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Research Article

Some Identities on the Generalized q-Bernoulli Numbers and Polynomials Associated with q-Volkenborn Integrals

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We give some interesting equation of p-adic q-integrals on $\mathbb{Z}_p$. From those p-adic q-integrals, we present a systemic study of some families of extended Carlitz type q-Bernoulli numbers and polynomials in p-adic number field.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integer, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume that $|q| < 1$, and if $q \in \mathbb{C}_p$, we normally assume that $|1-q|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.1}$$

The q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \tag{1.2}$$

and the Gaussian q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}, \tag{1.3}$$

(see [1]). Note that

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.4}$$

From (1.3), we easily see that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n+1-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \tag{1.5}$$

(see [2, 3]). For a fixed positive integer f, $(f,p)=1$, let

$$\begin{gathered} X = X_f = \varprojlim_N \left(\frac{\mathbb{Z}}{fp^N\mathbb{Z}}\right), \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} (a + fp\mathbb{Z}_p), \quad a + fp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\operatorname{mod} fp^N\right)\right\}, \end{gathered} \tag{1.6}$$

where $a \in \mathbb{Z}$ and $0 \le a < fp^N$ (see [1–14]).

We say that f is a uniformly differential function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y} \tag{1.7}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x = \sum_{0\le x<p^N} f(x)\mu_q\left(x + p^N\mathbb{Z}_p\right), \tag{1.8}$$

representing a q-analogue of the Riemann sums for f, (see [1–3, 11–18]). The integral of f

on $\mathbb{Z}_p$ is defined as the limit ($N \longrightarrow \infty$) of the sums (if exists). The p-adic q-integral (= q-Volkenborn integral) of $f \in UD(\mathbb{Z}_p))$ is defined by

$$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\longrightarrow\infty} \frac{1}{[p^N]_q} \sum_{0\leq x<p^N} f(x)q^x, \tag{1.9}$$

(see [12]). Carlitz's q-Bernoull numbers $\beta_{k,q}$ can be defined recursively by $\beta_{0,q} = 1$ and by the rule that

$$q(q\beta^* + 1)^k - \beta^*_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.10}$$

with the usual convention of replacing $(\beta^*)^i$ by $\beta^*_{i,q}$, (see [1–13]).

It is well known that

$$\begin{aligned} \beta^*_{n,q} &= \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x), \quad n \in \mathbb{Z}_+, \\ \beta^*_{n,q}(x) &= \int_{\mathbb{Z}_p} [y+x]_q^n d\mu_q(y) = \int_X [y+x]_q^n d\mu_q(y), \quad n \in \mathbb{Z}_+, \end{aligned} \tag{1.11}$$

(see [1]), where $\beta^*_{n,q}(x)$ are called the nth Carlitz's q-Bernoulli polynomials (see [1, 12, 13]).

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$, then the generalized Carlitz's q-Bernoulli numbers attached to χ are defined as follows:

$$\beta^*_{n,\chi,q} = \int_X \chi(x)[x]_q^n d\mu_q(x), \tag{1.12}$$

(see [13]). Recently, many authors have studied in the different several areas related to q-theory (see [1–13]). In this paper, we present a systemic study of some families of multiple Carlitz's type q-Bernoulli numbers and polynomials by using the integral equations of p-adic q-integrals on $\mathbb{Z}_p$. First, we derive some interesting equations of p-adic q-integrals on $\mathbb{Z}_p$. From these equations, we give some interesting formulae for the higher-order Carlitz's type q-Bernoulli numbers and polynomials in the p-adic number field.

2. On the Generalized Higher-Order q-Bernoulli Numbers and Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We first consider the q-extension of Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} d\mu_q(y) = -t\sum_{m=0}^{\infty} e^{[x+m]_q t} q^{x+m}. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned}
\beta_{n,q}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\frac{l}{[l]_q} \\
&= \frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\left(\frac{l}{1-q^l}\right) \\
&= \frac{n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}q^{(l+1)x}\left(\frac{1}{1-q^{l+1}}\right)(-1)^{l+1} \\
&= \frac{-n}{(1-q)^{n-1}}\sum_{m=0}^{\infty}q^{m+x}\sum_{l=0}^{n-1}\binom{n-1}{l}q^{l(x+m)} \\
&= -n\sum_{m=0}^{\infty}q^{m+x}[x+m]_q^{n-1}.
\end{aligned} \tag{2.2}$$

Note that

$$\lim_{q\to 1}\beta_{n,q}(x) = -n\sum_{m=0}^{\infty}(x+m)^{n-1} = B_n(x), \tag{2.3}$$

where $B_n(x)$ are called the n th ordinary Bernoulli polynomials. In the special case, $x = 0$, $\beta_{n,q}(0) = \beta_{n,q}$ are called the n th q-Bernoulli numbers.

By (2.2), we have the following lemma.

Lemma 2.1. *For $n \geq 0$, one has*

$$\begin{aligned}
\beta_{n,q}(x) &= \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n d\mu_q(y) = -n\sum_{m=0}^{\infty}q^{m+x}[x+m]_q^{n-1} \\
&= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\frac{l}{[l]_q}.
\end{aligned} \tag{2.4}$$

Now, one considers the q-Bernoulli polynomials of order $r \in \mathbb{N}$ as follows:

$$\sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(x)\frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ \text{times}} q^{-(x_1+\cdots+x_r)}e^{[x+x_1+\cdots+x_r]_q t}d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{2.5}$$

By (2.5), *one sees that*

$$\beta_{n,q}^{(r)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{-(x_1+\cdots+x_r)} [x + x_1 + \cdots + x_r]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r)$$
$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \left(\frac{l}{[l]_q} \right)^r. \tag{2.6}$$

In the special case, $x = 0$*, the sequence* $\beta_{n,q}^{(r)}(0) = \beta_{n,q}^{(r)}$ *is refereed to as the* q*-extension of Bernoulli numbers of order* r*. For* $f \in \mathbb{N}$*, one has*

$$\beta_{n,q}^{(r)}(x) = \underbrace{\int_{X} \cdots \int_{X}}_{r \text{ times}} q^{-(x_1+\cdots+x_r)} [x + x_1 + \cdots + x_r]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r)$$
$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{a_1,\ldots,a_r=0}^{f-1} q^{l(x+a_1+\cdots+a_r)} \frac{l^r}{[lf]_q^r} \tag{2.7}$$
$$= [f]_q^{n-r} \sum_{a_1,\ldots,a_r=0}^{f-1} \beta_{n,q^f}^{(r)} \left(\frac{a_1 + \cdots + a_r + x}{f} \right).$$

By (2.5) and (2.7), one obtains the following theorem.

Theorem 2.2. *For* $r \in \mathbb{Z}_+$*,* $f \in \mathbb{N}$*, one has*

$$\beta_{n,q}^{(r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \sum_{a_1,\ldots,a_r=0}^{f-1} \binom{n}{l} (-1)^l q^{l(a_1+\cdots+a_r+x)} \frac{l^r}{[lf]_q^r}$$
$$= [f]_q^{n-r} \sum_{a_1,\ldots,a_r=0}^{f-1} \beta_{n,q^f}^{(r)} \left(\frac{a_1 + \cdots + a_r + x}{f} \right). \tag{2.8}$$

Let χ *be the primitive Dirichlet's character with conductor* $f \in \mathbb{N}$*, then the generalized* q*-Bernoulli polynomials attached to* χ *are defined by*

$$\sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!} = \int_X \chi(y) q^{-y} e^{[x+y]_q t} d\mu_q(y). \tag{2.9}$$

From (2.9)*, one derives*

$$\begin{aligned}\beta_{n,\chi,q}(x) &= \int_X \chi(y) q^{-y}[x+y]_q^n d\mu_q(y)\\ &= \sum_{a=0}^{f-1}\chi(a)\lim_{N\longrightarrow\infty}\frac{1}{[fp^N]_q}\sum_{y=0}^{fp^N-1}[a+x+fy]_q^n\\ &= \frac{1}{(1-q)^n}\sum_{a=0}^{f-1}\chi(a)\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(x+a)}\frac{l}{[lf]_q}\\ &= \sum_{a=0}^{f-1}\chi(a)\sum_{m=0}^{\infty}\left(-n[x+a+mf]_q^{n-1}\right)\\ &= -n\sum_{m=0}^{\infty}\chi(m)[x+m]_q^{n-1}.\end{aligned} \tag{2.10}$$

By (2.9) *and* (2.10)*, one can give the generating function for the generalized q-Bernoulli polynomials attached to χ as follows:*

$$F_{\chi,q}(x,t) = -t\sum_{m=0}^{\infty}\chi(m)e^{[x+m]_q t} = \sum_{n=0}^{\infty}\beta_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.11}$$

From (1.3)*,* (2.10)*, and* (2.11)*, one notes that*

$$\begin{aligned}\beta_{n,\chi,q}(x) &= \frac{1}{[f]_q}\sum_{a=0}^{f-1}\chi(a)\int_{\mathbb{Z}_p} q^{-fy}[a+x+fy]_q^n d\mu_{q^f}(y)\\ &= [f]_q^{n-1}\sum_{a=0}^{f-1}\chi(a)\beta_{n,q^f}\left(\frac{a+x}{f}\right).\end{aligned} \tag{2.12}$$

In the special case, $x=0$, the sequence $\beta_{n,\chi,q}(0)=\beta_{n,\chi,q}$ are called the n th generalized q-Bernoulli numbers attached to χ.

Let one consider the higher-order q-Bernoulli polynomials attached to χ as follows:

$$\underbrace{\int_X\cdots\int_X}_{r\ \text{times}}\left(\prod_{i=1}^{r}\chi(x_i)\right)e^{[x+x_1+\cdots+x_r]_q t}q^{-(x_1+\cdots+x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r) = \sum_{n=0}^{\infty}\beta_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}, \tag{2.13}$$

where $\beta_{n,\chi,q}^{(r)}(x)$ are called the n th generalized q-Bernoulli polynomials of order r attaches to χ.

By (2.13), *one sees that*

$$\begin{aligned}\beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l \sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)q^{l\sum_{i=1}^{r}a_i}\frac{l^r}{[lf]_q^r}\\ &= [f]_q^{n-r}\sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)\beta_{n,q^f}^{(r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).\end{aligned}\tag{2.14}$$

In the special case, $x = 0$, the sequence $\beta_{n,\chi,q}^{(r)}(0) = \beta_{n,\chi,q}^{(r)}$ are called the n th generalized q-Bernoulli numbers of order r attaches to χ.

By (2.13) and (2.14), one obtains the following theorem.

Theorem 2.3. *Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$. For $n \in \mathbb{Z}_+, r \in \mathbb{N}$, one has*

$$\begin{aligned}\beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l \sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)q^{l\sum_{i=1}^{r}a_i}\frac{l^r}{[lf]_q^r}\\ &= [f]_q^{n-r}\sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)\beta_{n,q^f}^{(r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).\end{aligned}\tag{2.15}$$

For $h \in \mathbb{Z}$, and $r \in \mathbb{N}$, one introduces the extended higher-order q-Bernoulli polynomials as follows:

$$\beta_{n,q}^{(h,r)}(x) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ \text{times}} q^{\sum_{j=1}^{r}(h-j-1)x_j}[x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r).\tag{2.16}$$

From (2.16), *one notes that*

$$\beta_{n,q}^{(h,r)}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l+h-1}{r}}{\binom{l+h-1}{r}_q}\frac{r!}{[r]_q!},\tag{2.17}$$

and

$$\beta_{n,q}^{(h,r)}(x) = [f]_q^{n-r}\sum_{a_1,\ldots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j}\beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).\tag{2.18}$$

In the special case, $x = 0$, $\beta_{n,q}^{(h,r)}(0) = \beta_{n,q}^{(h,r)}$ are called the n th (h,q)-Bernoulli numbers of order r.

By (2.17), one obtains the following theorem.

Theorem 2.4. *For $h \in \mathbb{Z}, r \in \mathbb{N}$, one has*

$$\beta_{n,q}^{(h,r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \frac{\binom{l+h-1}{r}}{\binom{l+h-1}{r}_q} \frac{r!}{[r]_q!},$$
$$\beta_{n,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1,\ldots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right). \tag{2.19}$$

Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$, then one considers the generalized (h,q)-Bernoulli polynomials attached to χ of order r as follows:

$$\beta_{n,\chi,q}^{(h,r)}(x) = \underbrace{\int_X \cdots \int_X}_{r\ \text{times}} q^{\sum_{j=1}^{r}(h-j-1)x_j} \left(\prod_{j=1}^{r} \chi(x_j)\right) [x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{2.20}$$

By (2.20)*, one sees that*

$$\beta_{n,\chi,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1,\ldots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \left(\prod_{j=1}^{r} \chi(a_j)\right) \beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right). \tag{2.21}$$

In the special case, $x = 0$, $\beta_{n,\chi,q}^{(h,r)}(0) = \beta_{n,\chi,q}^{(h,r)}$ are called the n th generalized (h,q)-Bernoulli numbers attached to χ of order r.

From (2.20) *and* (2.21)*, one notes that*

$$\beta_{n,\chi,q}^{(h,r)} = (q-1)\beta_{n+1,\chi,q}^{(h-1,r)} + \beta_{n,\chi,q}^{(h-1,r)}. \tag{2.22}$$

By (2.16)*, it is easy to show that*

$$\begin{aligned}\beta_{n,\chi,q}^{(h,r)} &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\ \text{times}} [x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j-1)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1+\cdots+x_r]_q^n \left\{[x_1+\cdots+x_r]_q(q-1)+1\right\} q^{\sum_{j=1}^{r}(h-j-2)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r).\end{aligned} \tag{2.23}$$

Thus, one has

$$\beta_{n,q}^{(h,r)} = (q-1)\beta_{n+1,q}^{(h-1,r)} + \beta_{n,q}^{(h-1,r)}. \tag{2.24}$$

From (2.16) *and* (2.23), *one can also derive*

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}q^{n(x_1+\cdots+x_r)}q^{-x_1-2x_2-\cdots-rx_r}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]_q^l q^{-(x_1+\cdots+x_r)}q^{-x_1-2x_2-\cdots-rx_r}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)},\\
&\qquad\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r}\,d\mu_q(x_1)\cdots d\mu_q(x_r) = \frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}.
\end{aligned} \tag{2.25}$$

It is easy to see that

$$\begin{aligned}
\sum_{j=0}^{n}\binom{n}{j}(q-1)^j\int_{\mathbb{Z}_p}[x]_q^j q^{(h-2)x}\,d\mu_q(x) &= \int_{\mathbb{Z}_p}\left((q-1)[x]_q+1\right)^n q^{(h-2)x}\,d\mu_q(x)\\
&= \frac{n+h-1}{[n+h-1]_q}.
\end{aligned} \tag{2.26}$$

By (2.23), (2.25), and (2.26), one obtains the following theorem.

Theorem 2.5. *For $h \in \mathbb{Z}, r \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
\beta_{n,q}^{(h,r)} &= (q-1)\beta_{n+1,q}^{(h-1,r)} + \beta_{n,q}^{(h-1,r)},\\
\sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)} &= \frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}.
\end{aligned} \tag{2.27}$$

Furthermore, one gets

$$\sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(h,1)} = \frac{n+h-1}{[n+h-1]_q}. \tag{2.28}$$

Now, one considers the polynomials of $\beta_{n,q}^{(0,r)}(x)$ *by*

$$\begin{aligned}\beta_{n,q}^{(0,r)}(x) &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ \text{times}}[x+x_1+\cdots+x_r]_q^n q^{-2x_1-3x_2-\cdots-(r-1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l-1}{r}}{\binom{l-1}{r}_q}\frac{r!}{[r]_q!}.\end{aligned} \tag{2.29}$$

By (2.29), one obtains the following theorem.

Theorem 2.6. *For* $r\in\mathbb{N}$ *and* $n\in\mathbb{Z}_+$, *one has*

$$(1-q)^n\beta_{n,q}^{(0,r)}(x) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l-1}{r}}{\binom{l-1}{r}_q}\frac{r!}{[r]_q!}. \tag{2.30}$$

By using multivariate p-adic q-integral on $\mathbb{Z}_p$, *one sees that*

$$\begin{aligned}&q^{nx}\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}\\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ \text{times}} q^{nx+(n-2)x_1+\cdots+(n-r-1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left((q-1)[x+x_1+\cdots+x_r]_q+1\right)^n q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)}(x).\end{aligned} \tag{2.31}$$

Therefore, one obtains the following corollary.

Corollary 2.7. *For* $r\in\mathbb{N}$ *and* $n\in\mathbb{Z}_+$, *one has*

$$q^{nx}\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!} = \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)}(x). \tag{2.32}$$

It is easy to show that

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ times}[x+x_1+\cdots+x_r]_q^n q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=[f]_q^{n-r}\sum_{i_1,\ldots,i_r=0}^{f-1}q^{-\sum_{l=1}^r l i_l}\\
&\qquad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{-f\sum_{l=1}^r(l+1)x_l}\left[\frac{x+\sum_{l=1}^r i_l}{f}+\sum_{l=1}^r x_l\right]_{q^f}^n d\mu_{q^f}(x_1)\cdots d\mu_{q^f}(x_r).
\end{aligned}\tag{2.33}$$

From (2.33), *one notes that*

$$\beta_{n,q}^{(0,r)}(x)=[f]_q^{n-r}\sum_{i_1,\ldots,i_r=0}^{f-1}q^{-i_1-2i_2-\ldots-ri_r}\beta_{n,q^f}^{(0,r)}\left(\frac{x+i_1+\cdots+i_r}{f}\right).\tag{2.34}$$

From the multivariate p-adic q-integral on $\mathbb{Z}_p$, one has

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ times}[x+x_1+\cdots+x_r]_q^n q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([x]_q+q^x[x_1+\cdots+x_r]_q\right)^n q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^n\binom{n}{l}[x]_q^{n-l}q^{lx}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]_q^l q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r),
\end{aligned}\tag{2.35}$$

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\ times}[x+y+x_1+\cdots+x_r]_q^n q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^n\binom{n}{l}[y]_q^{n-l}q^{ly}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r).
\end{aligned}\tag{2.36}$$

By (2.35) and (2.36), one obtains the following corollary.

Corollary 2.8. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} \beta_{n,q}^{(0,r)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}^{(0,r)}, \\ \beta_{n,q}^{(0,r)}(x+y) &= \sum_{l=0}^{n} \binom{n}{l} [y]_q^{n-l} q^{ly} \beta_{l,q}^{(0,r)}(x). \end{aligned} \tag{2.37}$$

Now, one also considers the polynomial of $\beta_{n,q}^{(h,1)}(x)$. From the integral equation on $\mathbb{Z}_p$, one notes that

$$\begin{aligned} \beta_{n,q}^{(h,1)}(x) &= \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-2)} d\mu_q(x_1) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+h-1}{[l+h-1]_q}. \end{aligned} \tag{2.38}$$

By (2.38)*, one easily gets*

$$\begin{aligned} \beta_{n,q}^{(h,1)}(x) &= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx} l}{1-q^{l+h-1}} + \frac{h-1}{(1-q)^{n-1}} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{1-q^{l+h-1}} \\ &= \frac{-n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \frac{\binom{n-1}{l}(-1)^l q^{x} q^{lx}}{1-q^{l+h}} + \frac{h-1}{(1-q)^{n-1}} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{1-q^{l+h-1}} \\ &= -n \sum_{m=0}^{\infty} q^{hm+x} [x+m]_q^{n-1} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_q^n. \end{aligned} \tag{2.39}$$

Thus, one obtains the following theorem.

Theorem 2.9. *For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q}^{(h,1)}(x) = -n \sum_{m=0}^{\infty} q^{hm+x} [x+m]_q^{n-1} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_q^n. \tag{2.40}$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, one notes that

$$\begin{aligned} &\int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) \\ &\quad = \frac{1}{[f]_q} \sum_{i=0}^{f-1} q^{(h-1)i} [i]_q^n \int_{\mathbb{Z}_p} \left[\frac{x+i}{f} + x_1\right]_{q^f}^n q^{f(h-2)x_1} d\mu_{q^f}(x_1). \end{aligned} \tag{2.41}$$

Thus, one has

$$\beta_{n,q}^{(h,1)}(x) = \frac{1}{[f]_q}\sum_{i=0}^{f-1} q^{(h-1)i}[i]_q^n \beta_{n,q^f}^{(h,1)}\left(\frac{x+i}{f}\right). \tag{2.42}$$

By (2.38), *one easily gets*

$$\begin{aligned}&\int_{\mathbb{Z}_p}[x+x_1]_q^n q^{x_1(h-2)}d\mu_q(x_1)\\ &\quad= q^{-x}\int_{\mathbb{Z}_p}[x+x_1]_q^n\left\{[x+x_1]_q(q-1)+1\right\}q^{x_1(h-3)}d\mu_q(x_1).\end{aligned} \tag{2.43}$$

From (2.43), *one has*

$$\beta_{n,q}^{(h,1)}(x) = q^{-x}\left((q-1)\beta_{n+1,q}^{(h-1,1)}(x)+\beta_{n,q}^{(h-1,1)}(x)\right). \tag{2.44}$$

That is,

$$q^x\beta_{n,q}^{(h,1)}(x) = (q-1)\beta_{n+1,q}^{(h-1,1)}(x)+\beta_{n,q}^{(h-1,1)}(x). \tag{2.45}$$

By (2.38) *and* (2.43), *one easily sees that*

$$\begin{aligned}\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x+x_1]_q^n d\mu_q(x_1) &= \int_{\mathbb{Z}_p}q^{(h-2)x_1}\left([x]_q+q^x[x_1]_q\right)^n d\mu_q(x_1)\\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x_1]_q^l d\mu_q(x_1),\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}&q^{h-1}\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x_1+1+x]_q^n d\mu_q(x_1)-\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x+x_1]_q^n d\mu_q(x_1)\\ &\quad= q^x n[x]_q^{n-1}+h(q-1)[x]_q^n-(q-1)[x]_q^n.\end{aligned} \tag{2.47}$$

For $x = 0$, *this gives*

$$q^{h-1}\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x_1+1]_q^n d\mu_q(x_1)-\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x_1]_q^n d\mu_q(x_1) = \begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{2.48}$$

and

$$\beta_{0,q}^{(h,1)} = \int_{\mathbb{Z}_p}q^{(h-2)x_1}d\mu_q(x_1) = \frac{h-1}{[h-1]_q}. \tag{2.49}$$

From (2.46) *and* (2.48), *one can derive the recurrence relation for* $\beta_{n,q}^{(h,1)}$ *as follows:*

$$q^{h-1}\beta_{n,q}^{(h,1)}(1)-\beta_{n,q}^{(h,1)}=\delta_{n,1}, \tag{2.50}$$

where $\delta_{n,1}$ *is kronecker symbol.*

By (2.46), (2.48), and (2.50), one obtains the following theorem.

Theorem 2.10. *For* $h \in \mathbb{Z}$ *and* $n \in \mathbb{Z}_+$, *one has*

$$\begin{gathered}\beta_{n,q}^{(h,1)}(x)=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,q}^{(h,1)},\\ q^{h-1}\beta_{n,q}^{(h,1)}(x+1)-\beta_{n,q}^{(h,1)}=q^{x}n[x]_q^{n-1}+h(q-1)[x]_q^{n}-(q-1)[x]_q^{n}.\end{gathered} \tag{2.51}$$

Furthermore,

$$q^{h-2}(q-1)\beta_{n+1,q}^{(h-1,1)}(1)+q^{h-2}\beta_{n,q}^{(h-1,1)}(1)-\beta_{n,q}^{(h,1)}=\delta_{n,1}, \tag{2.52}$$

where $\delta_{n,1}$ *is kronecker symbol.*

From the definition of p*-adic* q*-integral on* $\mathbb{Z}_p$, *one notes that*

$$\begin{aligned}&\int_{\mathbb{Z}_p}q^{-(h-2)x_1}[1-x+x_1]_{q^{-1}}^{n}d\mu_{q^{-1}}(x_1)\\ &\quad=(-1)^n q^{n+h-2}\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x+x_1]_q^{n}d\mu_q(x_1).\end{aligned} \tag{2.53}$$

By (2.53), *one sees that*

$$\beta_{n,q^{-1}}^{(h,1)}(1-x)=(-1)^n q^{n+h-2}\beta_{n,q}^{(h,1)}(x). \tag{2.54}$$

Note that

$$B_n(1-x)=\lim_{q\longrightarrow1}\beta_{n,q^{-1}}^{(h,1)}(1-x)=\lim_{q\longrightarrow1}(-1)^n q^{n+h-2}\beta_{n,q}^{(h,1)}(x)=(-1)^n B_n(x), \tag{2.55}$$

where $B_n(x)$ *are the* n *th ordinary Bernoulli polynomials.*

In the special case, $x=1$, *one gets*

$$\beta_{n,q^{-1}}^{(h,1)}=(-1)^n q^{n+h-2}\beta_{n,q}^{(h,1)}(1)=(-1)^n q^{n-1}\beta_{n,q}^{(h,1)},\quad \textit{if } n>1. \tag{2.56}$$

It is not difficult to show that

$$\begin{aligned}
&[f]_q^{n-1}\sum_{l=0}^{f-1}q^{l(h-1)}\int_{\mathbb{Z}_p}\left[x+\frac{l}{f}+x_1\right]_{q^f}^n q^{f(h-2)x_1}d\mu_{q^f}(x_1)\\
&\quad=\int_{\mathbb{Z}_p}[fx+x_1]_q^n q^{(h-2)x_1}d\mu_q(x_1),\quad f\in\mathbb{N}.
\end{aligned}\tag{2.57}$$

That is,

$$[f]_q^{n-1}\sum_{l=0}^{f-1}q^{l(h-1)}\beta_{n,q^f}^{(h,1)}\left(x+\frac{l}{f}\right)=\beta_{n,q}^{(h,1)}(fx).\tag{2.58}$$

Let one consider Barnes' type multiple q-Bernoulli polynomials. For $w_1,w_2,\ldots,w_r\in\mathbb{Z}_p$, and $\delta_1,\delta_2,\ldots,\delta_r\in\mathbb{Z}$, one defines Barnes' type multiple q-Bernoulli polynomials as follows:

$$\begin{aligned}
&\beta_{n,q}^{(r)}(x\mid w_1,\ldots,w_r:\delta_1,\ldots,\delta_r)\\
&\quad=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[w_1x_1+\cdots+w_rx_r+x]_q^n q^{\sum_{j=1}^r(\delta_j-1)x_j}d\mu_q(x_1)\cdots d\mu_q(x_r).
\end{aligned}\tag{2.59}$$

From (2.59)*, one can easily derive the following equation:*

$$\begin{aligned}
&\beta_{n,q}^{(r)}(x\mid w_1,\ldots,w_r:\delta_1,\ldots,\delta_r)\\
&\quad=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{(lw_1+\delta_1)(lw_2+\delta_2)\cdots(lw_r+\delta_r)}{[lw_1+\delta_1]_q[lw_2+\delta_2]_q\cdots[lw_r+\delta_r]_q}.
\end{aligned}\tag{2.60}$$

Let $\delta_r=\delta_1+r-1$, then one has

$$\beta_{n,q}^{(r)}\left(x\mid\underbrace{w_1\cdots w_1}_{r\text{ times}}:\delta_1,\delta_1+1\ldots,\delta_1+r-1\right)=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{lw_1+\delta_1+r-1}{r}}{\binom{lw_1+\delta_1+r-1}{r}_q}\frac{r!}{[r]_q!}.\tag{2.61}$$

Therefore, one obtains the following theorem.

Theorem 2.11. *For $w_1 \in \mathbb{Z}_p, r \in \mathbb{N}$, and $\delta_1 \in \mathbb{Z}$, one has*

$$\beta_{n,q}^{(r)}\left(x \mid \underbrace{w_1 \cdots w_1}_{r\ times} : \delta_1, \delta_1 + 1 \ldots, \delta_1 + r - 1 \right) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{\binom{lw_1+\delta_1+r-1}{r}}{\binom{lw_1+\delta_1+r-1}{r}_q} \frac{r!}{[r]_q!}. \tag{2.62}$$

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