathcal{L}(\mathcal{H})\), denote by \(D_T^{\alpha} = (T^*T)^\alpha - (TT^*)^\alpha\). We denote that \(C_p(\mathcal{H})\), \(1 \leq p < \infty\), the ideal of operators in the Schatten \(p\)-class [2]. Although, for \(0 < p < 1\), the usual definition of \(\| \cdot \|_p\) does not satisfy the triangle inequality, nevertheless \((C_p, \| \cdot \|_p)\) is closed and \(\|TK\|_p \leq \|T\| \cdot \|K\|_p\), when \(T \in \mathcal{L}(\mathcal{H})\) and \(K \in C_p(\mathcal{H})\). An operator \(T\) in \(\mathcal{L}(\mathcal{H})\) is \((C_p, \alpha)\)-normal if \(D_T^{\alpha} \in C_p(\mathcal{H})\), and denote the class of \((C_p, \alpha)\)-normal operators by \(\mathcal{N}_p^{\alpha}(\mathcal{H})\). An operator \(T\) in \(\mathcal{L}(\mathcal{H})\) will be called \((C_p, \alpha)\)-hyponormal if \(D_T^{\alpha} = P + K\), where \(P\) is a positive semidefinite operator (\(P \geq 0\)) and \(K \in C_p(\mathcal{H})\). The class of \((C_p, \alpha)\)-hyponormal operators will be denoted by \(\mathcal{N}_p^{\alpha}(\mathcal{H})\). In particular, an operator \(T\) in \(\mathcal{L}(\mathcal{H})\) will be called almost hyponormal. Furthermore, an operator \(T\) in \(\mathcal{L}(\mathcal{H})\) whose \(D_T^{\alpha}\) is positive semidefinite is called \(\alpha\)-hyponormal (notation: \(T \in \mathcal{H}_p^{\alpha}(\mathcal{H})\)).

In this paper, we first study the inclusion relation between these classes under various hypotheses for \(p\) and \(\alpha\) in Section 2. Then we study the Aluthge transform \(\tilde{T}_{s,t} = |T|^s|U|T|^t\) and \(T^2\) of \((C_p, \alpha)\)-hyponormal operators in Section 3.

Before proceeding, we will make use of the following inequality.
Theorem F (See Furuta inequality in [3]). If $A \geq B \geq 0$, then, for each $r \geq 0$,

\[
\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geq \left(B^{r/2}B^pB^{r/2}\right)^{1/q},
\]

\[
\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geq \left(A^{r/2}B^pA^{r/2}\right)^{1/q},
\]

as long as real numbers $p, r, q$ satisfy

\[
p \geq 0, q \geq 1 \text{ with } (1 + r)q \geq p + r.
\]

Lemma 1.1 (see [1]). Let $A \in \mathcal{L}(\mathcal{K})$, $A \geq 0$, $\alpha \in (0,1]$, $p \geq \alpha$, and $K \in \mathcal{C}_p(\mathcal{K})$, such that $A + K \geq 0$. Then $(A + K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_{p/\alpha}(\mathcal{K})$. If in addition $K \geq 0$, then $K_1 \geq 0$.

Lemma 1.2 (see [1]). Let $A \in \mathcal{L}(\mathcal{K})$, $A \geq 0$, $p \geq 1$, and $K \in \mathcal{C}_p(\mathcal{K})$, such that $A + K \geq 0$, and let $\alpha \in [1, +\infty)$. Then $(A + K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_p(\mathcal{K})$.

2. Some Inclusions

According to Löwner-Heinz (L-H) inequality [4, 5] that $A \geq B \geq 0$ ensures that $A^\alpha \geq B^\alpha$ for each $\alpha \in [0,1]$, we obtain $\mathcal{N}_0^{\alpha}(\mathcal{K}) \supseteq \mathcal{N}_0^{\beta}(\mathcal{K})$ when $\alpha \leq \beta$. However, the similar inclusions for the classes $\mathcal{N}_0^{\alpha}(\mathcal{K})$ and $\mathcal{N}_0^{\beta}(\mathcal{K})$ are less obvious. In this section, we will examine various inclusions between these classes of operators. (1) of Theorem 2.1 has been already shown in [1]. But we will give a proof for the readers’ convenience.

Theorem 2.1. Let $\alpha > 0$, $p \geq 1$, and let $T$ be in $\mathcal{N}_p^{\alpha}(\mathcal{K})$.

(1) If $\beta \geq \alpha$, then $T$ belongs to $\mathcal{N}_p^{\beta}(\mathcal{K})$, and therefore $\mathcal{N}_p^{\alpha}(\mathcal{K}) \subseteq \mathcal{N}_p^{\beta}(\mathcal{K})$.

(2) If $0 < \beta \leq \alpha$, then $T$ belongs to $\mathcal{N}_p^{\alpha/\beta}(\mathcal{K})$, and therefore $\mathcal{N}_p^{\alpha}(\mathcal{K}) \subseteq \mathcal{N}_p^{\alpha/\beta}(\mathcal{K})$.

Proof. Let $\alpha$, $p$, and $T$ be as in the hypotheses and let $T = U|T|$ be the polar decomposition of $T$.

For $T \in \mathcal{N}_p^{\alpha}(\mathcal{K})$, we have

\[
D_T^\alpha = (T^*T)^\alpha = |T|^\alpha - |T^*|\alpha = K,
\]

with $K \in \mathcal{C}_p(\mathcal{K})$. Then we obtain

\[
|T|^{2\alpha} = |T^*|^{2\alpha} + K \geq 0.
\]

(1) First we consider the case $\beta \geq \alpha$. According to Lemma 1.2, we obtain

\[
|T|^{2\beta} = (|T|^{2\alpha} + K)^{\beta/\alpha} = |T^*|^{2\beta} + K_1,
\]

with $K_1 \in \mathcal{C}_p(\mathcal{K})$. Then $T \in \mathcal{N}_p^{\beta}(\mathcal{K})$. 
(2) Next we consider the case $0 < \beta \leq \alpha$. According to Lemma 1.1, we obtain

$$|T|^{2\beta} = \left(|T^*|^{2\alpha} + K\right)^{\beta/\alpha} = |T^*|^{2\beta} + K_1,$$

with $K_1 \in C_{ap/\beta}(\mathcal{H})$. Then $T \in \mathcal{N}_{ap/\beta}^t(\mathcal{H})$. \hfill \Box

The following corollary is a consequence of Theorem 2.1.

**Corollary 2.2.** Let $\alpha > 0$, $p \geq 1$, then, for $0 < \beta \leq \alpha$,

$$\mathcal{N}_p^\beta(\mathcal{H}) \subseteq \mathcal{N}^\alpha_p(\mathcal{H}) \subseteq \mathcal{H}_{ap/\beta}(\mathcal{H}) \subseteq \mathcal{N}_{ap/\beta}^t(\mathcal{H}). \tag{2.5}$$

**Theorem 2.3.** Let $\alpha > 0$, $p \geq 1$, and let $T$ be in $\mathcal{H}^t_p(\mathcal{H})$. If $0 < \beta \leq \alpha$, then $T$ belongs to $\mathcal{H}_{ap/\beta}^\beta(\mathcal{H})$, and therefore $\mathcal{H}^t_p(\mathcal{H}) \subseteq \mathcal{H}_{ap/\beta}^\beta(\mathcal{H})$.

**Proof.** Let $\alpha$, $p$, and $T$ be as in the hypotheses and let $T = U|T|$ be the polar decomposition of $T$.

For $T \in \mathcal{H}^t_p(\mathcal{H})$, we have

$$D_t^\alpha = (T^*T)^\alpha - (TT^*)^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = P + K,$$

with $P \geq 0$, $K \in C_p(\mathcal{H})$. Then we obtain

$$|T|^{2\alpha} = |T^*|^{2\alpha} + P + K \geq 0. \tag{2.7}$$

For $0 < \beta \leq \alpha$, according to Lemma 1.1 and L-H inequality, we obtain

$$|T|^{2\beta} = \left(|T^*|^{2\alpha} + P + K\right)^{\beta/\alpha} = \left(|T^*|^{2\alpha} + P\right)^{\beta/\alpha} + K_1 \geq |T^*|^{2\beta} + K_1,$$

with $K_1 \in C_{ap/\beta}(\mathcal{H})$. Then we obtain $T \in \mathcal{H}_{ap/\beta}^\beta(\mathcal{H})$. \hfill \Box

### 3. Some Properties of $(C_p, \alpha)$-Hyponormal Operators

Let $T = U|T|$ be the polar decomposition of an operator $T$ on a Hilbert space $\mathcal{H}$, where $U$ is a partial isometry operator. Recently, Lauric [1] shows some theorems on the Aluthge transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ of $(C_p, \alpha)$-hyponormal operators. In this section, we will show some sufficient conditions for the generalized Aluthge transform $\tilde{T}_{s,t} = |T|^{s}U|T|^t(s, t > 0)$ and
$T^2$ of $(C_p, \alpha)$-hyponormal operators to be $(C_p, \alpha)$-hyponormal. Aluthge transform $\tilde{T}_{s,t}$ arose in the study of $p$-hyponormal operators [6, 7] and has since been studied in many different contexts [8–15].

Let $T$ belong to $\mathcal{M}_p^a(\mathcal{A})$, for some $\alpha > 0, p > 0$, such that $D_T^a = P + K$ with $P \geq 0$, $K \in C_p(\mathcal{A})$. Since $K = K^* = K_+ - K_-$ and $K_+, K_- \geq 0$ are $C_p$-class operators, in what follows we will assume that $D_T^a = P_1 - K_1$ with $P_1 \geq 0$ and $K_1 \geq 0, K_1 \in C_p(\mathcal{A})$.

**Theorem 3.1.** Let $p \geq 1, \alpha \geq \max\{s, t, \}$, and $T \in \mathcal{M}_p^a(\mathcal{A})$ such that $D_T^a = P - K$ with $P, K \geq 0$, $K \in C_p(\mathcal{A})$, and let $\epsilon \in (0, 1/2)$ such that $\alpha + \epsilon \leq 1$. Then $\tilde{T}_{s,t} \in \mathcal{M}_p^{(\alpha + \epsilon)s}(\mathcal{A})$.

**Proof.** We may assume that $T = U|T|$ with $U$ being unitary. The equality $D_T^a = P - K$ with $P, K \geq 0$ implies that $|T|^{2a} + K \geq U|T|^{2a}U^*$. Multiplying this inequality by $U^*$ to the left and by $U$ to the right, we obtain

$$A = U^*|T|^{2a}U + U^*KU \geq |T|^{2a} = B.$$ (3.1)

According to Lemma 1.1,

$$A^{s/\alpha} = \left(U^*\left(|T|^{2a} + K\right)U\right)^{s/\alpha} = U^*\left(|T|^{2a} + K\right)^{s/\alpha}U = U^*\left(|T|^{2a} + K_1\right)U,$$ (3.2)

with $K_1 \in C_{ap/s}(\mathcal{A})$. Setting $K_2 = |T|^sU^*K_1U|T|^s$, by Theorem F we have

$$\left(\tilde{T}_{s,t}^* \tilde{T}_{s,t} + K_2\right)^{a+\epsilon} = \left(|T|^sU^*\left(|T|^{2a} + K_1\right)U\right)^{a+\epsilon}$$

$$= \left(|T|^sU^*\left(|T|^{2a} + K\right)U\right)^{s/\alpha}|T|^{s} = \left(B^{s/2a} A^{s/\alpha} B^{t/2a}\right)^{a+\epsilon}$$

$$\geq B^{(s+t)(\alpha+\epsilon)/\alpha}$$

$$= |T|^{2(s+t)(\alpha+\epsilon)}.$$ (3.3)

On the other hand, according to Lemma 1.1,

$$\left(\tilde{T}_{s,t}^* \tilde{T}_{s,t} + K_2\right)^{a+\epsilon} = \left(\tilde{T}_{s,t}^* \tilde{T}_{s,t}\right)^{a+\epsilon} + K_3,$$ (3.4)

with $K_3 \in C_{ap/(\alpha+\epsilon)s}(\mathcal{A})$. Then we have

$$\left(\tilde{T}_{s,t}^* \tilde{T}_{s,t}\right)^{a+\epsilon} + K_3 \geq |T|^{2(s+t)(\alpha+\epsilon)}.$$ (3.5)

According to the following inequality

$$C = |T|^{2a} + K \geq U|T|^{2a}U^* = D,$$ (3.6)
by Theorem F, we have
\[
\left( C^{s/2} D^{l/2} C^{s/2} \right)^{a+\varepsilon} \leq C^{(s+l)(a+\varepsilon)/a}.
\]  
(3.7)

Again, according to Lemma 1.1,
\[
C^{s/2} = \left( |T|^2 + K \right)^{s/2} = |T|^s + K_4,
\]  
(3.8)

with \( K_4 \in C_{2p/s}(\mathcal{H}) \).

Next, obviously,
\[
D^{l/2} = \left( U |T|^2 U^* \right)^{l/2} = U |T|^l U^*.
\]  
(3.9)

Then we have
\[
\left( C^{s/2} D^{l/2} C^{s/2} \right)^{a+\varepsilon} = \left\{ (|T|^s + K_4) \left( U |T|^2 U^* \right) (|T|^s + K_4) \right\}^{a+\varepsilon}
\]
\[
= \left( |T|^s U |T|^2 U^* |T|^s + K_5 \right)^{a+\varepsilon}
\]
\[
= \left( \tilde{T}_{s,l} \tilde{T}_{s,l}^* + K_5 \right)^{a+\varepsilon}
\]  
(3.10)

\[
= \left( \tilde{T}_{s,l} \tilde{T}_{s,l}^* \right)^{a+\varepsilon} + K_6,
\]

with \( K_5 \in C_{2p/s}(\mathcal{H}) \), \( K_6 \in C_{2p/(a+\varepsilon)s}(\mathcal{H}) \).
(1) First we consider the case $0 \leq ((s + t) / \alpha) \leq 1$. According to Lemma 1.1, we have

\[
\left( C^{s+t/\alpha} \right)^{a+\epsilon} = \left\{ |T|^{2\alpha} + K \right\}^{s+t/\alpha}
\]

\[
= \left( |T|^{2(s+t)} + K_{7}\right)^{a+\epsilon}
\]

\[
= |T|^{2(s+t)(a+\epsilon)} + K_{8},
\]

with $K_{7} \in \mathcal{C}_{ap/(s+t)}(\mathcal{E})$ and $K_{8} \in \mathcal{C}_{ap/(a+\epsilon)(s+t)}(\mathcal{E})$.

Then by (3.7) and (3.10), set $K_{9} = K_{6} - K_{8} \in \mathcal{C}_{2ap/(a+\epsilon)s}(\mathcal{E})$, and

\[
|T|^{2(s+t)(a+\epsilon)} \geq \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} + K_{9}. \tag{3.12}
\]

Combining (3.5) and (3.12), we obtain

\[
\left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} - \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} \geq K_{10}, \tag{3.13}
\]

where $K_{10} = K_{9} - K_{3} \in \mathcal{C}_{2ap/(a+\epsilon)s}(\mathcal{E})$.

(2) Next we consider the case $(s + t / \alpha) > 1$. According to Lemmas 1.1 and 1.2,

\[
\left( C^{s+t/\alpha} \right)^{a+\epsilon} = \left\{ |T|^{2\alpha} + K \right\}^{s+t/\alpha}
\]

\[
= \left( |T|^{2(s+t)} + K_{7}^{'} \right)^{a+\epsilon}
\]

\[
= |T|^{2(s+t)(a+\epsilon)} + K_{8}^{'}
\]

with $K_{7}^{'} \in \mathcal{C}_{p}(\mathcal{E})$ and $K_{8}^{'} \in \mathcal{C}_{p/a+\epsilon}(\mathcal{E})$, and

Then by (3.7) and (3.10), set $K_{9}^{'} = K_{6} - K_{8}^{'} \in \mathcal{C}_{2ap/(a+\epsilon)s}(\mathcal{E})$,

\[
|T|^{2(s+t)(a+\epsilon)} \geq \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} + K_{9}^{'} \tag{3.15}
\]

Combining (3.5) and (3.15), we obtain

\[
\left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} - \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^{*} \right)^{a+\epsilon} \geq K_{10}^{'} \tag{3.16}
\]

where $K_{10}^{'} = K_{9}^{'} - K_{3} \in \mathcal{C}_{2ap/(a+\epsilon)s}(\mathcal{E})$.

By (3.13) and (3.16), we obtain $\tilde{T}_{s,t} \in \mathcal{E}_{2ap/(a+\epsilon)s}(\mathcal{E})$. □
Remark 3.2. The main theorem of [1] was considered in the case \( \alpha \in [1/2, 1] \). Apparently, Theorem 3.1 implies (Theorems 13 in [1]) when \( s = t = 1/2 \). And we also obtain the following theorem.

**Theorem 3.3.** Let \( p \geq 1, 0 < \alpha \leq \min\{s, t\} \), and \( T \in \mathcal{A}_p^\alpha(\mathcal{H}) \) such that \( D_T^\alpha = P - K \) with \( P, K \geq 0, K \in C_p(\mathcal{H}) \), and let \( \varepsilon \geq 0 \) such that \( \alpha + \varepsilon \leq 2\alpha/(s + t) \).

1. If \( s \geq 2\alpha \), then \( \tilde{T}_{s,t} \in \mathcal{A}_p^{(\alpha+\varepsilon)}(\mathcal{H}) \).

2. If \( 0 < s < 2\alpha \), then \( \tilde{T}_{s,t} \in \mathcal{A}_{2p/(\alpha+\varepsilon)}\mathcal{A}_p(\mathcal{H}) \).

**Proof.** The proof of Theorem 3.3 is similar to the proof of Theorem 3.1. \( \Box \)

**Corollary 3.4.** Let \( p \geq 1, T \in \mathcal{A}_p^\alpha(\mathcal{H}) \) such that \( D_T^\alpha = P - K \) with \( P, K \geq 0, K \in C_p(\mathcal{H}) \), and let \( \varepsilon \in (0, 1/2] \).

1. If \( \alpha \in (0, 1/4] \), then \( \tilde{T} \in \mathcal{A}_p^{(\alpha+\varepsilon)}(\mathcal{H}) \).

2. If \( \alpha \in (1/4, 1/2] \), then \( \tilde{T} \in \mathcal{A}_{4p/(\alpha+\varepsilon)}\mathcal{A}_p(\mathcal{H}) \).

**Proof.** Put \( s = t = 1/2 \) in Theorem 3.3.

1. When \( \alpha \in (0, 1/4] \), we have \( s \geq 2\alpha \). According to (1) of Theorem 3.3, then \( \tilde{T} \in \mathcal{A}_p^{(\alpha+\varepsilon)}(\mathcal{H}) \).

2. When \( \alpha \in (1/4, 1/2] \), we have \( 0 < s < 2\alpha \). According to (2) of Theorem 3.3, then \( \tilde{T} \in \mathcal{A}_{4p/(\alpha+\varepsilon)}\mathcal{A}_p(\mathcal{H}) \). \( \Box \)

Next, we will study \( T^2 \) of \((C_p, \alpha)\)-hyponormal operators. And first we will prove the following lemma.

**Lemma 3.5.** Let \( p \geq 1, \alpha \in (0, 1] \), and \( T \in \mathcal{A}_p^\alpha(\mathcal{H}) \) such that \( D_T^\alpha = P + K \) with \( P \geq 0, K \in C_p(\mathcal{H}) \), and \( D_T^\alpha = P_1 - K_1 \) with \( P_1 \geq 0, K_1 \geq 0, K_1 \in C_p(\mathcal{H}) \). Then if \( |T|^{2\alpha} - P \geq 0 \), one has the following inequalities

1. There exists \( K' \in C_{2p/\alpha}^\alpha(\mathcal{H}) \) such that \( (|T||T^*|)^{\alpha/2} + K' \leq |T|^{2\alpha} \).

2. There exists \( K'' \in C_{2p/\alpha}^\alpha(\mathcal{H}) \) such that \( (|T^*||T^*|)^{\alpha/2} + K'' \geq |T|^{2\alpha} \).

**Proof.** Let \( \alpha, p, \) and \( T \) be as in the hypotheses and let \( T = U|T| \) be the polar decomposition of \( T \). Then we have

\[
D_T^\alpha = (T^*T)^\alpha - (TT^*)^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = P + K, \tag{3.17}
\]

with \( P \geq 0, K \in C_p(\mathcal{H}) \).

\[
D_T^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = P_1 - K_1, \tag{3.18}
\]

with and \( P_1 \geq 0, K_1 \geq 0, \) and \( K_1 \in C_p(\mathcal{H}) \).
By (3.17), we have
\[ A_1 = |T|^{2a} \geq |T^*|^{2a} + K = B_1 \geq 0. \] (3.19)

And according to Lemma 1.2,
\[ B_1^{1/\alpha} = \left( |T^*|^{2a} + K \right)^{1/\alpha} = |T^*|^2 + K_2, \] (3.20)
with \( K_2 \in \mathcal{C}_p(\mathcal{A}) \). Setting \( K_3 = |T||K_2|T| \), by Theorem F we have
\[
\left( \frac{|T||T^*|^2}{|T|} + K_3 \right)^{a/2} = \left\{ |T| \left( |T^*|^2 + K_2 \right) |T| \right\}^{a/2} \\
= \left( A_1^{1/2a} B_1^{1/2a} A_1^{1/2a} \right)^{a/2} \\
\leq A_1 \\
= |T|^{2a}.
\] (3.21)

By (3.18), we have
\[ A_2 = |T|^{2a} + K_1 \geq |T^*|^{2a} = B_2. \] (3.22)

And according to Lemma 1.2,
\[ A_2^{1/\alpha} = \left( |T|^{2a} + K_1 \right)^{1/\alpha} = |T|^2 + K_4, \] (3.23)
with \( K_4 \in \mathcal{C}_p(\mathcal{A}) \). Setting \( K_5 = |T^*||K_4||T^*| \), by Theorem F we have
\[
\left( |T^*||T|^2|T^*| + K_5 \right)^{a/2} = \left\{ |T^*| \left( |T|^2 + K_4 \right) |T^*| \right\}^{a/2} \\
= \left( B_2^{1/2a} A_2^{1/2a} B_2^{1/2a} \right)^{a/2} \\
\geq B_2 \\
= |T^*|^{2a}.
\] (3.24)

On the other hand, by Lemma 1.1,
\[
\left( \frac{|T||T^*|^2}{|T|} + K_3 \right)^{a/2} = \left( \frac{|T^*|^2}{|T|} \right)^{a/2} + K', \\
\left( |T^*||T|^2|T^*| + K_5 \right)^{a/2} = \left( |T^*|^2 |T^*| \right)^{a/2} + K'',
\] (3.25)
with \( K', K'' \in \mathcal{C}_{2p/\alpha}(\mathcal{A}) \).
Then by (3.21) and (3.24), we obtain

\[
\left( |T||T^*|^2 |T| \right)^{a/2} + K' \leq |T|^{2a}, \quad \left( |T^*||T^2|T^*| \right)^{a/2} + K'' \geq |T^*|^{2a},
\]

(3.26)

with \( K', K'' \in C_{2p/\alpha}(\mathcal{E}) \).

**Theorem 3.6.** Let \( p \geq 1, \alpha \in (0, 1], and T \in \mathcal{E}_{p}^{a}(\mathcal{E}) \) such that \( D^a T = P + K \) with \( P \geq 0, K \in \mathcal{C}_p(\mathcal{E}) \), and \( D^a T = P_1 - K_1 \) with \( P_1 \geq 0, K_1 \geq 0, and K_1 \in \mathcal{C}_p(\mathcal{E}) \). Then if \( |T|^{2a} - P \geq 0 \), one has \( T^2 \in \mathcal{E}_{2p/\alpha}^{2/\alpha}(\mathcal{E}) \).

**Proof.** Let \( \alpha, p, \) and \( T \) be as in the hypotheses. We may assume that \( T = U|T| \) with \( U \) being unitary. Then obviously,

\[
\left\{ T^2 \left(T^2 \right)^* \right\}^{a/2} = U \left( |T||T^*|^2 |T| \right)^{a/2} U^*,
\]

(3.27)

\[
\left\{ \left(T^2 \right)^* T^2 \right\}^{a/2} = \left( |T^*||T^2|T^*| \right)^{a/2} U^* \left( |T^*||T^2|T^*| \right)^{a/2} U.
\]

(3.28)

By Lemma 3.5, there exits \( K', K'' \in C_{2p/\alpha}(\mathcal{E}) \) such that

\[
\left( |T||T^*|^2 |T| \right)^{a/2} + K' \leq |T|^{2a},
\]

(3.29)

\[
\left( |T^*||T^2|T^*| \right)^{a/2} + K'' \geq |T^*|^{2a}.
\]

(3.30)

Multiplying (3.29) by \( U \) to the left and by \( U^* \) to the right, we obtain

\[
U \left( |T||T^*|^2 |T| \right)^{a/2} U^* + UK' U^* \leq U |T|^{2a} U^* = |T^*|^{2a}.
\]

(3.31)

Multiplying (3.30) by \( U^* \) to the left and by \( U \) to the right, we obtain

\[
U^* \left( |T^*||T^2|T^*| \right)^{a/2} U + U^* K'' U \geq U^* |T^*|^{2a} U = |T|^{2a}.
\]

(3.32)

By (3.27) and (3.31), we have

\[
\left\{ T^2 \left(T^2 \right)^* \right\}^{a/2} + UK' U^* \leq |T^*|^{2a}.
\]

(3.33)

By (3.28) and (3.32), we have

\[
\left\{ \left(T^2 \right)^* T^2 \right\}^{a/2} + U^* K'' U \geq |T|^{2a}.
\]

(3.34)
Setting $K_2 = U K' U^* - U^* K'' U$, $K_2 \in C_{2p/\alpha}(\mathcal{H})$, we have

\[
\left\{ \left( T^2 \right)^* T^2 \right\}^{\alpha/2} - \left\{ T^2 \left( T^2 \right)^* \right\}^{\alpha/2} \geq |T|^{2\alpha} - |T^*|^{2\alpha} + K_2.
\] (3.35)

Therefore, for $K_3 = K + K_2, K_3 \in C_{2p/\alpha}(\mathcal{H})$, we have

\[
\left\{ \left( T^2 \right)^* T^2 \right\}^{\alpha/2} - \left\{ T^2 \left( T^2 \right)^* \right\}^{\alpha/2} \geq P + K_3.
\] (3.36)

Then the proof of Theorem 3.6 is finished. \qed

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**References**


