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Research Article

On Schur Convexity of Some Symmetric Functions

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For $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ and $r \in \{1, 2, \ldots, n\}$, the symmetric function $F_n(x, r)$ is defined as $F_n(x,r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^{r} ((1+x_{i_j})/(1-x_{i_j}))$, where $i_1, i_2, \ldots, i_n$ are positive integers. In this paper, the Schur convexity, Schur multiplicative convexity, and Schur harmonic convexity of $F_n(x, r)$ are discussed. As consequences, several inequalities are established by use of the theory of majorization.

1. Introduction

Throughout this paper, we use the following notation system.

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, let

$$
\begin{gathered}
x + y = \left(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\right), \\
xy = \left(x_1 y_1, x_2 y_2, \ldots, x_n y_n\right), \\
\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n), \\
A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \\
G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \\
\alpha + x = (\alpha + x_1, \alpha + x_2, \ldots, \alpha + x_n).
\end{gathered}
\tag{1.1}
$$

If $x_i > 0,\ i = 1, 2, \ldots, n$, then we denote by

$$\frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)$$
$$\log x = (\log x_1, \log x_2, \ldots, \log x_n). \tag{1.2}$$

Next we introduce some definitions and well-known results.

Definition 1.1. Let $E \subseteq \mathbb{R}^n$ be a set, and a real-valued function F on E is called a Schur convex function if

$$F(x_1, x_1, \ldots, x_n) \leqslant F(y_1, y_2, \ldots, y_n) \tag{1.3}$$

for each pair of n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in E with $x \prec y$, that is,

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1,$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{1.4}$$

where $x_{[i]}$ denotes the ith largest component of x. A function F is called Schur concave if $-F$ is Schur convex.

Definition 1.2. Let $E \subseteq (0, \infty)^n$ be a set, and a function $F : E \to (0, \infty)$ is called a Schur multiplicatively convex function on E if

$$F(x_1, x_2, \ldots, x_n) \leqslant F(y_1, y_2, \ldots, y_n) \tag{1.5}$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E with $\log x \prec \log y$. F is called Schur multiplicatively concave if $1/F$ is Schur multiplicatively convex.

Definition 1.3. Let $E \subseteq (0, \infty)^n$ be a set. A function $F : E \to \mathbb{R}$ is called a Schur harmonic convex (or Schur harmonic concave, resp.) function on E if

$$F(x_1, x_2, \ldots, x_n) \leqslant (\text{or} \geqslant, \text{resp.}) F(y_1, y_2, \ldots, y_n) \tag{1.6}$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E with $1/x \prec 1/y$.

Schur convexity was introduced by Schur [1] in 1923 and it has many applications in analytic inequalities [2, 3], extended mean values [4, 5], graphs and matrices [6], and other related fields. Recently, the Schur multiplicative convexity was investigated in [7–9] and the Schur hamonic convexity was discussed in [10].

For $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ $(n \geqslant 2)$ and $r \in \{1, 2, \ldots, n\}$, the symmetric function $F_n(x, r)$ is defined as

$$F_n(x,r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \leqslant i_1 < i_2 \cdots < i_r \leqslant n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{1 - x_{i_j}}, \tag{1.7}$$

where $i_1, i_2, \ldots, i_n$ are positive integers.

The aim of this article is to discuss the Schur convexity, Schur multiplicative convexity, and Schur harmonic convexity of the symmetric function $F_n(x, r)$.

Lemma 1.4 (see [11]). *Let $f : (0,1)^n \to \mathbb{R}$ be a continuous symmetric function. If f is differentiable in $(0,1)^n$, then f is Schur convex in $(0,1)^n$ if and only if*

$$(x_1 - x_2)\left(\frac{\partial f(x)}{\partial x_1} - \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \tag{1.8}$$

for all $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$.

Lemma 1.5 (see [7]). *Let $f : (0,1)^n \to (0,\infty)$ be a continuous symmetric function. If f is differentiable in $(0,1)^n$, then f is Schur multiplicatively convex in $(0,1)^n$ if and only if*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial f(x)}{\partial x_1} - x_2 \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \tag{1.9}$$

for all $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$.

Lemma 1.6 (see [10]). *Let $f : (0,1)^n \to (0,\infty)$ be a continuous symmetric function. If f is differentiable in $(0,1)^n$, then f is Schur harmonic convex in $(0,1)^n$ if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial f(x)}{\partial x_1} - x_2^2 \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \tag{1.10}$$

for all $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$.

Lemma 1.7 (see [12]). *Let $x = (x_1, x_2, \ldots, x_n) \in (0,\infty)^n$ and $\sum_{i=1}^{n} x_i = s$. If $c \geqslant s$, then*

$$\frac{c - x}{nc/s - 1} = \left(\frac{c - x_1}{nc/s - 1}, \frac{c - x_2}{nc/s - 1}, \ldots, \frac{c - x_n}{nc/s - 1}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{1.11}$$

Lemma 1.8 (see [12]). *Let $x = (x_1, x_2, \ldots, x_n) \in (0,\infty)^n$ and $\sum_{i=1}^{n} x_i = s$. If $c \geqslant 0$, then*

$$\frac{c + x}{nc/s + 1} = \left(\frac{c + x_1}{nc/s + 1}, \frac{c + x_2}{nc/s + 1}, \ldots, \frac{c + x_n}{nc/s + 1}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{1.12}$$

Lemma 1.9 (see [13]). *Suppose that* $x = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ *and* $\sum_{i=1}^n x_i = s$. *If* $0 \leqslant \lambda \leqslant 1$, *then*

$$\frac{s - \lambda x}{n - \lambda} = \left(\frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \ldots, \frac{s - \lambda x_n}{n - \lambda}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{1.13}$$

2. Main Result

Theorem 2.1. *The symmetric function* $F_n(x, r)$ *is Schur convex, Schur multiplicatively convex, and Schur harmonic convex in* $(0, 1)^n$ *for all* $r = 1, 2, \ldots, n$.

Proof. According to Lemmas 1.4–1.6 we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial F_n(x, r)}{\partial x_1} - \frac{\partial F_n(x, r)}{\partial x_2}\right) \geqslant 0, \tag{2.1}$$

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x, r)}{\partial x_1} - x_2 \frac{\partial F_n(x, r)}{\partial x_2}\right) \geqslant 0, \tag{2.2}$$

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x, r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, r)}{\partial x_2}\right) \geqslant 0, \tag{2.3}$$

for all $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$ and $r \in \{1, 2, \ldots, n\}$.

We divided the proof into seven cases.

Case 1. If $r = 1$ and $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$, then (1.7) leads to

$$\begin{aligned} F_n(x, 1) &= \sum_{i=1}^n \frac{1 + x_i}{1 - x_i}, \\ \frac{\partial F_n(x, 1)}{\partial x_i} &= \frac{2}{(1 - x_i)^2}, \quad i = 1, 2, \end{aligned} \tag{2.4}$$

$$(x_1 - x_2)\left(\frac{\partial F_n(x, 1)}{\partial x_1} - \frac{\partial F(x, 1)}{\partial x_2}\right) = \frac{2(x_1 - x_2)^2(2 - x_1 - x_2)}{(1 - x_1)^2(1 - x_2)^2} \geqslant 0, \tag{2.5}$$

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x, 1)}{\partial x_1} - x_2 \frac{\partial F_n(x, 1)}{\partial x_2}\right) = \frac{2(\log x_1 - \log x_2)(x_1 - x_2)(1 - x_1 x_2)}{(1 - x_1)^2(1 - x_2)^2} \geqslant 0, \tag{2.6}$$

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x, 1)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, 1)}{\partial x_2}\right) = \frac{2(x_1 - x_2)^2(x_1 + x_2 - 2x_1 x_2)}{(1 - x_1)^2(1 - x_2)^2} \geqslant 0. \tag{2.7}$$

Case 2. If $n = r = 2$ and $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then (1.7) yields

$$F_2(x,2) = \frac{(1+x_1)(1+x_2)}{(1-x_1)(1-x_2)},$$
$$\frac{\partial F_2(x,2)}{\partial x_i} = \frac{2F_2(x,2)}{(1-x_i)(1+x_i)}, \quad i = 1,2, \tag{2.8}$$

$$(x_1 - x_2)\left(\frac{\partial F_2(x,2)}{\partial x_1} - \frac{\partial F_2(x,2)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2(x_1+x_2)}{(1-x_1)^2(1-x_2)^2} \geqslant 0, \tag{2.9}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_2(x,2)}{\partial x_1} - x_2\frac{\partial F_2(x,2)}{\partial x_2}\right) = \frac{2(\log x_1 - \log x_2)(x_1-x_2)(1+x_1x_2)}{(1-x_1)^2(1-x_2)^2} \geqslant 0, \tag{2.10}$$

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_2(x,2)}{\partial x_1} - x_2^2\frac{\partial F_2(x,2)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2(x_1+x_2)F_2(x,2)}{(1-x_1)^2(1-x_2)^2} \geqslant 0. \tag{2.11}$$

Case 3. If $n = 3$, $r = 2$, and $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then from (1.7) we clearly see that

$$F_3(x,2) = \frac{(1+x_1)(1+x_2)}{(1-x_1)(1-x_2)} + \left(\frac{1+x_1}{1-x_1} + \frac{1+x_2}{1-x_2}\right)\frac{1+x_3}{1-x_3},$$
$$\frac{\partial F_3(x,2)}{\partial x_i} = \frac{2(1+x_{3-i})}{(1-x_i)^2(1-x_{3-i})} + \frac{2(1+x_3)}{(1-x_i)^2(1-x_3)}, \quad i = 1,2, \tag{2.12}$$

$$(x_1 - x_2)\left(\frac{\partial F_3(x,2)}{\partial x_1} - \frac{\partial F_3(x,2)}{\partial x_2}\right)$$
$$= \frac{2(x_1-x_2)^2}{(1-x_1)^2(1-x_2)^2}\left[(x_1+x_2) + (2-x_1-x_2)\frac{1+x_3}{1-x_3}\right] \geqslant 0, \tag{2.13}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_3(x,2)}{\partial x_1} - x_2\frac{\partial F_3(x,2)}{\partial x_2}\right)$$
$$= \frac{2(\log x_1 - \log x_2)(x_1-x_2)}{(1-x_1)^2(1-x_2)^2}\left[1 + x_1x_2 + (1-x_1x_2)\frac{1+x_3}{1-x_3}\right] \geqslant 0, \tag{2.14}$$

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_3(x,2)}{\partial x_1} - x_2^2\frac{\partial F_3(x,2)}{\partial x_2}\right)$$
$$= \frac{2(x_1-x_2)^2}{(1-x_1)^2(1-x_2)^2}\left[(x_1+x_2) + (x_1+x_2-2x_1x_2)\frac{1+x_3}{1-x_3}\right] \geqslant 0. \tag{2.15}$$

Case 4. If $n \geqslant 4$, $r = 2$, and $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then (1.7) implies that

$$F_n(x,2) = \frac{(1+x_1)(1+x_2)}{(1-x_1)(1-x_2)} + \left(\frac{1+x_1}{1-x_1} + \frac{1+x_2}{1-x_2}\right)\sum_{i=3}^{n}\frac{1+x_i}{1-x_i} + F_{n-2}(x_3, x_4, \ldots, x_n; 2), \tag{2.16}$$

$$\frac{\partial F_n(x,2)}{\partial x_j} = \frac{2(1+x_{3-j})}{(1-x_j)^2(1-x_{3-j})} + \frac{2}{(1-x_j)^2}\sum_{i=3}^{n}\frac{1+x_i}{1-x_i}, \quad j = 1,2,$$

$$(x_1 - x_2)\left(\frac{\partial F_n(x,2)}{\partial x_1} - \frac{\partial F_n(x,2)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2}{(1-x_1)^2(1-x_2)^2}\left[(x_1+x_2) + (2-x_1-x_2)\sum_{i=3}^{n}\frac{1+x_i}{1-x_i}\right] \geqslant 0, \tag{2.17}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,2)}{\partial x_1} - x_2\frac{\partial F_n(x,2)}{\partial x_2}\right) = \frac{2(\log x_1 - \log x_2)(x_1-x_2)}{(1-x_1)^2(1-x_2)^2}\left[1 + x_1x_2 + (1-x_1x_2)\sum_{i=3}^{n}\frac{1+x_i}{1-x_i}\right] \geqslant 0, \tag{2.18}$$

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_n(x,2)}{\partial x_1} - x_2^2\frac{\partial F_n(x,2)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2}{(1-x_1)^2(1-x_2)^2}\left[(x_1+x_2) + (x_1+x_2-2x_1x_2)\sum_{i=3}^{n}\frac{1+x_i}{1-x_i}\right] \geqslant 0. \tag{2.19}$$

Case 5. If $n = r \geqslant 3$, and $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then (1.7) leads to

$$F_n(x,n) = \prod_{i=1}^{n}\frac{1+x_i}{1-x_i}, \qquad \frac{\partial F_n(x,n)}{\partial x_i} = \frac{2}{(1-x_i)(1+x_i)}F_n(x,n), \quad i = 1,2, \tag{2.20}$$

$$(x_1 - x_2)\left(\frac{\partial F_n(x,n)}{\partial x_1} - \frac{\partial F_n(x,n)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2(x_1+x_2)}{(1-x_1^2)(1-x_2^2)}F_n(x,n) \geqslant 0, \tag{2.21}$$

$$\begin{aligned}&(\log x_1-\log x_2)\left(x_1\frac{\partial F_n(x,n)}{\partial x_1}-x_2\frac{\partial F_n(x,n)}{\partial x_2}\right)\\&\quad=\frac{2(\log x_1-\log x_2)(x_1-x_2)}{(1-x_1^2)(1-x_2^2)}(1+x_1x_2)F_n(x,n)\geqslant 0,\end{aligned}\tag{2.22}$$

$$\begin{aligned}&(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,n)}{\partial x_1}-x_2^2\frac{\partial F_n(x,n)}{\partial x_2}\right)\\&\quad=\frac{2(x_1-x_2)^2(x_1+x_2)}{(1-x_1^2)(1-x_2^2)}F_n(x,n)\geqslant 0.\end{aligned}\tag{2.23}$$

Case 6. If $n\geqslant 4$, $r=n-1$, and $x=(x_1,x_2,\dots,x_n)\in(0,1)^n$, then (1.7) yields

$$\begin{aligned}F_n(x,n-1)&=\frac{(1+x_1)(1+x_2)}{(1-x_1)(1-x_2)}F_{n-2}(x_3,x_4,\dots,x_n;n-3)\\&\quad+\left(\frac{1+x_1}{1-x_1}+\frac{1+x_2}{1-x_2}\right)\prod_{i=3}^{n}\frac{1+x_i}{1-x_i},\\\frac{\partial F_n(x,n-1)}{\partial x_j}&=\frac{2(1+x_{3-j})}{(1-x_j)^2(1-x_{3-j})}F_{n-2}(x_3,x_4,\dots,x_n;n-3)\\&\quad+\frac{2}{(1-x_j)^2}\prod_{i=3}^{n}\frac{1+x_i}{1-x_i},\quad j=1,2,\end{aligned}\tag{2.24}$$

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial F_n(x,n-1)}{\partial \mathrm{x}_1}-\frac{\partial F_n(x,n-1)}{\partial x_2}\right)\\&\quad=\frac{2(x_1-x_2)^2}{(1-x_1)^2(1-x_2)^2}\Bigg[(x_1+x_2)F_{n-2}(x_3,x_4,\dots,x_n;n-3)\\&\qquad\qquad+(2-x_1-x_2)\prod_{i=3}^{n}\frac{1+x_i}{1-x_i}\Bigg]\geqslant 0,\end{aligned}\tag{2.25}$$

$$\begin{aligned}&(\log x_1-\log x_2)\left(x_1\frac{\partial F_n(x,n-1)}{\partial x_1}-x_2\frac{\partial F_n(x,n-1)}{\partial x_2}\right)\\&\quad=\frac{2(\log x_1-\log x_2)(x_1-x_2)}{(1-x_1)^2(1-x_2)^2}\Bigg[(1+x_1x_2)F_{n-2}(x_3,x_4,\dots,x_n;n-3)\\&\qquad\qquad+(1-x_1x_2)\prod_{i=3}^{n}\frac{1+x_i}{1-x_i}\Bigg]\geqslant 0,\end{aligned}\tag{2.26}$$

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x, n-1)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, n-1)}{\partial x_2}\right)$$
$$= \frac{2(x_1 - x_2)^2}{(1 - x_1)^2(1 - x_2)^2}\left[(x_1 + x_2)F_{n-2}(x_3, x_4, \ldots, x_n; n-3) + (x_1 + x_2 - 2x_1x_2)\prod_{i=3}^{n}\frac{1 + x_i}{1 - x_i}\right] \geqslant 0. \tag{2.27}$$

Case 7. If $n \geqslant 5$, $3 \leqslant r \leqslant n-2$, and $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then (1.7) implies

$$\begin{aligned} F_n(x, r) &= \frac{(1 + x_1)(1 + x_2)}{(1 - x_1)(1 - x_2)} F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ &+ \left(\frac{1 + x_1}{1 - x_1} + \frac{1 + x_2}{1 - x_2}\right) F_{n-2}(x_3, x_4, \ldots, x_n; r-1) \\ &+ F_{n-2}(x_3, x_4, \ldots, x_n; r), \end{aligned} \tag{2.28}$$

$$\begin{aligned} \frac{\partial F_n(x, r)}{\partial x_i} &= \frac{2(1 + x_{3-i})}{(1 - x_i)^2(1 - x_{3-i})} F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ &+ \frac{2}{(1 - x_i)^2} F_{n-2}(x_3, x_4, \ldots, x_n; r-1), \quad i = 1, 2, \end{aligned}$$

$$\begin{aligned} &(x_1 - x_2)\left(\frac{\partial F_n(x, r)}{\partial x_1} - \frac{\partial F_n(x, r)}{\partial x_2}\right) \\ &\quad = \frac{2(x_1 - x_2)^2}{(1 - x_1)^2(1 - x_2)^2}[(x_1 + x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ &\qquad + (2 - x_1 - x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-1)] \geqslant 0, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x, r)}{\partial x_1} - x_2 \frac{\partial F_n(x, r)}{\partial x_2}\right) \\ &\quad = \frac{2(\log x_1 - \log x_2)(x_1 - x_2)}{(1 - x_1)^2(1 - x_2)^2}[(1 + x_1x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ &\qquad + (1 - x_1x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-1)] \geqslant 0, \end{aligned} \tag{2.30}$$

$$\begin{aligned} &(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x, r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, r)}{\partial x_2}\right) \\ &\quad = \frac{2(x_1 - x_2)^2}{(1 - x_1)^2(1 - x_2)^2}[(x_1 + x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ &\qquad + (x_1 + x_2 - 2x_1x_2)F_{n-2}(x_3, x_4, \ldots, x_n; r-1)] \geqslant 0. \end{aligned} \tag{2.31}$$

Therefore, inequality (2.1) follows from inequalities (2.5), (2.9), (2.13), (2.17), (2.21), (2.25), and (2.29), inequality (2.2) follows from inequalities (2.6), (2.10), (2.14),(2.18), (2.22), (2.26), and (2.30), and inequality (2.3) follows from inequalities (2.7), (2.11), (2.15), (2.19), (2.23), (2.27), and (2.31). □

3. Applications

In this section, we establish several inequalities by use of Theorem 2.1 and the theory of majorization.

It follows from Lemmas 1.7, 1.8, 1.9, and Theorem 2.1 that Theorem 3.1 is obvious.

Theorem 3.1. *If* $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, $s = \sum_{i=1}^{n} x_i$, *and* $r \in \{1, 2, \ldots, n\}$, *then*

$$\begin{aligned} &(1) \quad F_n(x,r) \geqslant F_n\left(\frac{c-x}{nc/s-1}, r\right) \quad \text{for } c \geqslant s, \\ &(2) \quad F_n(x,r) \geqslant F_n\left(\frac{c+x}{nc/s+1}, r\right) \quad \text{for } c \geqslant 0, \\ &(3) \quad F_n(x,r) \geqslant F_n\left(\frac{s-\lambda x}{n-\lambda}; r\right) \quad \text{for } 0 \leqslant \lambda \leqslant 1. \end{aligned} \tag{3.1}$$

Theorem 3.2. *If* $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, $s = \sum_{i=1}^{n} x_i$, $r \in \{1, 2, \ldots, n\}$, *and* $0 \leqslant \lambda \leqslant 1$, *then*

$$F_n(x,r) \geqslant F_n\left(\frac{s+\lambda x}{n+\lambda}; r\right). \tag{3.2}$$

Proof. Theorem 3.2 follows from Theorem 2.1 and the fact that

$$\frac{s+\lambda x}{n+\lambda} = \left(\frac{s+\lambda x_1}{n+\lambda}, \frac{s+\lambda x_2}{n+\lambda}, \ldots, \frac{s+\lambda x_n}{n+\lambda}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{3.3}$$

If we take $r = 1$ and $s = 1$ in Theorem 3.1(3) and Theorem 3.2, respectively, then we get the following. □

Corollary 3.3. *If* $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ *with* $\sum_{i=1}^{n} x_i = 1$ *and* $0 \leqslant \lambda \leqslant 1$, *then*

$$\begin{aligned} &(1) \quad \sum_{i=1}^{n} \frac{1+x_i}{1-x_i} \geqslant \sum_{i=1}^{n} \frac{n+1-\lambda(1+x_i)}{n-1-\lambda(1-x_i)}, \\ &(2) \quad \sum_{i=1}^{n} \frac{1+x_i}{1-x_i} \geqslant \sum_{i=1}^{n} \frac{n+1+\lambda(1+x_i)}{n-1+\lambda(1-x_i)}. \end{aligned} \tag{3.4}$$

If we take $r = n$ *and* $s = 1$ *in Theorem 3.1(3) and Theorem 3.2, respectively, then one gets the following.*

Corollary 3.4. *If $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ with $\sum_{i=1}^n x_i = 1$ and $0 \leqslant \lambda \leqslant 1$, then*

$$\begin{aligned} &(1) \quad \prod_{i=1}^n \frac{1+x_i}{1-x_i} \geqslant \prod_{i=1}^n \frac{n+1-\lambda(1+x_i)}{n-1-\lambda(1-x_i)}, \\ &(2) \quad \prod_{i=1}^n \frac{1+x_i}{1-x_i} \geqslant \prod_{i=1}^n \frac{n+1+\lambda(1+x_i)}{n-1+\lambda(1-x_i)}. \end{aligned} \tag{3.5}$$

Remark 3.5. If we take $\lambda = 0$ in Corollaries 3.3 and 3.4, then we have

$$\sum_{i=1}^n \frac{1+x_i}{1-x_i} \geqslant \frac{n(n+1)}{n-1}, \qquad \prod_{i=1}^n \frac{1+x_i}{1-x_i} \geqslant \left(\frac{n+1}{n-1}\right)^n \tag{3.6}$$

for $0 < x_i < 1$ and $\sum_{i=1}^n x_i = 1$.

Theorem 3.6. *If $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then*

$$\sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \prod_{j=1}^r \frac{1+x_{i_j}}{1-x_{i_j}} \geqslant \frac{n!}{r!(n-r)!} \left[\frac{A_n(1+x)}{A_n(1-x)}\right]^r. \tag{3.7}$$

Proof. Theorem 3.6 follows from Theorem 2.1 and (1.7) together with the fact that

$$(A_n(x), A_n(x), \ldots, A_n(x)) \prec (x_1, x_2, \ldots, x_n) = x. \tag{3.8}$$

□

If we take $r = n$ in Theorem 3.6, then we have the following.

Corollary 3.7. *If $x = (x_1, x_2, \ldots, x_n) \in (0,1)^n$, then*

$$\frac{G_n(1+x)}{A_n(1+x)} \geqslant \frac{G_n(1-x)}{A_n(1-x)}. \tag{3.9}$$

Theorem 3.8. *Let $\mathcal{A} = A_1A_2 \cdots A_{n+1}$ be an n-dimensional simplex in $\mathbb{R}^n$ and let P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$, $i = 1, 2, \ldots, n+1$, then*

$$\begin{aligned} &(1) \quad \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n+1} \prod_{j=1}^r \frac{A_{i_j}B_{i_j} + PB_{i_j}}{PA_{i_j}} \geqslant \frac{(n+1)!}{r!(n-r+1)!} \left(\frac{n+2}{n}\right)^r, \\ &(2) \quad \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n+1} \prod_{j=1}^r \frac{A_{i_j}B_{i_j} + PA_{i_j}}{PB_{i_j}} \geqslant \frac{(n+1)!}{r!(n-r+1)!} (2n+1)^r. \end{aligned} \tag{3.10}$$

Proof. It is easy to see that $\sum_{i=1}^{n+1}(PB_i/A_iB_i) = 1$ and $\sum_{i=1}^{n+1}(PA_i/A_iB_i) = n$, and these identities imply that

$$\begin{gathered}\left(\frac{1}{n+1},\frac{1}{n+1},\ldots,\frac{1}{n+1}\right)\prec\left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\ldots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right),\\ \left(\frac{n}{n+1},\frac{n}{n+1},\ldots,\frac{n}{n+1}\right)\prec\left(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2},\ldots,\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right).\end{gathered}\tag{3.11}$$

Therefore, Theorem 3.8 follows from Theorem 2.1 and (1.7) together with (3.11). □

Theorem 3.9. *Suppose that $A \in M_n(C)$ $(n \geqslant 2)$ is a complex matrix, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. If A is a positive definite Hermitian matrix, then*

$$\begin{aligned}&(1)\quad \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\prod_{j=1}^{r}\frac{\operatorname{tr}A+\lambda_{i_j}}{\operatorname{tr}A-\lambda_{i_j}}\geqslant\frac{n!}{r!(n-r)!}\left(\frac{n+1}{n-1}\right)^r,\\ &(2)\quad \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\prod_{j=1}^{r}\frac{\operatorname{tr}A+\lambda_{i_j}}{\operatorname{tr}A-\lambda_{i_j}}\geqslant\frac{n!}{r!(n-r)!}\left(\frac{\operatorname{tr}A+\sqrt[n]{\det A}}{\operatorname{tr}A-\sqrt[n]{\det A}}\right)^r,\\ &(3)\quad \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\prod_{j=1}^{r}\left(1+\frac{2}{\lambda_{i_j}}\right)\geqslant\frac{n!}{r!(n-r)!}\left(\frac{\sqrt[n]{\det(I+A)}+1}{\sqrt[n]{\det(I+A)}-1}\right)^r,\\ &(4)\quad \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\prod_{j=1}^{r}\left(1+\frac{2}{\lambda_{i_j}}\right)\geqslant\frac{n!}{r!(n-r)!}\left(1+\frac{2n}{\operatorname{tr}A}\right)^r.\end{aligned}\tag{3.12}$$

Proof. We clearly see that $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$, and

$$\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)\prec\left(\frac{\lambda_1}{\operatorname{tr}A},\frac{\lambda_2}{\operatorname{tr}A},\ldots,\frac{\lambda_n}{\operatorname{tr}A}\right),\tag{3.13}$$

$$\log\left(\frac{\sqrt[n]{\det A}}{\operatorname{tr}A},\frac{\sqrt[n]{\det A}}{\operatorname{tr}A},\ldots,\frac{\sqrt[n]{\det A}}{\operatorname{tr}A}\right)\prec\log\left(\frac{\lambda_1}{\operatorname{tr}A},\frac{\lambda_2}{\operatorname{tr}A},\ldots,\frac{\lambda_n}{\operatorname{tr}A}\right),\tag{3.14}$$

$$\log\left(\frac{1}{\sqrt[n]{\det(I+A)}},\frac{1}{\sqrt[n]{\det(I+A)}},\ldots,\frac{1}{\sqrt[n]{\det(I+A)}}\right)\prec\log\left(\frac{1}{1+\lambda_1},\frac{1}{1+\lambda_2},\ldots,\frac{1}{1+\lambda_n}\right),\tag{3.15}$$

$$\left(\frac{n+\operatorname{tr}A}{n},\frac{n+\operatorname{tr}A}{n},\ldots,\frac{n+\operatorname{tr}A}{n}\right)\prec(1+\lambda_1,1+\lambda_2,\ldots,1+\lambda_n).\tag{3.16}$$

Therefore, Theorem 3.9(1) follows from (1.7), (3.13), and the Schur convexity of $F_n(x,r)$, Theorems 3.9(2) and (3) follow from (3.14) and (3.15) together with the Schur multiplitively convexity of $F_n(x,r)$, and Theorem 3.9(4) follows from (3.16) and the Schur harmonic convexity of $F_n(x,r)$. □

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