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Research Article

Some Priori Estimates about Solutions to Nonhomogeneous A-Harmonic Equations

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We deal with the nonhomogeneous A-harmonic equation $d^*A(x, g + du) = d^*h$ and the related conjugate A-harmonic equation $A(x, g + du) = h + d^*v$. Some priori estimates about solutions to these equations are obtained, which generalize some existing results. Particularly, we obtain the same estimate given by Theorem 1 of Iwaniec (1992) for the weak solution to the first equation under weaker conditions by a simpler method.

1. Introduction

The A-harmonic equation and the related conjugate A-harmonic equation for differential forms originated from the Laplace equation $\Delta u = 0$ and Cauchy-Riemann equation $\nabla u = (\partial v/\partial y, -\partial v/\partial x)$ for functions u and v in the plane $\mathbf{R}^2$, which are the characteristics of analytic functions $f(x) = u + iv$ in the two-dimensional plane. Their general forms are p-harmonic equations and A-harmonic equations that have been playing a significant role in the development of the theory of quasiconformal and quasiregular mappings, being generalized from analytic functions. Many classic partial differential equations concerned with physical problems may be formulated compactly as A-harmonic equations for differential forms. So the exploration of these kinds of equations has unique interests and meanings, which are referred to [1–9].

Let $H : \Omega \to L(\Lambda^l)$ be a bounded measure function on $\Omega \subset \mathbf{R}^n$ with values in symmetric linear transformations of $\Lambda^l = \Lambda^l(\mathbf{R}^n)$, the linear space of l-covectors in $\mathbf{R}^n$ for $l = 1, 2, \ldots, n$. Assume that

$$\lambda^{-1}|\xi| \leq \langle H(x)\xi, \xi\rangle^{1/2} \leq \lambda|\xi| \tag{1.1}$$

for $(x,\xi) \in \Omega \times \Lambda^l$, where λ is a constant independent of x and ξ. The nonlinear mapping $A(x,\xi) : \Omega \times \Lambda^l \to \Lambda^l$ is formulated by

$$A(x,\xi) = \langle H(x)\xi, \xi \rangle^{(p-2)/2} H(x)\xi \tag{1.2}$$

for $(x,\xi) \in \Omega \times \Lambda^l$. The problem of weak solutions defined as follows, concerned with $A(x,\xi)$, which was considered in [10] to give the priori estimate for weak solutions.

Definition 1.1 (see [10]). Let $g \in L^s(\Omega, \Lambda^l), s \geq \max\{1, p-1\}$, and $h \in L^{s/(p-1)}(\Omega, \Lambda^l)$. A differential form $u \in D'(\Omega, \Lambda^{l-1})$ is said to be a weak solution of equation

$$d^* A(x, g + du) = d^* h \tag{1.3}$$

if (1) $du \in L^s(\Omega, \Lambda^l)$ and (2) $\int_\Omega \langle A(x, g + du), d\alpha \rangle = \int_\Omega \langle h, d\alpha \rangle$ for each test form $\alpha \in L_1^{s/(s-p+1)}(\Omega, \Lambda^{l-1})$.

Theorem 1.2 (see [10]). *For each A-harmonic equation* (1.3) *there exist* $\nu = \nu(n,p,\lambda) \in (0, p-1)$ *and a constant* $C(p,\lambda) > 0$ *such that every weak solution* u*, with* $du \in L^s(\Omega, \Lambda^l)$ *and* $p-\nu \leq s \leq p+\nu$*, satisfies*

$$\int_\Omega |du|^s \leq C(p,\lambda) \left[\int_\Omega |g|^s + \int_\Omega |h|^{s/(p-1)} \right]. \tag{1.4}$$

It is easy to see that the mapping $A(x,\xi)$ given by (1.2) satisfies the following conditions

$$\langle A(x,\xi), \xi \rangle \geq \frac{1}{\lambda^p} |\xi|^p, \qquad |A(x,\xi)| \leq M \lambda^{p-2} |\xi|^{p-1}, \tag{1.5}$$

where M is the bound of H in Ω, that is, $|H(x)| \leq M$ for all $x \in \Omega$. In this paper we obtain the same result of Theorem 1.2 under the weaker hypotheses (1.5).

On the other hand, it is interesting to investigate the conjugate A-harmonic equation related to (1.3)

$$A(x, g + du) = h + d^* v \tag{1.6}$$

with the conditions (1.5). A series of norm comparison theorems for a pair of solutions to (1.6) were established in [5]. The following is the fundamental conclusion there, which will be extended in this paper to the situation that the conjugateness of p and q is not required.

Theorem 1.3 (see [5]). *Let u and v be a pair of solutions to* (1.3) *in a domain $\Omega \subset \mathbf{R}^n$. If $g \in L^p(\Omega, \Lambda^l)$ and $h \in L^q(\Omega, \Lambda^l)$, then $du \in L^p(\Omega, \Lambda^l)$ if and only if $d^*v \in L^q(\Omega, \Lambda^l)$. Moreover, there exist constants C_1, C_2, independent of u and v, such that*

$$\begin{aligned} \|d^*v\|_{q,B}^q &\le C_1\Big(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|du\|_{p,B}^p\Big), \\ \|du\|_{p,B}^p &\le C_2\Big(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|d^*v\|_{q,B}^q\Big) \end{aligned} \tag{1.7}$$

for all balls B with $B \subset \Omega \subset \mathbf{R}^n$. Here $1/p + 1/q = 1$.

As the extension of some results mentioned above, we give their weighted forms by the A_r weight function in the final section.

2. Some Preliminaries about Differential Forms

The majority of notations and preliminaries used throughout this paper can be found in [1]. For the sake of convenience we list them briefly in this section.

Let $e_1, e_2, \ldots, e_n$ denote the standard orthogonal basis of $\mathbf{R}^n$. Suppose that $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ is the linear space of l-covectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l), 1 \le i_1 < i_2 < \cdots i_l \le n$, and $l = 0, 1, \ldots, n$. The Grassmann algebra $\Lambda = \bigoplus_{l=0}^n \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \Sigma \alpha^I e_I \in \Lambda$ and $\beta = \Sigma \beta^I e_I \in \Lambda$, the inner product in Λ is given by $\langle \alpha, \beta \rangle = \Sigma \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator $\star : \Lambda \to \Lambda$ by

$$\star\omega = \operatorname{sign}(\pi)\alpha_{i_1,i_2,\ldots,i_k}(x_1, x_2, \ldots, x_n) dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}, \tag{2.1}$$

where $\omega = \alpha_{i_1,i_2,\ldots,i_k}(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ is a k-form, $\pi = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, 2, \ldots, n)$, and $\operatorname{sign}(\pi)$ is the signature of the permutation. The norm of $\alpha \in \Lambda$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbf{R}$.

Now and later on the notation, Ω stands for a ball or cube in $\mathbf{R}^n$, even though we do not always need this strong restriction on it. A differential l-form ω is a Schwartz distribution on Ω with values in $\Lambda^l(\mathbf{R}^n)$. We use $D'(\Omega, \Lambda^l)$ to denote the space of all deferential l-forms, and $L^p(\Omega, \Lambda^l)$ to denote the l-forms

$$\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1,i_2,\ldots,i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} \tag{2.2}$$

with all coefficients $\omega_I \in L^p(\Omega, \mathbf{R})$. Thus $L^p(\Omega, \Lambda^l)$, $p \ge 1$, is a Banach space with norm

$$\|\omega\|_p = \|\omega(x)\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p\right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}. \tag{2.3}$$

The space $L_1^p(\Omega, \Lambda^l)$ is the subspace of $D'(\Omega, \Lambda^l)$ with the condition

$$\|\alpha\|_{L_1^p(\Omega)} = \left(\int_\Omega \left(\sum_{i=1}^n \left|\frac{\partial \alpha}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p} < \infty. \tag{2.4}$$

The Sobolev space $W^{1,p}(\Omega, \Lambda^l)$ of l-forms is $W^{1,p}(\Omega, \Lambda^l) = L^p(\Omega, \Lambda^l) \cap L_1^p(\Omega, \Lambda^l)$.

We denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n-1$, which means

$$d\omega(x) = \sum_{k=1}^n \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial \omega_{i_1, i_2, \ldots, i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}. \tag{2.5}$$

Its formal adjoint operator is defined by

$$d^* = (-1)^{nl+1} \star d \star : D'\left(\Omega, \Lambda^{l+1}\right) \longrightarrow D'\left(\Omega, \Lambda^l\right), \quad l = 0, 1, \ldots, n-1, \tag{2.6}$$

which is called the Hodge codifferential.

Theorem 2.1 (Hodge decomposition [10]). *For each $\omega \in L^p(\Omega, \Lambda^l), 1 < p < \infty$, there exist differential forms $\alpha \in \ker d^* \cap L_1^p(\Omega, \Lambda^{l-1})$ and $\beta \in \ker d \cap L_1^p(\Omega, \Lambda^{l+1})$ such that*

$$\omega = d\alpha + d^*\beta. \tag{2.7}$$

The forms $d\alpha$ and $d^*\beta$ are unique and satisfy the uniform estimate

$$\|\alpha\|_{L_1^p(\Omega, \Lambda^{l-1})} + \|\beta\|_{L_1^p(\Omega, \Lambda^{l+1})} \le C_p(n) \|\omega\|_p \tag{2.8}$$

for some constant $C_p(n)$ independent of ω.

It is noticeable that the Hodge decomposition (2.7) corresponds to two bounded linear operators A and B from $L^p(\Omega, \Lambda^l)$ to $L^p(\Omega, \Lambda^l)$, defined by

$$A\omega = d^*\beta, \qquad B\omega = d\alpha \tag{2.9}$$

for $\omega \in L^p(\Omega, \Lambda^l)$, $1 < p < \infty$.

To consider priori estimates for the nonhomogeneous A-harmonic equation we need the bounds of $d\alpha$ and $d^*\beta$ in the sense of the L^p-norm for some special differential forms ω. The following interpolation theorem plays a key role in dealing with this problem. Let (X, μ) be a measure space and let E be a complex Hilbert space. The notation $\|T\|_r$ denotes the norm of bounded linear operators $T : L^r(X, E) \to L^r(X, E)$ for all $r \in [r_1, r_2]$, where $1 \le r_1 \le r_2 \le \infty$.

Theorem 2.2 (see [9]). *Suppose that $r/r_2 \le 1 + \varepsilon \le r/r_1$. Then*

$$\left\|T\left(|f|^\varepsilon f\right)\right\|_{r/(1+\varepsilon)} \le K|\varepsilon| \|f\|_r^{1+\varepsilon} \tag{2.10}$$

for each $f \in L^r(X,E) \cap \ker T$, *where*

$$K = \frac{2r(r_2 - r_1)}{(r - r_1)(r_2 - r)} \left(\|T\|_{r_1} + \|T\|_{r_2}\right). \tag{2.11}$$

3. Priori Estimates for Solutions

For convenience of estimates we reformulate the condition (1.5). Let the mapping $A : \Omega \times \Lambda^l(\mathbf{R}^n) \to \Lambda^l(\mathbf{R}^n)$ satisfy the following conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad \langle A(x,\xi), \xi\rangle \ge |\xi|^p \tag{3.1}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbf{R}^n)$.

Theorem 3.1. *Let* $g \in L^s(\Omega, \Lambda^l)$, $s \ge \max\{1, p-1\}$, *and* $h \in L^{s/(p-1)}(\Omega, \Lambda^l)$. *If* $u \in L^s(\Omega, \Lambda^{l-1})$ *is a weak solution to the equation*

$$d^* A(x, g + du) = d^* h \tag{3.2}$$

with the conditions (3.1), *then there exist* $\varepsilon = \varepsilon(n,a,p) \in (0, p-1)$ *and* $C = C(n,a,p)$ *such that*

$$\int_\Omega |du|^s \le C \int_\Omega \left(|g|^s + |h|^{s/(p-1)}\right) \tag{3.3}$$

for $p - \varepsilon \le s \le p + \varepsilon$.

Lemma 3.2. *For the* g *and* u *in Theorem 3.1 there exists a constant* K, *independent of* g *and* u, *such that*

$$\begin{aligned} \|d^*\beta\|_{s/(s-p+1)} &\le K|s-p| \, \|g + du\|_s^{s-p+1}, \\ \|d\alpha\|_{s/(s-p+1)} &\le (1 + K|s-p|) \, \|g + du\|_s^{s-p+1}, \end{aligned} \tag{3.4}$$

where $d\alpha$ *and* $d^*\beta$ *are given by the Hodge decomposition* $|g + du|^{s-p}(g + du) = d\alpha + d^*\beta$.

This lemma can be directly deduced from Theorem 2.2 (so-called interpolation theorem) as shown in [9]. For the sake of completeness we give its proof which displays how to use the Hodge decomposition and the interpolation theorem.

Proof. For $\omega \in L^r(\Omega, \Lambda)$ $(r > 1)$ and its Hodge decomposition $\omega = d\alpha + d^*\beta$ we can define a bounded linear operator T from $L^r(\Omega, \Lambda)$ to $L^r(\Omega, \Lambda)$ by $T\omega = d^*\beta$. In view of the restriction for s and p we have $1 < s/(s-p+1) < \infty$. So taking $1 < r_1 < r_2$ such that $s/(s-p+1) \in (r_1, r_2)$ and choosing $r = s$, $\varepsilon = s - p$, and $f = g + du$ in Theorem 2.2 yield

$$\|d^*\beta\|_{s/(s-p+1)} = \left\|T\left(|g + du|^{s-p}(g + du)\right)\right\|_{s/(s-p+1)} \le K|s-p| \, \|g + du\|_s^{s-p+1}, \tag{3.5}$$

where $K = K(s,p)$ does not depend on g and u. The second inequality is an immediate result from the first one. The proof is complete. □

Besides, Young's inequality and Hölder's inequality play a very important role in a variety of estimates throughout this paper and are listed as follows.

Lemma 3.3 (Young's inequality [11]). *If $a \geq 0$, $b \geq 0$, $p > 1$, and $p^{-1} + q^{-1} = 1$, then*

$$ab \leq \frac{\mu^{-p/q} a^p}{p} + \frac{\mu b^q}{q} \leq \mu^{-p/q} a^p + \mu b^q \tag{3.6}$$

for any positive numbers μ.

Lemma 3.4 (Hölder's inequality [1]). *Let $0 < p < \infty, 0 < q < \infty$, and $p^{-1} + q^{-1} = s^{-1}$. If f and g are measure functions on $E \subset \mathbf{R}^n$, then*

$$\|fg\|_{s,E} \leq \|f\|_{p,E} \|g\|_{q,E}. \tag{3.7}$$

Proof of Theorem 3.1. In view of (3.1) we have

$$\begin{aligned} |g + du|^s &\leq \left\langle A(x, g + du), |g + du|^{s-p}(g + du) \right\rangle \\ &= \langle A(x, g + du), d\alpha \rangle + \langle A(x, g + du), d^*\beta \rangle, \end{aligned} \tag{3.8}$$

where $d\alpha$ and $d^*\beta$ are given by Lemma 3.2. Taking $d\alpha$ as the test form in the definition of weak solutions and integrating (3.8) over Ω, and using (3.1) again, we have

$$\begin{aligned} \int_\Omega |g + du|^s &\leq \int_\Omega \langle A(x, g + du), d\alpha \rangle + \int_\Omega \langle A(x + g + du), d^*\beta \rangle \\ &= \int_\Omega \langle h, d\alpha \rangle + \int_\Omega \langle A(x, g + du), d^*\beta \rangle \\ &\leq \int_\Omega |h||d\alpha| + a \int_\Omega |g + du|^{p-1} |d^*\beta|. \end{aligned} \tag{3.9}$$

Using Hölder's inequality and Lemma 3.2 yields

$$\begin{aligned} \|g + du\|_s^s &= \int_\Omega |g + du|^s \\ &\leq \|h\|_{s/(p-1)} \|d\alpha\|_{s/(s-p+1)} + \left\| |g + du|^{p-1} \right\|_{s/(p-1)} \|d^*\beta\|_{s/(s-p+1)} \\ &\leq (1 + K|s - p|) \|h\|_{s/(p-1)} \|g + du\|_s^{s-p+1} + K|s - p| \|g + du\|_s^s. \end{aligned} \tag{3.10}$$

Applying Young's inequality to the first term of the right-hand side in (3.10), we have

$$\|g+du\|_s^s \le (1+K|s-p|)\left[\mu^{-(s-p+1)/(p-1)}\|h\|_{s/(p-1)}^{s/(p-1)}+\mu\|g+du\|_s^s\right]+K|s-p|\|g+du\|_s^s. \tag{3.11}$$

Choosing proper $\mu > 0$ and $\varepsilon > 0$ leads to

$$\|g+du\|_s^s \le C_1\|h\|_{s/(p-1)}^{s/(p-1)} \tag{3.12}$$

whenever $|s-p| \le \varepsilon$, where $C_1 = C_1(n,p,a)$. Thus, by the elementary inequality $(a+b)^s \le 2^{s-1}(a^s+b^s)$ $(s \ge 1)$ and (3.12), we can conclude that

$$\begin{aligned}\int_\Omega |du|^s &\le 2^{s-1}\int_\Omega \left(|g+du|^s+|g|^s\right)\\ &\le 2^{s-1}C_1\int_\Omega |h|^{s/(p-1)}+2^{s-1}\int_\Omega |g|^s\\ &\le C\left(\int_\Omega |h|^{s/(p-1)}+\int_\Omega |g|^s\right),\end{aligned} \tag{3.13}$$

where $C = C(n,p,a)$ is independent of g and u, which finishes the proof. □

Another kind of restrictive conditions about $A(x,\xi)$ was given in [9], where the same result as Theorem 3.1 was obtained under the following hypotheses (H1) and (H2). Let $p > 1$ be a constant and $A : \Omega \times \Lambda^l(\mathbf{R}^n) \to \Lambda^l(\mathbf{R}^n)$ be a nonlinear operator satisfying

(H1) $|A(x,\xi)-A(x,\zeta)| \le b|\xi-\zeta|(|\xi|+|\zeta|)^{p-2}$,

(H2) $\langle A(x,\xi)-A(x,\zeta),\xi-\zeta\rangle \ge a|\xi-\zeta|^2(|\xi|+|\zeta|)^{p-2}$,

(H3) $A(x,\lambda\zeta) = |\lambda|^{p-2}\lambda A(x,\zeta)$

for almost every $x \in \Omega, \lambda \in \mathbf{R}$ and $\xi,\zeta \in \Lambda^l(\mathbf{R}^n)$.

Notice that (3.1) and both (H1) and (H2) are not mutual of inclusion. But all (H1)–(H3) may lead to (3.1) except for constants. Since the main results with (H3) in [9] were based on the conclusion of Theorem 3.1, we can obtain responding results on a larger scale. Taking Lemma 2 in [9] for example, we can establish the following theorem.

Theorem 3.5. *Let ε be the same as in Theorem 3.1. Suppose that u is a weak solution for some $s \in (p-\varepsilon,p)$ to the homogeneous A-harmonic equation*

$$d^* A(x,du) = 0 \tag{3.14}$$

with the assumptions (H1)–(H3). Then for any concentric cubes $Q \subset 2Q \subset \Omega$ one has

$$\left(⨍_Q |du|^s\right)^{1/s} \le C(n,p,a)\left(⨍_{2Q} |du|^r\right)^{1/r}, \tag{3.15}$$

where $r = \max\{ns/(n+s-1), ns/(np-n+s-p+1)\}$ and $⨍_D$ stands for the integral mean over D.

Now we consider the nonhomogeneous conjugate A-harmonic equation

$$A(x, g + du) = h + d^*v \tag{3.16}$$

and establish the norm comparison theorem for du and d^*v. It generalizes Theorem 1.3 because here s and $s/(p-1)$ do not generally satisfy the conjugate condition which is demanded there.

Theorem 3.6. *Let u and v be a pair of solutions to the nonhomogeneous A-harmonic equation* (3.16) *with the condition* (3.1) *in the domain $\Omega \subset \mathbf{R}^n$. If $g \in L^s(\Omega, \Lambda^l)$ and $h \in L^{s/(p-1)}(\Omega, \Lambda^l)$, where $s \geq \max\{1, p-1\}$, then $du \in L^s(\Omega, \Lambda^l)$ if and only if $d^*v \in L^{s/(p-1)}(\Omega, \Lambda^l)$. Moreover, there exist constants C_1 and C_2, independent of u and v, such that*

$$\|d^*v\|_{s/(p-1)}^{s/(p-1)} \leq C_1\left(\|h\|_{s/(p-1)}^{s/(p-1)} + \|g\|_s^s + \|du\|_s^s\right), \tag{3.17}$$

$$\|du\|_s^s \leq C_2\left(\|g\|_s^s + \|h\|_{s/(p-1)}^{s/(p-1)} + \|d^*v\|_{s/(p-1)}^{s/(p-1)}\right). \tag{3.18}$$

Proof. It is enough to check both (3.17) and (3.18). First, from (3.1) and (3.16), we have

$$|d^*v| \leq |h| + |A(x, g + du)| \leq |h| + a|g + du|^{p-1}. \tag{3.19}$$

Applying the elementary inequality $(a+b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda)$ $(\lambda \geq 1)$ to the above inequality leads to

$$\begin{aligned} |d^*v|^{s/(p-1)} &\leq 2^{s/(p-1)-1}\left(|h|^{s/(p-1)} + a|g + du|^s\right) \\ &\leq 2^{(s-p+1)/(p-1)}\left(|h|^{s/(p-1)} + a2^{s-1}\left(|g|^s + |du|^s\right)\right) \\ &\leq C_1\left(|h|^{s/(p-1)} + |g|^s + |du|^s\right), \end{aligned} \tag{3.20}$$

where C_1 does not depend on u and v. Integrating (3.20) over Ω, we get (3.17).

Next, we use the trick used in the proof of Theorem 3.1 to check (3.18). Notice that

$$\begin{aligned} |g + du|^s &\leq |\langle A(x, g + du), g + du\rangle||g + du|^{s-p} = |\langle h + d^*v, g + du\rangle||g + du|^{s-p} \\ &\leq |h + d^*v||g + du|^{s-p+1}. \end{aligned} \tag{3.21}$$

Integrating (3.21) over Ω and then using Hölder's inequality, we have

$$\begin{aligned} \int_\Omega |g + du|^s &\leq \left(\int_\Omega |h + d^*v|^{s/(p-1)}\right)^{(p-1)/s}\left(\int_\Omega |g + du|^{(s-p+1)s/(s-p+1)}\right)^{(s-p+1)/s} \\ &= \|h + d^*v\|_{s/(p-1)}\|g + du\|_s^{s-p+1}. \end{aligned} \tag{3.22}$$

Using Young's inequality to (3.22), we get

$$\|g + du\|_s^s \le \mu^{-(s-p+1)/(p-1)} \|h + d^*v\|_{s/(p-1)}^{s/(p-1)} + \mu \|g + du\|_s^s. \tag{3.23}$$

Taking $\mu > 0$ small enough and using the elementary inequality $(a+b)^\lambda \le 2^{\lambda-1}(a^\lambda + b^\lambda)$ $(\lambda \ge 1)$, we obtain

$$\|g + du\|_s^s \le C_3 \|h + d^*v\|_{s/(p-1)}^{s/(p-1)} \le C_4 \left(\|h\|_{s/(p-1)}^{s/(p-1)} + \|d^*v\|_{s/(p-1)}^{s/(p-1)} \right), \tag{3.24}$$

where C_3 and C_4 are constants independent of u and v. It is not difficult to get (3.18) from (3.24), and so the proof is complete. □

Combining Theorems 3.1 and 3.6, we obtain immediately the norm estimate for d^*v by means of g and h, which can be viewed as the symmetrical result to (3.3).

Corollary 3.7. *If u and v simultaneously satisfy the hypothesis of both Theorem 3.1 and Theorem 3.6, then there is a constant C, independent of u and v, such that*

$$\|d^*v\|_{s/(p-1)}^{s/(p-1)} \le C \left(\|h\|_{s/(p-1)}^{s/(p-1)} + \|g\|_s^s \right). \tag{3.25}$$

4. Some Weighted Estimates

In this section we give the weighted estimates for some results obtained in the front part. A function $w(x)$ is called a weight if $w > 0$ a.e. and $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Among all weights the A_r function is one of the most important weights and is widely applied to the theory of harmonic analysis, quasiconformal mappings, differential forms, and so on.

Definition 4.1. A weight $w(x)$ is called A_r weight, where $r > 1$, and we write $w \in A_r$ if

$$\sup_B \left(\frac{1}{|B|} \int_B w \, dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} < \infty, \tag{4.1}$$

where the supremum is over all balls $B \subset \mathbf{R}^n$ and $|B|$ is the Lebesgue measure of B.

The A_r weight and the related Radon measure have many interesting properties; see [12, 13] for details. In order to express weighted integrals briefly, we introduce the notation $\|f\|_{p,B,w^\alpha}$ as follows:

$$\|f\|_{p,B,w^\alpha} = \left(\int_B f^p(x) w^\alpha(x) dx \right)^{1/p}. \tag{4.2}$$

Theorem 4.2. *Let u be a weak solution to the nonhomogeneous A-harmonic equation* (3.2) *in a ball $B \subset \mathbf{R}^n$, and let $w \in A_r$ for some $r > 1$. Then, for $\sigma \in ((1-1/r)s, s)$, there exists a constant C, independent of u, such that*

$$\|du\|_{\sigma,B,w^{\alpha}} \le C|B|^{r\kappa}\left(\|g\|_{\lambda,B,w^{\kappa\lambda}} + \left\||h|^{1/(p-1)}\right\|_{\lambda,B,w^{\kappa\lambda}}\right), \tag{4.3}$$

where $\alpha = 1-\sigma/s$, $\kappa = (s-\sigma)/\sigma s$, and $\lambda = s/(1-(s/\sigma-1)(r-1))$.

Proof. Let $p' = s$, and $1/p' + 1/q' = 1/\sigma$, that is, $q' = s\sigma/(s-\sigma) = \sigma/\alpha$. Using Hölder's inequality, we have

$$\begin{aligned}
\|du\|_{\sigma,B,w^{\alpha}} &= \left(\int_B \left(|du|w^{\alpha/\sigma}\right)^{\sigma}\right)^{1/\sigma} \le \left(\int_B |du|^{p'}\right)^{1/p'}\left(\int_B w^{\alpha q'/\sigma}\right)^{1/q'} \\
&= \left(\int_B |du|^{s}\right)^{1/s}\left(\int_B w^{\alpha s/(s-\sigma)}\right)^{(s-\sigma)/s\sigma} \\
&= \left(\int_B |du|^{s}\right)^{1/s}\left(\int_B w\right)^{\kappa} = \|du\|_s\left(\int_B w\right)^{\kappa}.
\end{aligned} \tag{4.4}$$

Next, we estimate $\|g\|_s$ by Hölder's inequality with $p' = \lambda$ and $q' = (\lambda - s)/\lambda s = rs/(s-\sigma)(r-1)$. Noticing $\lambda > s$ and $\kappa q' = 1/(r-1)$, we have

$$\begin{aligned}
\|g\|_s &= \left(\int_B \left(|g|w^{\kappa}\frac{1}{w^{\kappa}}\right)^{s}\right)^{1/s} \le \left(\int_B \left(|g|w^{\kappa}\right)^{p'}\right)^{1/p'}\left(\int_B \left(\frac{1}{w}\right)^{\kappa q'}\right)^{1/q'} \\
&= \left(\int_B |g|^{\lambda}w^{\kappa\lambda}\right)^{1/\lambda}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\right)^{\kappa(r-1)} = \|g\|_{\lambda,B,w^{\kappa\lambda}}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\right)^{\kappa(r-1)}.
\end{aligned} \tag{4.5}$$

With the same method shown above, we can get the weighted estimate for $\|h\|_{s/(p-1)}^{1/(p-1)}$. But, as a matter of fact, we have a shortcut to obtain the same result. Since $\|h\|_{s/(p-1)}^{1/(p-1)} = \||h|^{1/(p-1)}\|_s$, replacing g in (4.5) with $|h|^{1/(p-1)}$, we get right away

$$\|h\|_{s/(p-1)}^{1/(p-1)} = \left\||h|^{1/(p-1)}\right\|_s \le \left\||h|^{1/(p-1)}\right\|_{\lambda,B,w^{\kappa\lambda}}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\right)^{\kappa(r-1)}. \tag{4.6}$$

In view of the elementary inequality $(a+b)^{\mu} < a^{\mu} + b^{\mu}$ for $a, b > 0$ and $\mu \in (0,1)$, from (3.3), we have

$$\|du\|_s \le C_1\left(\|g\|_s + \|h\|_{s/(p-1)}^{1/(p-1)}\right). \tag{4.7}$$

Using (4.5) and (4.6) to plug (4.7) and then applying to (4.4), we have

$$\|du\|_{\sigma,B,w^{\alpha}} \le C_1\left(\|g\|_{\lambda,B,w^{\kappa\lambda}} + \left\| |h|^{1/(p-1)} \right\|_{\lambda,B,w^{\kappa\lambda}} \right)\left(\int_B w\right)^{\kappa}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\right)^{\kappa(r-1)}. \tag{4.8}$$

Since $w \in A_r$, we have from (4.1)

$$\left(\int_B w\right)\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\right)^{r-1} < C_2|B|^r. \tag{4.9}$$

Thus applying this to (4.8), we can finish the proof. □

Corollary 4.3. *Let v satisfy the conditions of Corollary 3.7 in a ball $B \subset \mathbf{R}^n$ and $w \in A_r$ for some $r > 1$. Then for $\sigma \in ((1-1/r)s/(p-1), s/(p-1))$, there exists a constant C, independent of v, such that*

$$\|d^* v\|_{\sigma,B,w^{\alpha'}} \le C|B|^{r\kappa'}\left(\|h\|_{\lambda',B,w^{\alpha'\kappa'}} + \left\| |g|^{p-1} \right\|_{\lambda',B,w^{\alpha'\kappa'}} \right), \tag{4.10}$$

where $\alpha' = 1 - (\sigma/s)(p-1)$, $\kappa' = s - \sigma(p-1)/\sigma s$, and $\lambda' = s\sigma/(s - rs + (p-1)\sigma r)$.

Proof. Putting $s' = s/(p-1)$, that is, $s = (p-1)s'$ into (3.25), we have

$$\|d^* v\|_{s'}^{s'} \le C\left(\|h\|_{s'}^{s'} + \|g\|_{s'(p-1)}^{s'(p-1)} \right) = C\left(\|h\|_{s'}^{s'} + \left\| |g|^{p-1} \right\|_{s'}^{s'} \right). \tag{4.11}$$

Taking a notice to another form of (3.3), we have

$$\|du\|_s^s \le C\left(\|g\|_s^s + \left\| |h|^{1/(p-1)} \right\|_s^s \right) \tag{4.12}$$

which is the source of (4.3). Making a comparison between (4.11) and (4.12), we can obtain the conditions by means of α', τ', and λ' that guarantee (4.10) to hold. Specifically, noticing $s' = s/(p-1)$, we have $\sigma \in ((1-1/r)s', s') = ((1-1/r)s/(p-1), s/(p-1))$ and

$$\begin{aligned} \alpha' &= 1 - \frac{\sigma}{s'} = 1 - \frac{\sigma}{s}(p-1), \\ \kappa' &= \frac{s' - \sigma}{\sigma s'} = \frac{s - \sigma(p-1)}{\sigma s}, \\ \lambda' &= \frac{s'}{1 - (s'/\sigma - 1)(r-1)} = \frac{s\sigma}{s - rs + (p-1)\sigma r}. \end{aligned} \tag{4.13}$$

Based on this approach it is easy to get the weighted forms of (3.17) and (3.18) if the domain Ω in Theorem 3.6 is replaced by a ball $B \subset \mathbf{R}^n$. For example, under the same hypotheses of Theorem 4.2, we have the weighted form of (3.18) as follows:

$$\|du\|_{\sigma,B,w^{\alpha}} \leq C|B|^{r\kappa}\left(\|g\|_{\lambda,B,w^{\kappa\lambda}} + \left\||h|^{1/(p-1)}\right\|_{\lambda,B,w^{\kappa\lambda}} + \left\||d^{*}v|^{1/(p-1)}\right\|_{\lambda,B,w^{\kappa\lambda}}\right). \tag{4.14}$$

□

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