Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 510838, 15 pages
doi:10.1155/2010/510838

Research Article

Stability Analysis for Higher-Order Adjacent Derivative in Parametrized Vector Optimization

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Received 29 March 2010; Accepted 3 August 2010

Academic Editor: Jong Kim

By virtue of higher-order adjacent derivative of set-valued maps, relationships between higher-order adjacent derivative of a set-valued map and its profile map are discussed. Some results concerning stability analysis are obtained in parametrized vector optimization.

1. Introduction

Research on stability and sensitivity analysis is not only theoretically interesting but also practically important in optimization theory. A number of useful results have been obtained in scalar optimization (see [1, 2]). Usually, by stability, we mean the qualitative analysis, which is the study of various continuity properties of the perturbation (or marginal) function (or map) of a family of parametrized optimization problems. On the other hand, by sensitivity, we mean the quantitative analysis, which is the study of derivatives of the perturbation function.

Some authors have investigated the sensitivity of vector optimization problems. In [3], Tanino studied some results concerning the behavior of the perturbation map by using the concept of contingent derivative of set-valued maps for general multiobjective optimization problems. In [4], Shi introduced a weaker notion of set-valued derivative (TP-derivative) and investigated the behavior of contingent derivative for the set-valued perturbation maps in a nonconvex vector optimization problem. Later on, Shi also established sensitivity analysis for a convex vector optimization problem (see [5]). In [6], Kuk et al. investigated the relationships between the contingent derivatives of the perturbation maps (i.e., perturbation map, proper perturbation map, and weak perturbation map) and those of feasible set map in the objective space by virtue of contingent derivative, TP-derivative and Dini derivative. Considering convex vector optimization problems, they also investigated the behavior of the above three kinds of perturbation maps under some convexity assumptions (see [7]).

On the other hand, some interesting results have been proved for stability analysis in vector optimization problems. In [8], Tanino studied some qualitative results concerning the behavior of the perturbation map in convex vector optimization. In [9], Li investigated the continuity and the closedness of contingent derivative of the marginal map in multiobjective optimization. In [10], Xiang and Yin investigated some continuity properties of the mapping which associates the set of efficient solutions to the objective function by virtue of the additive weight method of vector optimization problems and the method of essential solutions.

To the best of our knowledge, there is no paper to deal with the stability of higher-order adjacent derivative for weak perturbation maps in vector optimization problems. Motivated by the work reported in [3–9], in this paper, by higher-order adjacent derivative of set-valued maps, we first discuss some relationships between higher-order adjacent derivative of a set-valued map and its profile map. Then, by virtue of the relationships, we investigate the stability of higher-order adjacent derivative of the perturbation maps.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions. In Section 3, after recalling the concept of higher-order adjacent derivative of set-valued maps, we provide some relationships between the higher-order adjacent derivative of a set-valued map and its profile map. In Section 4, we discuss some stability results of higher-order adjacent derivative for perturbation maps in parametrized vector optimization.

2. Preliminaries

Throughout this paper, let X and Y be two finite dimensional spaces, and let $K \subseteq Y$ be a pointed closed convex cone with a nonempty interior $\operatorname{int} K$, where K is said to be pointed if $K \cap (-K) = \{0\}$. Let $F : X \rightrightarrows Y$ be a set-valued map. The domain and the graph of F are defined by $\operatorname{Dom}(F) = \{x \in X : F(x) \neq \emptyset\}$ and $\operatorname{Graph}(F) = \{(x, y) \in X \times Y : y \in F(x), x \in \operatorname{Dom}(F)\}$, respectively. The so-called profile map $F + K : X \rightrightarrows Y$ is defined by $(F + K)(x) := F(x) + K$, for all $x \in \operatorname{Dom}(F)$.

At first, let us recall some important definitions.

Definition 2.1 (see [11]). Let Q be a nonempty subset of Y. An elements $\hat{y} \in Q$ is said to be a minimal point (resp. weakly minimal point) of Q if $(Q - \hat{y}) \cap (-K) = \{0\}$(resp., $(Q - \hat{y}) \cap (-\operatorname{int} K) = \emptyset$). The set of all minimal points (resp., weakly minimal point) of Q is denoted by $\operatorname{Min}_K Q$ (resp., $\operatorname{WMin}_K Q$).

Definition 2.2 (see [12]). A base for K is a nonempty convex subset B of $K$with $0 \notin B$ such that every $k \in K$, $k \neq 0$ has a unique representation $k = \alpha b$, where $b \in B$ and $\alpha > 0$.

Definition 2.3 (see [13]). The weak domination property is said to hold for a subset H of Y if $H \subseteq \operatorname{WMin}_K H + \operatorname{int} K \cup \{0\}$.

Definition 2.4 (see [14]). Let F be a set-valued map from X to Y.

(i) F is said to be lower semicontinuous (l.s.c) at $\overline{x} \in X$ if for any generalized sequence $\{x_n\}$ with $x_n \to \overline{x}$ and $\overline{y} \in F(\overline{x})$, there exists a generalized sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that $y_n \to \overline{y}$.

(ii) F is said to be upper semicontinuous (u.s.c) at $\overline{x} \in X$ if for any neighborhood $N(F(\overline{x}))$ of $F(\overline{x})$, there exists a neighborhood $N(\overline{x})$ of $\overline{x}$ such that $F(x) \subseteq N(F(\overline{x}))$, for all $x \in N(\overline{x})$.

(iii) F is said to be closed at $\overline{x} \in X$ if for any generalized sequence $(x_n, y_n) \in \mathrm{Graph}(F)$, $(x_n, y_n) \to (\overline{x}, \overline{y})$, it yields $(\overline{x}, \overline{y}) \in \mathrm{Graph}(F)$.

We say that F is l.s.c (resp., u.s.c, closed) on X if it is l.s.c (resp., u.s.c, closed) at each $x \in X$. F is said to be continuous on X if it is both l.s.c and u.s.c on X.

Definition 2.5 (see [14]). F is said to be Lipschitz around $\overline{x} \in X$ if there exist a real number $M > 0$ and a neighborhood $N(\overline{x})$ of $\overline{x}$ such that

$$F(x_1) \subseteq F(x_2) + M\|x_1 - x_2\|B_Y, \quad \forall x_1, x_2 \in N(\overline{x}), \tag{2.1}$$

where B_Y denotes the closed unit ball of the origin in Y.

Definition 2.6 (see [14]). F is said to be uniformly compact near $\overline{x} \in X$ if there exists a neighborhood $N(\overline{x})$ of $\overline{x}$ such that $\bigcup_{x \in N(\overline{x})} F(x)$ is a compact set.

3. Higher-Order Adjacent Derivatives of Set-Valued Maps

In this section, we recall the concept of higher-order adjacent derivative of set-valued maps and provide some basic properties which are necessary in the following section. Throughout this paper, let m be an integer number and $m > 1$.

Definition 3.1 (see [15]). Let $\overline{x} \in C \subseteq X$ and $u_1, \ldots, u_{m-1}$ be elements of X. The set $T_C^{b(m)}(\overline{x}, u_1, \ldots, u_{m-1})$ is called the mth-order adjacent set of C at $(\overline{x}, u_1, \ldots, u_{m-1})$, if and only if, for any $x \in T_C^{b(m)}(\overline{x}, u_1, \ldots, u_{m-1})$, for any sequence $\{h_n\} \subseteq R_+ \setminus \{0\}$ with $h_n \to 0$, there exists a sequence $\{x_n\} \subseteq X$ with $x_n \to x$ such that

$$\overline{x} + h_n u_1 + h_n^2 u_2 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n \in C, \quad \forall n. \tag{3.1}$$

Definition 3.2 (see [15]). Let $(\overline{x}, \overline{y}) \in \mathrm{Graph}(F)$ and $(u_i, v_i) \in X \times Y$, $i = 1, 2, \ldots, m-1$. The mth-order adjacent derivative $D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at $(\overline{x}, \overline{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\mathrm{Graph}\Big(D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\Big) = T^{b(m)}_{\mathrm{Graph}(F)}(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}). \tag{3.2}$$

Proposition 3.3. *Let $(\overline{x}, \overline{y}) \in Graph(F)$ and $(u_i, v_i) \in X \times Y$, $i = 1, 2, \ldots, m-1$. Then, for any $x \in \mathrm{Dom}(D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}))$,*

$$D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) + K \subseteq D^{b(m)}(F + K)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.3}$$

Proof. The proof follows on the lines of Proposition 2.1 in [3] by replacing contingent derivative by mth-order adjacent derivative. □

Note that the converse inclusion of (3.3) may not hold. The following example explains the case where we only take $m = 2, 3$.

Example 3.4. Let $X = Y = R$ and $K = R_+$, let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x \leq 0, \\ \{-1, x^3\} & \text{if } x > 0. \end{cases} \tag{3.4}$$

Let $(\overline{x}, \overline{y}) = (0,0) \in \text{Graph}(F)$ and $(u_1, v_1) = (u_2, v_2) = (1,0)$. For any $x > 0$, we have

$$\begin{aligned} D^{b(2)}F(\overline{x}, \overline{y}, u_1, v_1)(x) &= \{0\}, \qquad D^{b(2)}(F+K)(\overline{x}, \overline{y}, u_1, v_1)(x) = R, \\ D^{b(3)}F(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) &= \{1\}, \qquad D^{b(3)}(F+K)(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) = R. \end{aligned} \tag{3.5}$$

Thus, for any $x > 0$, we have

$$\begin{aligned} D^{b(2)}(F+K)(\overline{x}, \overline{y}, u_1, v_1)(x) &\not\subseteq D^{b(2)}F(\overline{x}, \overline{y}, u_1, v_1)(x) + K, \\ D^{b(3)}(F+K)(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) &\not\subseteq D^{b(3)}F(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) + K. \end{aligned} \tag{3.6}$$

Proposition 3.5. *Let $(\overline{x}, \overline{y}) \in \text{Graph}(F)$ and $(u_i, v_i) \in X \times Y$, $i = 1, 2, \ldots, m-1$. Assume that K has a compact base. Then, for any $x \in \text{Dom}(D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}))$,*

$$\text{WMin}_K D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \subseteq D^{b(m)}F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.7}$$

where $\widetilde{K}$ is a closed convex cone contained in $(\text{int}\, K) \cup \{0\}$.

Proof. If $\text{WMin}_K D^{b(m)}(F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) = \emptyset$, the inclusion holds trivially. Thus, we suppose that $\text{WMin}_K D^{b(m)}(F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \neq \emptyset$. Let $y_0 \in \text{WMin}_K D^{b(m)}(F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$. Then,

$$y_0 \in D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.8}$$

Since $\widetilde{K} \subseteq \text{int}\, K \cup \{0\}$,

$$\begin{aligned} &\text{WMin}_K D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \\ &\quad \subseteq \text{Min}_{\widetilde{K}} D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x), \end{aligned} \tag{3.9}$$

then it follows that

$$y_0 \in \text{Min}_{\widetilde{K}} D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.10}$$

From (3.8) and the definition of mth-order adjacent derivative, we have that for any sequence $\{h_n\} \subseteq R_+ \setminus \{0\}$ with $h_n \to 0$, there exist sequences $\{(x_n, y_n)\}$ with $(x_n, y_n) \to (x, y_0)$ and $\{\widetilde{k}_n\} \subseteq \widetilde{K}$ such that

$$\overline{y} + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m y_n - \widetilde{k}_n \in F\left(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n\right). \tag{3.11}$$

Since $\widetilde{K}$ is a closed convex cone contained in $(\operatorname{int} K) \cup \{0\}$, $\widetilde{K}$ has a compact base. It is clear that $B \cap \widetilde{K}$ is a compact base for $\widetilde{K}$, where B is a compact base for K. In this proposition, we assume that $\widetilde{B}$ is a compact base of $\widetilde{K}$. Since $\widetilde{k}_n \in \widetilde{K}$, there exist $\alpha_n > 0$ and $b_n \in \widetilde{B}$ such that $\widetilde{k}_n = \alpha_n b_n$. Since $\widetilde{B}$ is compact, we may assume without loss of generality that $b_n \to b \in \widetilde{B}$.

Now, we show that $\alpha_n / h_n^m \to 0$. Suppose that $\alpha_n / h_n^m \nrightarrow 0$. Then, for some $\varepsilon > 0$, we may assume, without loss of generality, that $\alpha_n / h_n^m \geq \varepsilon$, for all n. Let $\overline{k}_n = (\varepsilon h_n^m / \alpha_n) \widetilde{k}_n \in \widetilde{K}$. Then, we have

$$\widetilde{k}_n - \overline{k}_n \in \widetilde{K}. \tag{3.12}$$

By (3.11) and (3.12), we obtain that

$$\overline{y} + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m y_n - \overline{k}_n \in F\left(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n\right) + \widetilde{K}. \tag{3.13}$$

From (3.13) and $\overline{k}_n / h_n^m = (\varepsilon / \alpha_n) \widetilde{k}_n = \varepsilon b_n \to \varepsilon b \neq 0$, we have

$$y_0 - \varepsilon b \in D^{b(m)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x), \tag{3.14}$$

which contradicts (3.10). Therefore, $\alpha_n / h_n^m \to 0$ and $y_n - \widetilde{k}_n / h_n^m \to y_0$. Thus, it follows from (3.11) that $y_0 \in D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$, and the proof is complete. □

Remark 3.6. The inclusion of

$$\operatorname{WMin}_K D^{b(m)}(F + K)(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \subseteq D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \tag{3.15}$$

may not hold under the assumptions of Proposition 3.5. The following example explains the case where we only take $m = 2, 3$.

Example 3.7. Let $X = R$ and $Y = R^2$, let $K = R_+^2$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \left\{ y \in R^2 : y = \left(x^3, x^3\right) \right\}. \tag{3.16}$$

Suppose that $(\overline{x},\overline{y}) = (0,(0,0)) \in \mathrm{Graph}(F)$, $(u_1,v_1) = (u_2,v_2) = (1,(0,0))$. Then, for any $x \in X$,

$$\begin{aligned}
D^{b(2)}F(\overline{x},\overline{y},u_1,v_1)(x) &= \{(0,0)\},\\
D^{b(3)}F(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \{(1,1)\},\\
D^{b(2)}(F+K)(\overline{x},\overline{y},u_1,v_1)(x) &= \left\{(y_1,y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\right\},\\
D^{b(3)}(F+K)(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \left\{(y_1,y_2) \in R^2 : y_1 \geq 1, y_2 \geq 1\right\}.
\end{aligned} \tag{3.17}$$

Naturally, we have

$$\begin{aligned}
\mathrm{WMin}_K D^{b(2)}(F+K)(\overline{x},\overline{y},u_1,v_1)(x) &= \left\{(y_1,y_2) \in R^2 : y_1y_2 = 0, y_1 \geq 0, y_2 \geq 0\right\},\\
\mathrm{WMin}_K D^{b(3)}(F+K)(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \left\{(y_1,y_2) \in R^2 : y_1 \geq 1, y_2 = 1\right\}\\
&\quad \cup \left\{(y_1,y_2) \in R^2 : y_1 = 1, y_2 \geq 1\right\}.
\end{aligned} \tag{3.18}$$

Thus, for any $x \in X$,

$$\begin{aligned}
\mathrm{WMin}_K D^{b(2)}(F+K)(\overline{x},\overline{y},u_1,v_1)(x) &\not\subseteq D^{b(2)}F(\overline{x},\overline{y},u_1,v_1)(x),\\
\mathrm{WMin}_K D^{b(3)}(F+K)(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &\not\subseteq D^{b(3)}F(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x).
\end{aligned} \tag{3.19}$$

Proposition 3.8. *Let* $(\overline{x},\overline{y}) \in \mathrm{Graph}(F)$, *and* $(u_i,v_i) \in X \times Y$, $i = 1,2,\ldots,m-1$, *and let* K *has a compact base. Suppose that* $P(x) := D^{b(m)}(F+\widetilde{K})(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$ *fulfills the weak domination property for any* $x \in \mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$. *Then, for any* $x \in \mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$,

$$\begin{aligned}
&\mathrm{WMin}_K D^{b(m)}\left(F+\widetilde{K}\right)(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\\
&\quad = \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x),
\end{aligned} \tag{3.20}$$

where $\widetilde{K}$ *is a closed convex cone contained in* $(\mathrm{int}\,K) \cup \{0\}$.

Proof. Let $y_0 \in \mathrm{WMin}_K D^{b(m)}(F+\widetilde{K})(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$. Then,

$$y_0 \in D^{b(m)}\left(F+\widetilde{K}\right)(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x). \tag{3.21}$$

By Proposition 3.5, we also have $y_0 \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$.

Suppose that $y_0 \notin \mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$. Then, there exists $y' \in D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ such that

$$y_0 - y' \in \operatorname{int} K. \tag{3.22}$$

From $y' \in D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ and Proposition 3.3, we have

$$y' \in D^{b(m)} \left(F + \widetilde{K} \right) (\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.23}$$

So, by (3.21), (3.22), and (3.23), $y_0 \notin \mathrm{WMin}_K D^{b(m)} (F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$, which leads to a contradiction. Thus, $y_0 \in \mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$.

Conversely, let $y_0 \in \mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$. Then,

$$y_0 \in D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \subseteq D^{b(m)} \left(F + \widetilde{K} \right) (\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.24}$$

Suppose that $y_0 \notin \mathrm{WMin}_K D^{b(m)} (F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$. Then, there exists $y' \in D^{b(m)} (F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ such that

$$y_0 - y' = k \in \operatorname{int} K. \tag{3.25}$$

Since $P(x)$ fulfills the weak domination property for any $x \in \mathrm{Dom}(D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}))$, there exists $k' \in \operatorname{int} K \cup \{0\}$ such that

$$y' - k' \in \mathrm{WMin}_K D^{b(m)} \left(F + \widetilde{K} \right) (\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.26}$$

From (3.25) and (3.26), we have

$$y_0 - k - k' \in \mathrm{WMin}_K D^{b(m)} \left(F + \widetilde{K} \right) (\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x). \tag{3.27}$$

It follows from Proposition 3.5 and (3.27) that

$$y_0 - k - k' \in D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x), \tag{3.28}$$

which contradicts $y_0 \in \mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$. Thus, $y_0 \in \mathrm{WMin}_K D^{b(m)} (F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$, and the proof is complete. □

Obviously, Example 3.4 can also show that the weak domination property of $P(x)$ is essential for Proposition 3.8.

Remark 3.9. From Example 3.7, the equality of

$$\begin{aligned} &\mathrm{WMin}_K D^{b(m)}(F+K)(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(x) \\ &\quad = \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(x) \end{aligned} \tag{3.29}$$

may still not hold under the assumptions of Proposition 3.8.

Proposition 3.10. *Let $(\overline{x},\overline{y}) \in Graph(F)$ and $(u_i,v_i) \in X \times Y$, $i = 1,2,\dots,m-1$. Suppose that F is Lipschitz at $\overline{x}$. Then, $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})$ is continuous on* $\mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1}))$.

Proof. Since F is Lipschitz at $\overline{x}$, there exist a real number $M > 0$ and a neighborhood $N(\overline{x})$ of $\overline{x}$ such that

$$F(x_1) \subseteq F(x_2) + M\|x_1 - x_2\|B_Y, \quad \forall x_1, x_2 \in N(\overline{x}). \tag{3.30}$$

First, we prove that $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})$ is l.s.c. at $\widehat{x} \in \mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1}))$. Indeed, for any $\widehat{y} \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(\widehat{x})$. From the definition of mth-order adjacent derivative, we have that for any sequence $\{h_n\} \subseteq R_+ \setminus \{0\}$ with $h_n \to 0$, there exists a sequence $\{(\widehat{x}_n,\widehat{y}_n)\}$ with $(\widehat{x}_n,\widehat{y}_n) \to (\widehat{x},\widehat{y})$ such that

$$\overline{y} + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m \widehat{y}_n \in F\Big(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \widehat{x}_n\Big). \tag{3.31}$$

Take any $x \in X$ and $x_n \to x$. Obviously, $\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n$, $\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \widehat{x}_n \in N(\overline{x})$, for any n sufficiently large. Therefore, by (3.30), we have

$$\begin{aligned} &F\Big(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \widehat{x}_n\Big) \\ &\quad \subseteq F\Big(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n\Big) + M h_n^m \|\widehat{x}_n - x_n\| B_Y. \end{aligned} \tag{3.32}$$

So, with (3.31), there exists $-b_n \in B_Y$ such that

$$\overline{y} + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m (\widehat{y}_n + M\|\widehat{x}_n - x_n\| b_n) \in F\Big(\overline{x} + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n\Big). \tag{3.33}$$

We may assume, without loss of generality, that $b_n \to b \in B_Y$. Thus, by (3.33),

$$\widehat{y} + M\|\widehat{x} - x\| b \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(x). \tag{3.34}$$

It follows from (3.34) that for any sequence $\{x_k\}$ with $x_k \to \widehat{x}$, $\widehat{y} \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(\widehat{x})$, there exists a sequence $\{y_k\}$ with

$$y_k := \widehat{y} + M\|\widehat{x} - x_k\| b \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(x_k). \tag{3.35}$$

Obviously, $y_k \to \hat{y}$. Hence, $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is l.s.c. on $\mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$.

We will prove that $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is u.s.c. on $\hat{x}\in\mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$. In fact, for any $\varepsilon>0$, we consider the neighborhood $\hat{x}+(\varepsilon/M)B_X$ of $\hat{x}$. Let $x\in\hat{x}+(\varepsilon/M)B_X$ and $y\in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$. From the definition of $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$, we have that for any sequence $\{h_n\}\subseteq R_+\setminus\{0\}$ with $h_n\to 0$, there exists a sequence $\{(x_n,y_n)\}$ with $(x_n,y_n)\to(x,y)$ such that

$$\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^my_n\in F\Big(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\Big). \tag{3.36}$$

Take any $\hat{x}_n\to\hat{x}$. Obviously, $\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n,\ \overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^m\hat{x}_n\in N(\overline{x})$, for any n sufficiently large. Therefore, by (3.30), we have

$$\begin{aligned}F\Big(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\Big)\subseteq{}&F\Big(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^m\hat{x}_n\Big)\\&+Mh_n^m\|x_n-\hat{x}_n\|B_Y.\end{aligned} \tag{3.37}$$

Similar to the proof of l.s.c., there exists $b\in B_Y$ such that

$$y+M\|x-\hat{x}\|b\in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\hat{x}). \tag{3.38}$$

Thus, $y\in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\hat{x})+\varepsilon B_Y$. Hence, $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is u.s.c. on $\mathrm{Dom}(D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$, and the proof is complete. □

4. Continuity of Higher-Order Adjacent Derivative for Weak Perturbation Map

In this section, we consider a family of parametrized vector optimization problems. Let F be a set-valued map from U to Y, where U is the Banach space of perturbation parameter vectors, Y is the objective space, and F is considered as the feasible set map in the objective space. In the optimization problem corresponding to each parameter valued x, our aim is to find the set of weakly minimal points of the feasible objective valued set $F(x)$. Hence, we define another set-valued map S from U to Y by

$$S(x)=\mathrm{WMin}_KF(x),\quad \text{for any } x\in U. \tag{4.1}$$

The set-valued map S is called the weak perturbation map. Throughout this section, we suppose that $\widetilde{K}$ is a closed convex cone contained in $(\mathrm{int}\,K)\cup\{0\}$.

Definition 4.1 (see [11]). F is said to be K-minicomplete by S near $\overline{x}$ if $F(x)\subseteq S(x)+K$, for any $x\in N(\overline{x})$, where $N(\overline{x})$ is a neighborhood of $\overline{x}$.

Remark 4.2. Since $S(x) \subseteq F(x)$, the K-minicompleteness of F by S near $\overline{x}$ implies that

$$S(x) + K = F(x) + K, \quad \text{for any } x \in N(\overline{x}). \tag{4.2}$$

Hence, if F is K-minicomplete by S near $\overline{x}$, then, for any $\overline{y} \in S(\overline{x})$

$$D^{b(m)}(F + K)(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}) = D^{b(m)}(S + K)(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}). \tag{4.3}$$

The following lemma palys a crucial role in this paper.

Lemma 4.3. *Let $(\overline{x}, \overline{y}) \in \mathrm{Graph}(S)$ and $(u_i, v_i) \in U \times Y$, $i = 1, 2, \dots, m-1$, and let K have a compact base. Suppose that the following conditions are satisfied:*

(i) *$P(x) := D^{b(m)}(F + \widetilde{K})(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$ fulfills the weak domination property for any $x \in \mathrm{Dom}(D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}))$;*

(ii) *F is Lipschitz at $\overline{x}$;*

(iii) *F is $\widetilde{K}$-minicomplete by S near $\overline{x}$.*

Then, for any $x \in \mathrm{Dom}(D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}))$,

$$D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) = \mathit{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x). \tag{4.4}$$

Proof. We first prove that

$$\mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \subseteq D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x). \tag{4.5}$$

In fact, from Proposition 3.5, Proposition 3.8, and the $\widetilde{K}$-minicompleteness of F by S near $\overline{x}$, we have

$$\begin{aligned}
&\mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\
&\quad = \mathrm{WMin}_K D^{b(m)} \left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\
&\quad = \mathrm{WMin}_K D^{b(m)} \left(S + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\
&\quad \subseteq D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x).
\end{aligned} \tag{4.6}$$

Thus, result (4.5) holds.

Now, we prove that

$$D^{b(m)} S(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \subseteq \mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x). \tag{4.7}$$

In fact, assume that $y \in D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$. Then, for any sequence $\{h_n\} \subseteq R_+ \setminus \{0\}$ with $h_n \to 0$, there exists a sequence $\{(x_n,y_n)\}$ with $(x_n,y_n) \to (x,y)$ such that

$$\begin{aligned}\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^my_n &\in S\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\right)\\ &\subseteq F\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\right).\end{aligned}\tag{4.8}$$

Suppose that $y \notin \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$. Then, there exists $\widehat{y} \in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$ such that $y-\widehat{y} \in \operatorname{int} K$. Thus, for the preceding sequence $\{h_n\}$, there exists a sequence $\{(\widehat{x}_n,\widehat{y}_n)\}$ with $(\widehat{x}_n,\widehat{y}_n) \to (x,y)$ such that

$$\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^m\widehat{y}_n \in F\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^m\widehat{x}_n\right).\tag{4.9}$$

Obviously, $\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n, \overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^m\widehat{x}_n \in N(\overline{x})$, for any n sufficiently large. Therefore, by (ii), we have

$$\begin{aligned}F\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^m\widehat{x}_n\right) &\subseteq F\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\right)\\ &\quad+Mh_n^m\|\widehat{x}_n-x_n\|B_Y.\end{aligned}\tag{4.10}$$

So, with (4.9), there exists $-b_n \in B_Y$ such that

$$\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^m\left(\widehat{y}_n+M\|\widehat{x}_n-x_n\|b_n\right) \in F\left(\overline{x}+h_nu_1+\cdots+h_n^{m-1}u_{m-1}+h_n^mx_n\right).\tag{4.11}$$

Since $y_n-(\widehat{y}_n+M\|\widehat{x}_n-x_n\|b_n) \to y-\widehat{y}$ and $y-\widehat{y} \in \operatorname{int} K$, $y_n-(\widehat{y}_n+M\|\widehat{x}_n-x_n\|b_n) \in \operatorname{int} K$, for n sufficiently large. Then, we have

$$\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^my_n-\left(\overline{y}+h_nv_1+\cdots+h_n^{m-1}v_{m-1}+h_n^m\left(\widehat{y}_n+M\|\widehat{x}_n-x_n\|b_n\right)\right) \in \operatorname{int} K,\tag{4.12}$$

which contradicts (4.8). Then, y $\in$ $\mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)$. This completes the proof. □

The following example shows that the $\widetilde{K}$-minicompleteness of F is essential in Lemma 4.3, where we only take $m=2,3$.

Example 4.4 (F is not $\widetilde{K}$-minicomplete by S near $\overline{x}$). Let $U=R$, $Y=R^2$ and $K=R^2_+$, and let $F: U \rightrightarrows Y$ be defined by

$$F(x)=\left\{(y_1,y_2)\in R^2 : y_1\geq 0, y_2\geq 0\right\}\cup\left\{(y_1,y_2)\in R^2 : y_2>\left|y_1\right|\right\}.\tag{4.13}$$

Then, for any $x \in U$,

$$F(x) + \widetilde{K} = F(x), \qquad S(x) = \left\{ (y_1, y_2) \in R^2 : y_1 \geq 0, y_2 = 0 \right\}. \tag{4.14}$$

Suppose that $(\overline{x}, \overline{y}) = (0, (0, 0)) \in \text{Graph}(S)$, $(u_1, v_1) = (u_2, v_2) = (1, (0, 0))$. Then, F is Lipschitz at $\overline{x}$, and for any $x \in U$,

$$\begin{aligned} D^{b(2)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1)(x) &= D^{b(3)}\left(F + \widetilde{K}\right)(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) \\ &= \left\{ (y_1, y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0 \right\} \\ &\quad \cup \left\{ (y_1, y_2) \in R^2 : y_2 \geq |y_1| \right\} \end{aligned} \tag{4.15}$$

fulfills the weak domination property. We also have

$$\begin{aligned} D^{b(2)}F(\overline{x}, \overline{y}, u_1, v_1)(x) &= D^{b(3)}F(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) \\ &= \left\{ (y_1, y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0 \right\} \\ &\quad \cup \left\{ (y_1, y_2) \in R^2 : y_2 \geq |y_1| \right\}. \end{aligned} \tag{4.16}$$

On the other hand,

$$\begin{aligned} D^{b(2)}S(\overline{x}, \overline{y}, u_1, v_1)(x) &= D^{b(3)}S(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) = S(x), \\ \text{WMin}_K D^{b(2)}F(\overline{x}, \overline{y}, u_1, v_1)(x) &= \text{WMin}_K D^{b(3)}F(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) \\ &= \left\{ (y_1, y_2) \in R^2 : y_1 \geq 0, y_2 = 0 \right\} \\ &\quad \cup \left\{ (y_1, y_2) \in R^2 : y_2 = -y_1, y_1 < 0 \right\}. \end{aligned} \tag{4.17}$$

Thus, for any $x \in X$,

$$\begin{aligned} D^{b(2)}S(\overline{x}, \overline{y}, u_1, v_1)(x) &\neq \text{WMin}_K D^{b(2)}F(\overline{x}, \overline{y}, u_1, v_1)(x), \\ D^{b(3)}S(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x) &\neq \text{WMin}_K D^{b(3)}F(\overline{x}, \overline{y}, u_1, v_1, u_2, v_2)(x). \end{aligned} \tag{4.18}$$

Theorem 4.5. *Let $(\overline{x}, \overline{y}) \in Graph(S)$ and $(u_i, v_i) \in U \times Y, i = 1, 2, \ldots, m-1$. Then, $D^{b(m)}S(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ is closed on* $\text{Dom}(D^{b(m)}S(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}))$.

Proof. From the definition of $D^{b(m)}S(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$, we have that

$$\text{Graph}\left(D^{b(m)}S(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\right) = T^{b(m)}_{\text{Graph}(S)}(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}). \tag{4.19}$$

Since $T^{b(m)}_{\mathrm{Graph}(S)}(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is closed set, $D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is closed on $\mathrm{Dom}(D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$, and the proof is complete. □

Theorem 4.6. *Let* $(\overline{x},\overline{y})\in \mathrm{Graph}(S)$ *and* $(u_i,v_i)\in U\times Y, i=1,2,\ldots,m-1$. *If* Y *is a compact space, then* $D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ *is u.s.c. on* $\mathrm{Dom}(D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$.

Proof. Since Y is a compact space, it follows from Corollary 9 of Chapter 3 in [14] and Theorem 4.5 that $D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is u.s.c. on $\mathrm{Dom}(D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$. Thus, the proof is complete. □

Theorem 4.7. *Let* $\widehat{x}\in \mathrm{Dom}(D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$. *Suppose that* $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\widehat{x})$ *is a compact set and the assumptions of Lemma 4.3 are satisfied. Then,* $D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ *is u.s.c. at* $\widehat{x}$.

Proof. It follows from Lemma 4.3 and Theorem 4.5 that $\mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is closed. By Proposition 3.10, we have that $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is u.s.c. at $\widehat{x}$. Since $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\widehat{x})$ is a compact set, it follows from Theorem 8 of Chapter 3 in [14] that

$$\begin{aligned} D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}) &= \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}) \\ &= \left[\mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})\right] \\ &\quad \cap D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}) \end{aligned} \tag{4.20}$$

is u.s.c. at $\widehat{x}$, and the proof is complete. □

Now, we give an example to illustrate Theorem 4.7, where we also take $m=2,3$.

Example 4.8. Let $U=[0,1]$, $Y=R^2$, and $K=R^2_+$, and let $F: U\rightrightarrows Y$ be defined by

$$F(x)=\left\{(y_1,y_2)\in R^2 : 0\le y_1\le x^3, 0\le y_2\le x^3\right\}. \tag{4.21}$$

Then, for any $x\in U$,

$$S(x)=\left\{(y_1,y_2)\in R^2 : 0\le y_1\le x^3, y_2=0\right\}\cup\left\{(y_1,y_2)\in R^2 : y_1=0, 0\le y_2\le x^3\right\}. \tag{4.22}$$

Suppose that $(\overline{x},\overline{y})=(0,(0,0))\in \mathrm{Graph}(S)$, $\widehat{x}=1/3$, $(u_1,v_1)=(u_2,v_2)=(1,(0,0))$ and $\widetilde{K}=\{(y_1,y_2)\in R^2_+ : (1/4)y_2\le y_1\le 4y_2\}$. Obviously, K has a compact base, F is Lipschitz at $\overline{x}$, and F is $\widetilde{K}$-minicomplete by S near $\overline{x}$. By direct calculating, for any $x\in U$,

$$\begin{aligned} D^{b(2)}\left(F+\widetilde{K}\right)(\overline{x},\overline{y},u_1,v_1)(x) &= \widetilde{K}, \\ D^{b(3)}\left(F+\widetilde{K}\right)(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \left\{(y_1,y_2)\in R^2 : 0\le y_1\le 4y_2+1, 0\le y_2\le 4y_1+1\right\} \end{aligned} \tag{4.23}$$

fulfill the weak domination property, which is a strong property for a set-valued map. We also have

$$\begin{aligned}
D^{b(2)}F(\overline{x},\overline{y},u_1,v_1)(x) &= \{(0,0)\},\\
D^{b(2)}S(\overline{x},\overline{y},u_1,v_1)(x) &= \{(0,0)\},\\
D^{b(3)}F(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \left\{(y_1,y_2)\in R^2 : 0\le y_1\le 1, 0\le y_2\le 1\right\},\\
D^{b(3)}S(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x) &= \left\{(y_1,y_2)\in R^2 : 0\le y_1\le 1, y_2=0\right\}\\
&\quad\cup\left\{(y_1,y_2)\in R^2 : y_1=0, 0\le y_2\le 1\right\}.
\end{aligned}\tag{4.24}$$

Thus, the conditions of Theorem 4.7 are satisfied. Obviously, both $D^{b(2)}S(\overline{x},\overline{y},u_1,v_1)(x)$ and $D^{b(3)}S(\overline{x},\overline{y},u_1,v_1,u_2,v_2)(x)$ are u.s.c at $\widehat{x}$.

Theorem 4.9. *Let $\widehat{x}\in \mathrm{Dom}(D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}))$. Suppose that $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is uniformly compact near $\widehat{x}$ and the assumptions of Lemma 4.3 are satisfied. Then, $D^{b(m)}S(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is l.s.c. at $\widehat{x}$.*

Proof. By Lemma 4.3, it suffices to prove that $\mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is l.s.c. at $\widehat{x}$. Let $x_n\to\widehat{x}$ and

$$\widehat{y}\in \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\widehat{x}).\tag{4.25}$$

By Proposition 3.10, we have that $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is l.s.c. at $\widehat{x}$. Then, there exists a sequence $\{y_n\}$ with $y_n\in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},\mathrm{v}_{m-1})(x_n)$ such that $y_n\to\widehat{y}$. Since $\widetilde{K}\subseteq(\mathrm{int}\,K)\cup\{0\}$,

$$\mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\subseteq \mathrm{Min}_{\widetilde{K}} D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x).\tag{4.26}$$

Then, for any sequence $\{y'_n\}$ with $y'_n\in \mathrm{WMin}_K D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x_n)$, we have

$$y'_n\in \mathrm{Min}_{\widetilde{K}} D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x_n),\tag{4.27}$$

then it follows that

$$y_n-y'_n\in\widetilde{K}.\tag{4.28}$$

Since $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is uniformly compact near $\widehat{x}$, we may assume, without loss of generality, that ${y'}_n\to y$. It follows from the closedness of $D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ that

$$y\in D^{b(m)}F(\overline{x},\overline{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(\widehat{x}).\tag{4.29}$$

From (4.28) and $\widetilde{K}$ is closed, we have $\widehat{y} - y \in \widetilde{K} \subseteq \operatorname{int} K \cup \{0\}$. Then, it follows from (4.25) and (4.29) that $y = \widehat{y}$. Thus, $\mathrm{WMin}_K D^{b(m)} F(\overline{x}, \overline{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ is l.s.c. at $\widehat{x}$, and the proof is complete. □

It is easy to see that Example 4.8 can also illustrate Theorem 4.9.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions which helped to improve the paper. This research was partially supported by the National Natural Science Foundation of China (Grant no. 10871216) and Chongqing University Postgraduates Science and Innovation Fund (Grant no. 201005B1A0010338).

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