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Research Article

On Several Matrix Kantorovich-Type Inequalities

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We present several matrix Kantorovich-type inequalities, which improve the results obtained in Liu and Neudecker (1996). Elementary methods suffice to prove the inequalities.

1. Introduction

Let $A \in M_n$ be a positive (semi-)definite Hermite matrix with eigenvalues contained in the interval $[m, M]$, where $0 < m < M$. Let V be $n \times r$ matrix, and let $\Re(A)$ denotes the column space of A.

A well-know matrix version of Kantorovich inequality asserts that (see[1–3])

$$V^* A^2 V \leq \frac{(m+M)^2}{4mM} (V^* A V)^2, \tag{1.1}$$

for $A > 0$ and $V^* V = I$, where V^* denotes the conjugate transpose of the matrix V.

Let B be an m-by-n matrix; the Moore-Penrose inverse B^+ of B is defined as the unique n-by-m matrix satisfying all of the following four criteria (see, e.g., [4]):

$$BB^+ B = B, \quad B^+ BB^+ = B^+, \quad (BB^+)^* = BB^+, \quad (B^+ B)^* = B^+ B. \tag{1.2}$$

It is not difficult to see that if $V^* V = I$, then $VV^* = VV^+ \leq I$; we can get $V^* AAV \geq V^* AVV^* AV$; thus, $V^* A^2 V - (V^* AV)^2 \geq 0$, for $A > 0$.

In paper [5], from $A^2 \le (m+M)A - mMI$ (which is equivalent to (13) in [6]), Liu and Neudecker presented the following so-called Kantorovich-type inequality:

$$V^*A^2V - (V^*AV)^2 \le \frac{(M-m)^2}{4} I \tag{1.3}$$

for $A > 0$ and $V^*V = I$, and the following inequality:

$$(V^*A^2V)^{1/2} \le \frac{(m+M)}{2\sqrt{mM}}(V^*AV) \tag{1.4}$$

for $A > 0$ and $V^*V = I$. Furthermore, in the same way, they obtained three more general versions.

$$VV^+A^2VV^+ - (VV^+AVV^+)^2 \le \frac{1}{4}(M-m)^2 VV^+, \tag{1.5}$$

$$V^*A^2V - V^*AVV^+AV \le \frac{1}{4}(M-m)^2 V^*V, \tag{1.6}$$

$$V^+A^2V^{+*} - V^+AVV^+AV^{+*} \le \frac{1}{4}(M-m)^2 VV^+ \tag{1.7}$$

for $A > 0$ and $V \in \Re(A)$.

In the next section, we shall present several similar matrix Kantorovich-type inequalities, which improve some results above.

2. New Matrix Kantorovich-Type Inequalities

We first introduce two lemmas.

Lemma 2.1. $0 \le (MI - V^*AV)(V^*AV - mI) \le (1/4)(M-m)^2 I$, *for* $A > 0$ *and* $V^*V = I$.

Proof. It is easy to see that if $mI \le A \le MI$, then $mI \le V^*AV \le MI$; thus, we have

$$\begin{aligned} 0 &\le (MI - V^*AV)(V^*AV - mI) \\ &= (m+M)V^*AV - mMI - (V^*AV)^2 \\ &= \frac{1}{4}(M-m)^2 I - \left[V^*AV - \frac{1}{2}(m+M)I\right]^2 \le \frac{1}{4}(M-m)^2 I, \end{aligned} \tag{2.1}$$

for $V^*V = I$. □

In [7], Dragomir defines a transform $C_{m,M}(A) = (A - mI)(MI - A)$; for this transform, we have the following lemma.

Lemma 2.2. *Let* $C(A,V) = V^*(A - mI)(MI - A)V$; *then*

$$C(A,V) = \frac{1}{4}(M-m)^2 I - V^*\left(A - \frac{1}{2}(m+M)I\right)^2 V; \tag{2.2}$$

thus

$$0 \leq C(A,V) \leq \frac{1}{4}(M-m)^2 I \tag{2.3}$$

for $A > 0$ *and* $V^*V = I$.

Proof.

$$\begin{aligned} C(A,V) &= V^*(A - mI)(MI - A)V \\ &= V^*\left(\frac{M-m}{2}I + \left(A - \frac{M+m}{2}I\right)\right)\left(\frac{M-m}{2}I - \left(A - \frac{M+m}{2}I\right)\right)V \\ &= \frac{1}{4}(M-m)^2 I - V^*\left(A - \frac{1}{2}(m+M)I\right)^2 V \\ &\leq \frac{1}{4}(M-m)^2\ I, \quad \text{for } V^*V = I. \end{aligned} \tag{2.4}$$

□

From Lemma 2.2, we can easily get the inequality (1.4).

Corollary 2.3. $(V^*A^2V)^{1/2} \leq (m+M)/(2\sqrt{mM})V^*AV$, *for* $A > 0$ *and* $V^*V = I$.

Proof. From $C(A,V) \geq 0$, we have

$$(m+M)V^*AV - V^*A^2V - mMI \geq 0; \tag{2.5}$$

then

$$(m+M)V^*AV \geq V^*A^2V + mMI \geq 2\sqrt{mM}(V^*A^2V)^{1/2}. \tag{2.6}$$

The proof is completed. □

Theorem 2.4. *$V^*A^2V-(V^*AV)^2 \le (1/4)(M-m)^2I-C(A,V)$ for $A>0$ and $V^*V=I$.*

Proof.

$$\begin{aligned}
&V^*A^2V-(V^*AV)^2\\
&\quad= V^*A^2V+mMI-(m+M)V^*AV-\left[(V^*AV)^2+mMI-(m+M)V^*AV\right] \\
&\quad= (MI-V^*AV)(V^*AV-mI)-V^*(A-mI)(MI-A)V.
\end{aligned} \tag{2.7}$$

From Lemmas 2.1 and 2.2, we have

$$V^*A^2V-(V^*AV)^2 \le \frac{1}{4}(M-m)^2I-C(A,V). \tag{2.8}$$

The proof of Theorem 2.4 is completed. □

Remark 2.5. It is not difficult to see that if $V^*A^2V-(V^*AV)^2 \le (1/4)(M-m)^2I-C(A,V) \le (1/4)(M-m)^2I$, then we conclude that Theorem 2.4 gives an improvement of the Kantorovich inequality (1.3).

Furthermore, in similar way we got Theorem 2.4, and we obtain three more general versions, which also improve the inequalities (1.5), (1.6), (1.7), respectively.

Theorem 2.6.

$$VV^+A^2VV^+-(VV^+AVV^+)^2 \le \frac{1}{4}(M-m)^2VV^+-C(A,V,V^+), \tag{2.9}$$

$$V^*A^2V-V^*AVV^+AV \le \frac{1}{4}(M-m)^2V^*V-C(A,V^*,V), \tag{2.10}$$

$$V^+A^2V^{+*}-V^+AVV^+AV^{+*} \le \frac{1}{4}(M-m)^2VV^{+*}-C(A,V,V^{+*}) \tag{2.11}$$

for $A>0$ and $V \in \mathfrak{R}(A)$, where $C(A,V,U)=VU(A-mI)(MI-A)VU, U \in C^{r\times n}$.

Proof. In fact, they are equivalent by noting $V^*=V^*VV^+$ and $V^+=V^+V^{+*}V^*$. For (2.9), pre- and postmultiplying by V^* and V, respectively, we get the inequality (2.10); similarly, for (2.10), pre- and postmultiplying by V^+V^{+*}, respectively, we get the inequality (2.11). So, we only prove the inequality (2.9).

Similarly, with Lemma 2.2, we have

$$
\begin{aligned}
&0 \le C(A,V,V^+) = \frac{1}{4}(M-m)^2 VV^+ - VV^+\left(A - \frac{1}{2}(m+M)I\right)^2 VV^+ \le \frac{1}{4}(M-m)^2 VV^+, \\
&VV^+A^2VV^+ - (VV^+AVV^+)^2 \\
&\qquad = VV^+A^2VV^+ + mMVV^+ - (m+M)VV^+AVV^+ \\
&\qquad\quad - \left[(VV^+AVV^+)^2 + mMVV^+ - (m+M)VV^+AVV^+\right] \\
&\quad = (MVV^+ - VV^+AVV^+)(VV^+AVV^+ - mVV^+) - VV^+(A-mI)(MI-A)VV^+, \\
&\quad \le \frac{1}{4}(M-m)^2\ VV^+ - C(A,V,V^+).
\end{aligned}
\tag{2.12}
$$

□

Remark 2.7. From the proof, it is easy to see that $VV^+A^2VV^+ - (VV^+AVV^+)^2 \le (1/4)(M-m)^2VV^+ - C(A,V,V^+) \le (1/4)(M-m)^2VV^+$; so, we conclude that the inequality (2.9) gives an improvement of the inequality (1.5), meanwhile, the inequalities (2.10) and (2.11) improve the inequalities (1.6) and (1.7), respectively.

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