Research Article

Multivariate Twisted $p$-Adic $q$-Integral on $\mathbb{Z}_p$
Associated with Twisted $q$-Bernoulli Polynomials
and Numbers

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Recently, many authors have studied twisted $q$-Bernoulli polynomials by using the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$. In this paper, we define the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$ and extend our result to the twisted $q$-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted $q$-Bernoulli polynomials.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < 1$.

For $n \in \mathbb{N}$, let $T_p$ be the $p$-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty},$$

(1.1)

where $C_{p^n} = \{ \xi \in \mathbb{C}_p | \xi^{p^n} = 1 \text{ for some } n \geq 0 \}$ is the cyclic group of order $p^n$. 


Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \).
For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N-1} f(x) q^x,
\]

(compare with [1–3]).

It is well known that the twisted \( q \)-Bernoulli polynomials of order \( k \) are defined as

\[
e^{xt}\left(\frac{t}{e^{t\zeta_q}-1}\right)^k = \sum_{n=0}^{\infty} \beta^{(k)}_{n,\zeta,q}(x) \frac{t^n}{n!}, \quad \zeta \in T_p,
\]

and \( \beta^{(k)}_{n,\zeta,q} \) are called the twisted \( q \)-Bernoulli numbers of order \( k \). When \( k = 1 \), the polynomials and numbers are called the twisted \( q \)-Bernoulli polynomials and numbers, respectively. When \( k = 1 \) and \( q = 1 \), the polynomials and numbers are called the twisted Bernoulli polynomials and numbers, respectively. When \( k = 1 \), \( q = 1 \), and \( \zeta = 1 \), the polynomials and numbers are called the ordinary Bernoulli polynomials and numbers, respectively.

Many authors have studied the twisted \( q \)-Bernoulli polynomials by using the properties of the \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) (cf. [4]). In this paper, we define the twisted \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) and extend our result to the twisted \( q \)-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted \( q \)-Bernoulli polynomials.

2. Multivariate Twisted \( p \)-Adic \( q \)-Integral on \( \mathbb{Z}_p \) Associated with Twisted \( q \)-Bernoulli Polynomials

In this section, we assume that \( q \in \mathbb{C}_p \) with \( |q-1|_p < 1 \). For \( \zeta \in T_p \), we define the \((q,\zeta)\)-numbers as

\[
[k]_{q,\zeta} = \frac{1 - q^k \zeta}{1 - q}, \quad \text{for } k \in \mathbb{Z}_p.
\]

Note that \([k]_q = [k]_{q,1} = (1 - q^k)/(1 - q)\).

Let us define

\[
\binom{n}{k}_{q,\zeta} = \frac{[n]_{q,\zeta}!}{[k]_{q,\zeta}![n-k]_{q,\zeta}!},
\]

where \([k]_{q,\zeta}! = [k]_{q,\zeta}[k-1]_{q,\zeta} \cdots [1]_{q,\zeta} \). Note that \( \binom{n}{k} = \binom{n}{k}_{1,1} = n!/(n-k)! \).
Now we construct the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$I_{q,ξ}(f) = \int_{\mathbb{Z}_p} f(x) dμ_{q,ξ}(x)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) μ_{q,ξ}(x + p^N\mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x ξ^x, \quad (2.3)$$

where $μ_{q,ξ}(x + p^N\mathbb{Z}_p) = q^x ξ^x /[p^N]_q$. From the definition of the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$, we can consider the twisted $q$-Bernoulli polynomials and numbers of order $k$ as follows:

$$β_{n,q,ξ}^{(k)}(x) = \int_{\mathbb{Z}_p} [x_1 + x_2 + \cdots + x_k + x]^n dμ_{q,ξ}(x_1) dμ_{q,ξ}(x_2) \cdots dμ_{q,ξ}(x_k)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q^{k_+}} \sum_{x_1,\ldots,x_k=0}^{p^N-1} (x_1 + x_2 + \cdots + x_k + x)^n q^{x_1+x_2+\cdots+x_k} ξ^{x_1+\cdots+x_k}$$

$$= \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \lim_{N \to \infty} \frac{1}{[p^N]_q^{k_+}} \sum_{x_1,\ldots,x_k=0}^{p^N-1} q^{l_1} q^{x_1+\cdots+x_k} ξ^{x_1+\cdots+x_k}$$

$$= \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{(l + 1)^k}{[l + 1]_q^k}. \quad (2.4)$$

In the special case $x = 0$ in (2.4), $β_{n,q,ξ}^{(k)}(0) = β_{n,ξ}^{(k)}$ are called the twisted $q$-Bernoulli numbers of order $k$.

If we take $k = 1$ and $ξ = 1$ in (2.4), we can easily see that

$$β_{n,q}(x) = \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l + 1}{[l + 1]_q} \quad (2.5)$$

compare with [4].

**Theorem 2.1.** For $k \in \mathbb{Z}_+$ and $ξ \in T_p$, we have

$$β_{n,q,ξ}^{(k)}(x) = \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{(l + 1)^k}{[l + 1]_q^k}. \quad (2.6)$$
Moreover, if we take $x = 0$ in Theorem 2.1, then we have the following identity for the twisted $q$-Bernoulli numbers

$$
\beta_{n,q,\zeta}^{(k)} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{(l+1)^k}{[l+1]_{q,\zeta}^k}.
$$

(2.7)

From the definition of multivariate twisted $p$-adic $q$-integral, we also see that

$$
\beta_{n,q,\zeta}^{(k)}(x) = \int \cdots \int_{\mathbb{Z}_p} [x_1 + x_2 + \cdots + x_k + x]^n q^x d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k)
$$

$$
= \sum_{l=0}^{n} \binom{n}{l} q^x [q^{-y}]_q^{n-l} \int \cdots \int_{\mathbb{Z}_p} [x_1 + x_2 + \cdots + x_k]^l q^x d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k)
$$

$$
= \sum_{l=0}^{n} \binom{n}{l} q^x [q^{-y}]_q^{n-l} \beta_{l,q,\zeta}^{(k)}.
$$

Corollary 2.2. For $k \in \mathbb{Z}_+$ and $\zeta \in T_p$, one obtains

$$
\beta_{n,q,\zeta}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^x [q^{-y}]_q^{n-l} \beta_{l,q,\zeta}^{(k)}.
$$

(2.9)

Note that

$$
q^{n(x_1 + \cdots + x_k)} = \sum_{l=0}^{n} \binom{n}{l} (q-1)^l [x_1 + \cdots + x_k]^l.
$$

(2.10)

We have

$$
\int \cdots \int_{\mathbb{Z}_p} q^{n(x_1 + \cdots + x_k)} d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k) = \sum_{l=0}^{n} \binom{n}{l} (q-1)^l \beta_{l,q,\zeta}^{(k)}.
$$

(2.11)

It is easy to see that

$$
\int \cdots \int_{\mathbb{Z}_p} q^{n(x_1 + \cdots + x_k)} d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k)
$$

$$
= \lim_{N \to \infty} \frac{1}{[p^N]_{q,\zeta}^k} \sum_{x_1+\cdots+x_k=0} q^{n(x_1 + \cdots + x_k)} q^{x_1 + \cdots + x_k} s^{x_1 + \cdots + x_k} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}.
$$

(2.12)

By (2.11) and (2.12), we obtain the following theorem.
Theorem 2.3. For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has
\[
\sum_{l=0}^{n} \binom{n}{l} (q-1)^l \mu_{l,q,\zeta}^{(k)} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}.
\] (2.13)

Now we consider the modified extension of the twisted $q$-Bernoulli polynomials of order $k$ as follows:
\[
B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{ix} \int_{\mathbb{Z}_p} q^{x_i (k-l+1)x_1 \mu_{q,\zeta}(x_1) \cdots \mu_{q,\zeta}(x_k)}. (2.14)
\]

In the special case $x = 0$, we write $B_{n,q,\zeta}^{(k)} = B_{n,q,\zeta}^{(k)}(0)$, which are called the modified extension of the twisted $q$-Bernoulli numbers of order $k$.

From (2.14), we derive that
\[
B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (i+k) \cdots (i+1) q^{ix} [i+k]_{q,\zeta} \cdots [i+1]_{q,\zeta}.
\] (2.15)

Therefore, we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has
\[
B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(i+k)_{q,\zeta}^k}{i+k} q^{ix}.
\] (2.16)

Now, we define $B_{n,q,\zeta}^{(-k)}(x)$ as follows:
\[
B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{ix} \int_{\mathbb{Z}_p} q^{x_i (k-l+1)x_1 \mu_{q,\zeta}(x_1) \cdots \mu_{q,\zeta}(x_k)}. (2.17)
\]

By (2.17), we can see that
\[
B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(i+k)_{q,\zeta}^k}{i+k} q^{ix}.
\] (2.18)

Therefore, we obtain the following theorem.
Theorem 2.5. For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{i+k}{k}_{q,\zeta} \frac{\binom{n+k}{k}_{q,\zeta}!}{(n+k)!} q^{ix}. \quad (2.19)$$

In (2.19), we can see the relations between the binomial coefficients and the modified extension of the twisted $q$-Bernoulli polynomials of order $k$.

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References


