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Research Article

Alon-Babai-Suzuki's Conjecture Related to Binary Codes in Nonmodular Version

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Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be sets of nonnegative integers. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ with $[F_i] \in K$ for each i and $|F_i \cap F_j| \in L$ for any $i \neq j$. Every subset F_e of $[n]$ can be represented by a binary code $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that $a_i = 1$ if $i \in F_e$ and $a_i = 0$ if $i \notin F_e$. Alon et al. made a conjecture in 1991 in modular version. We prove Alon-Babai-Sukuki's Conjecture in nonmodular version. For any K and L with $n \geq s + \max k_i$, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

1. Introduction

In this paper, $\mathcal{F}$ stands for a family of subsets of $[n] = \{1, 2, \ldots, n\}$, $K = \{k_1, \ldots, k_r\}$, and $L = \{l_1, \ldots, l_s\}$, where $|F_i| \in K$ for all $F_i \in \mathcal{F}$, $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathcal{F}, i \neq j$. The variable x will stand as a shorthand for the n-dimensional vector variable $(x_1, x_2, \ldots, x_n)$. Also, since these variables will take the values only 0 and 1, all the polynomials we will work with will be reduced modulo the relation $x_i^2 = x_i$. We define the characteristic vector $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$ of F_i such that $v_{ij} = 1$ if $j \in F_i$ and $v_{ij} = 0$ if $j \notin F_i$. We will present some results in this paper that give upper bounds on the size of $\mathcal{F}$ under various conditions. Below is a list of related results by others.

Theorem 1.1 (Ray-Chaudhuri and Wilson [1]). *If $K = \{k\}$, and L is any set of nonnegative integers with $k > \max l_j$, then $|\mathcal{F}| \leq \binom{n}{s}$.*

Theorem 1.2 (Alon et al. [2]). *If K and L are two sets of nonnegative integers with* $k_i > s - r$, *for every* i, *then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Theorem 1.3 (Snevily [3]). *If K and L are any sets such that* $\min k_i > \max l_j$, *then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

Theorem 1.4 (Snevily [4]). *Let K and L be sets of nonnegative integers such that* $\min k_i > \max l_j$. *Then,* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

Conjecture 1.5 (Snevily [5]). *For any K and L with* $\min k_i > \max l_j$, $|\mathcal{F}| \le \binom{n}{s}$.

In the same paper in which he stated the above conjecture, Snevily mentions that it seems hard to prove the above bound and states the following weaker conjecture.

Conjecture 1.6 (Snevily [5]). *For any K and L with* $\min k_i > \max l_j$, $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Hwang and Sheikh [6] proved the bound of Conjecture 1.6 when K is a consecutive set. The second theorem we prove is a special case of Conjecture 1.6 with the extra condition that $\bigcap_{i=1}^{m} F_i \neq \emptyset$. These two theorems are stated hereunder.

Theorem 1.7 (Hwang and Sheikh [6]). *Let* $K = \{k_1, k_2, \ldots, k_r\}$ *where* $k_i = k_1 + i - 1$, $k_1 > s - r$, *and* $L = \{l_1, l_2, \ldots, l_s\}$. *Let* $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ *be such that* $|F_i| \in K$ *for each* i, $|F_i| \notin L$, *and* $|F_i \cap F_j| \in L$ *for any* $i \neq j$. *Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Theorem 1.8 (Hwang and Sheikh [6]). *Let* $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$, *and* $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ *be such that* $|F_i| \in K$ *for each* i, $|F_i \cap F_j| \in L$ *for any* $i \neq j$, *and* $k_i > s - r$. *If* $\bigcap_{i=1}^{m} F_i \neq \emptyset$, *then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Theorem 1.9 (Alon et al. [2]). *Let K and L be subsets of* $\{0, 1, ..., p-1\}$ *such that* $K \cap L = \emptyset$, *where* p *is a prime and* $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ *a family of subsets of* $[n]$ *such that* $|F_i| (\text{mod } p) \in K$ *for all* $F_i \in \mathcal{F}$ *and* $|F_i \cap F_j| (\text{mod } p) \in L$ *for* $i \neq j$. *If* $r(s - r + 1) \le p - 1$, *and* $n \ge s + \max k_i$, *then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Conjecture 1.10 (Alon et al. [2]). *Let K and L be subsets of* $\{0, 1, \ldots, p-1\}$ *such that* $K \cap L = \emptyset$, *where* p *is a prime and* $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ *a family of subsets of* $[n]$ *such that* $|F_i| (\text{mod} p) \in K$ *for all* $F_i \in \mathcal{F}$ *and* $|F_i \cap F_j| (\text{mod } p) \in L$ *for* $i \neq j$. *If* $n \ge s + \max k_i$, *then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

In [2], Alon et al. proved their conjectured bound under the extra conditions that $r(s - r + 1) \le p - 1$ and $n \ge s + \max k_i$. Qian and Ray-Chaudhuri [7] proved that if $n > 2s - r$ instead of $n \ge s + \max k_i$, then the above bound holds.

We prove an Alon-Babai-Suzuki's conjecture in non-modular version.

Theorem 1.11. *Let* $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$ *be two sets of nonnegative integers and let* $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ *be such that* $|F_i| \in K$ *for each* i, $|F_i \cap F_j| \in L$ *for any* $i \neq j$, *and* $n \ge s + \max_i |F_i|$. *then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

2. Proof of Theorem

Proof of Theorem 1.11. For each $F_i \in \mathcal{F}$, consider the polynomial

$$f_i(x) = \prod_{\substack{j \\ l_j<|F_i|}} \left(v_i \cdot x - \left(k_i - l_j\right)\right), \tag{2.1}$$

where v_i is the characteristic vector of F_i and v_i^* is the characteristic vector of $F_i^* = F_i - \{1\}$. Let $\overline{v_i}$ the characteristic vector of F_i^c, and $\overline{v_i}^*$ be the characteristic vector of $(F_i^c)^*$.

We order $\{F_i\}$ by size of F_i, that is, $|F_j| \leq |F_k|$ if $j < k$. We substitute the characteristic vector $\overline{v_i}$ of F_i^c by order of size of F_i. Clearly, $f_i(\overline{v_i}) \neq 0$ for $1 \leq i \leq m$ and $f_i(\overline{v_j}) = 0$ for $1 \leq j < i \leq m$. Assume that

$$\sum_i \alpha_i f_i(x) = 0. \tag{2.2}$$

We prove that $\{f_i(x)\}$ is linearly independent. Assume that this is false. Let i_0 be the smallest index such that $\alpha_{i_0} \neq 0$. We substitute $\overline{v_{i_0}}$ into the above equation. Then we get $\alpha_{i_0} f_{i_0}(\overline{v_{i_0}}) = 0$. We get a contradiction. So $\{f_i(x)\}$ is linearly independent. Let $\mathcal{E} = \{E_1, \ldots, E_e\}$ be the family of subsets of $[n]$ with size at most $s - r$, which is ordered by size, that is, $|E_i| \leq |E_j|$ if $i < j$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$. Let u_i denote the characteristic vector of E_i. We define the multilinear polynomial g_i in n variables for each E_i:

$$g_i(x) = \prod_{l=1}^{r} \left(\sum_{t=1}^{n} x_t - (n - k_l) \right) \prod_{j \in E_i} x_j. \tag{2.3}$$

We prove that $\{g_i(x)\}$ is linearly independent. Assume that

$$\sum_i \beta_i g_i(x) = 0. \tag{2.4}$$

Choose the smallest size of E_i. Let u_i be the characteristic vector of E_i. We substitute u_i into the above equation. We know that $g_i(u_i) \neq 0$ and $g_j(u_i) = 0$ for any $i < j$. Since $n \geq s + \max k_i$, we get $\beta_i = 0$. If we follow the same process, then the family $\{g_i(x)\}$ is linearly independent. Next, we prove that $\{f_i(x), g_i(x)\}$ is linearly independent. Now, assume that

$$\sum_i \alpha_i f_i(x) + \sum_i \beta_i g_i(x) = 0. \tag{2.5}$$

Let F_1 be the smallest size of F_i. We substitute the characteristic vector $\overline{v_1}$ of F_1^c into the above equation. Since $|F_i^c| = n - k_l$, $g_i(\overline{v_1}) = 0$ for all i. We only get $\alpha_1 f_1(\overline{v_1}) = 0$. So $\alpha_1 = 0$. By the same way, choose the smallest size from $\{F_i\}$ after deleting F_1. We do the same process. We also can get $\alpha_2 = 0$. By the same process, we prove that all $\alpha_i = 0$. We prove that $\{f_i(x), g_i(x)\}$ is linearly independent.

Any polynomial in the set $\{f_i(x), g_i(x)\}$ can be represented by a linear combination of multilinear monomials of degree $\le s$. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s}\binom{n}{i}$. We found $|\mathcal{F}| + \sum_{i=0}^{s-r}\binom{n}{i}$ linearly independent polynomials with degree at most s. So $|\mathcal{F}| + \sum_{i=0}^{s-r}\binom{n}{i} \le \sum_{i=0}^{s}\binom{n}{i}$. Thus $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. □

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