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Research Article

On Lyapunov-Type Inequalities for Two-Dimensional Nonlinear Partial Systems

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We establish a new Laypunov-type inequality for two nonlinear systems of partial differential equations and the discrete analogue is also established. As application, boundness of the two-dimensional Emden-Fowler-type equation is proved.

1. Introduction

In a celebrated paper of 1893, Liapunov [1] proved the following well-known inequality: if y is a nontrivial solution of

$$y'' + q(t)y = 0, \tag{1.1}$$

on an interval containing the points a and b $(a < b)$ such that $y(a) = y(b) = 0$, then

$$4 < (b - a)\int_a^b |q(s)|\,ds. \tag{1.2}$$

Since the appearance of Liapunov's fundamental paper [1], considerable attention has been given to various extensions and improvements of the Lyapunov-type inequality from different viewpoints [2–7]. In particular, the Lyapunov-type inequalities for the following nonlinear system of differential equations were given in [8]

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), \\ u'(t) &= -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t). \end{aligned} \tag{1.3}$$

In this paper, we obtain new Lyapunov-type inequalities for the two-dimensional nonlinear system and discrete nonlinear system, respectively.

2. The Lyapunov-Type Integral Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned} \frac{\partial^2 x(s,t)}{\partial s \partial t} &= \alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t), \\ \frac{\partial^2 u(s,t)}{\partial s \partial t} &= -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t) - \alpha_1(s,t)u(s,t). \end{aligned} \tag{2.1}$$

We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (2.1), and furthermore, (2.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1$, $\beta > 1$ are real constants;

(ii) $\beta_1(s,t)$, $\beta_2(s,t) : [s_0, \infty) \times [t_0, \infty) \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $\beta_1(s,t) > 0$ for $(s,t) \in [s_0, \infty) \times [t_0, \infty)$;

(iii) $\alpha_1(s,t) : [s_0, \infty) \times [t_0, \infty) \to \mathbb{R}$ is a continuous function.

Theorem 2.1. *Let the hypotheses (i)–(iii) hold. If the nonlinear system* (2.1) *has a real solution* $(x(s,t), u(s,t))$ *such that* $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ *for* $(s,t) \in [a,b] \times [c,d]$, *and* $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t) + (\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)$ *and* $x(s,t)$ *is not identically zero on* $[a,b] \times [c,d]$, *where* $a, b, c, d \in \mathbb{R}$ *with* $a < b$, $c < d$, *then*

$$2 \le \int_a^b \int_c^d |\alpha_1(s,t)| dt\, ds + M^{\beta/\alpha - 1} \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t) dt\, ds \right)^{1/\alpha}, \tag{2.2}$$

where $(1/\alpha) + (1/\gamma) = 1$, $M = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)|$, *and* $\beta_2^+(s,t) = \max_{\substack{a<s<b \\ c<t<d}} \{\beta_2(s,t), 0\}$ *is the nonnegative part of* $\beta_2(s,t)$.

Proof. Since $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x(s,t)$ is not identically zero on $[a,b] \times [c,d]$, we can choose $(\tau, \sigma) \in (a,b) \times (c,d)$ such that $|x(\tau,\sigma)| = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)| > 0$.

Let $M = |x(\tau,\sigma)| > 0$. Integrating the first equation of system (2.1) over t from c to σ and over s from a to τ, respectively, we obtain

$$\int_a^\tau \int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s \partial t} dt\, ds = \int_a^\tau \int_c^\sigma \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
\int_a^\tau \int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s \partial t} dt\, ds &= \int_a^\tau \int_c^\sigma \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s}\right) dt\, ds \\
&= \int_a^\tau \left[\int_c^\sigma \frac{\partial x(s,t)}{\partial s}\bigg|_t dt\right] ds \\
&= \int_a^\tau \frac{\partial x(s,\sigma)}{\partial s} ds - \int_a^\tau \frac{\partial x(s,c)}{\partial s} ds \\
&= x(\tau,\sigma) - x(a,\sigma) - x(\tau,c) + x(a,c) \\
&= x(\tau,\sigma).
\end{aligned} \tag{2.4}$$

Hence,

$$x(\tau,\sigma) = \int_a^\tau \int_c^\sigma \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds, \tag{2.5}$$

and similarly, we have

$$x(\tau,\sigma) = \int_\tau^b \int_\sigma^d \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds. \tag{2.6}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \le \int_a^\tau \int_c^\sigma |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_a^\tau \int_c^\sigma \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds, \tag{2.7}$$

$$|x(\tau,\sigma)| \le \int_\tau^b \int_\sigma^d |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_\tau^b \int_\sigma^d \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds. \tag{2.8}$$

Summing (2.7) and (2.8), we obtain

$$2|x(\tau,\sigma)| \le \int_a^b \int_c^d |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds. \tag{2.9}$$

By using Hölder inequality on the second integral of the right side of (2.9) with indices α and γ, we have

$$\begin{aligned}
&\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1}dt\,ds \\
&\quad= \int_a^b\int_c^d \beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1}dt\,ds \\
&\quad\le \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}dt\,ds\right)^{1/\alpha} \\
&\quad= \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma}dt\,ds\right)^{1/\alpha},
\end{aligned} \tag{2.10}$$

where $(1/\alpha)+(1/\gamma)=1$.

Therefore, we obtain from (2.9)

$$\begin{aligned}
2|x(\tau,\sigma)| &\le \int_a^b\int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds \\
&\quad+\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma}dt\,ds\right)^{1/\alpha}.
\end{aligned} \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}(x(s,t)u(s,t)) &= \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s}\cdot u(s,t)+x(s,t)\cdot\frac{\partial u(s,t)}{\partial s}\right) \\
&= \frac{\partial^2 x(s,t)}{\partial s\partial t}\cdot u(s,t)+\frac{\partial x(s,t)}{\partial s}\cdot\frac{\partial u(s,t)}{\partial t} \\
&\quad+\frac{\partial x(s,t)}{\partial t}\cdot\frac{\partial u(s,t)}{\partial s}+x(s,t)\cdot\frac{\partial^2 u(s,t)}{\partial s\partial t}.
\end{aligned} \tag{2.12}$$

Multiplying the first equation of (2.1) by $u(s,t)$ and the second one by $x(s,t)$, adding the result, and noting $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t)+(\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)=0$, we have

$$\frac{\partial^2}{\partial s\partial t}[x(s,t)u(s,t)] = \beta_1(s,t)|u(s,t)|^{\gamma}-\beta_2(s,t)|x(s,t)|^{\beta}. \tag{2.13}$$

Integrating the left side of (2.13) over t from c to d and over s from a to b, respectively, we get

$$\begin{aligned}
&\int_a^b \int_c^d \frac{\partial^2}{\partial s \partial t}[x(s,t)u(s,t)]dt\,ds \\
&= \int_a^b \int_c^d \frac{\partial}{\partial t}\left[\frac{\partial(x(s,t)u(s,t))}{\partial s}\right]dt\,ds \\
&= \int_a^b \left[\int_c^d \frac{\partial(x(s,t)u(s,t))}{\partial s}\Big|_t dt\right]ds \\
&= \int_a^b \frac{\partial(x(s,d)u(s,d))}{\partial s}ds - \int_a^b \frac{\partial(x(s,c)u(s,c))}{\partial s}ds \\
&= x(b,d)u(b,d) - x(a,d)u(a,d) - x(b,c)u(b,c) + x(a,c)u(a,c).
\end{aligned} \tag{2.14}$$

Now integrating both sides of (2.13) over t from c to d and over s from a to b, respectively, and noting $x(a,t) = x(b,t) = 0$, we get

$$\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^\gamma dt\,ds = \int_a^b \int_c^d \beta_2(s,t)|x(s,t)|^\beta dt\,ds. \tag{2.15}$$

Substituting equality (2.15) by (2.11), we have

$$\begin{aligned}
2|x(\tau,\sigma)| \le &\int_a^b \int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds \\
&+ \left(\int_a^b \int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2(s,t)|x(s,t)|^\beta dt\,ds\right)^{1/\alpha}.
\end{aligned} \tag{2.16}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)| > 0$ and $\beta_2^+(s,t) = \max_{\substack{a<s<b \\ c<t<d}} \{\beta_2(s,t), 0\}$, we obtain

$$2 \le \int_a^b \int_c^d |\alpha_1(s,t)|dt\,ds + M^{\beta/\alpha - 1}\left(\int_a^b \int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t)dt\,ds\right)^{1/\alpha}. \tag{2.17}$$

The proof is complete. □

Remark 2.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (2.2), and with suitable changes, (2.2) changes to the following result:

$$2 \le \int_a^b |\alpha_1(t)|dt + M^{\beta/\alpha-1}\left(\int_a^b \beta_1(t)dt\right)^{1/\gamma}\left(\int_a^b \beta_2^+(t)dt\right)^{1/\alpha}. \tag{2.18}$$

This is just a new Lyapunov-type inequality which was given by Tiryaki et al. [8].

3. The Lyapunov-Type Discrete Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned} \Delta_1\Delta_2 x(s,t) &= \alpha_1(s,t)x(s+1,t+1) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t), \\ \Delta_1\Delta_2 u(s,t) &= -\beta_2(s,t)|x(s+1,t+1)|^{\beta-2}x(s+1,t+1) - \alpha_1(s,t)u(s,t), \end{aligned} \tag{3.1}$$

where $s,t \in \mathbb{Z}$, Δ_1 denotes the forward difference operator for s, that is, $\Delta_1 x(s,t) = x(s+1,t) - x(s,t)$, and Δ_2 denotes the forward difference operator for t, that is, $\Delta_2 x(s,t) = x(s,t+1) - x(s,t)$. We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (3.1), and furthermore, (3.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1, \beta > 1$ are real constants;

(ii) $\beta_1(s,t), \beta_2(s,t)$ are real-valued functions such that $\beta_1(s,t) > 0$ for all $s,t \in \mathbb{Z}$;

(iii) $\alpha_1(s,t)$ is a real-valued function for all $s,t \in \mathbb{Z}$.

Theorem 3.1. *Let the hypotheses* (*i*)–(*iii*) *hold. Assume* $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ *and* $n_1 < m_1 - 2, n_2 < m_2 - 2$. *If the nonlinear system* (3.1) *has a real solution* $(x(s,t), u(s,t))$ *such that* $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ *for all* $(s,t) \in [n_1,m_1] \times [n_2,m_2]$, *and* $\Delta_1 x(s,t+1) \cdot \Delta_2 u(s,t) + \Delta_2 x(s+1,t) \cdot \Delta_1 u(s,t) = 0$ *and* $x(s,t)$ *is not identically zero on* $[n_1,m_1] \times [n_2,m_2]$, *then*

$$2 \le \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)| + M^{\beta/\alpha-1}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)\right)^{1/\alpha}, \tag{3.2}$$

where $(1/\alpha) + (1/\gamma) = 1, M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1 \\ n_2+1<t<m_2-1}}|x(s,t)|$, *and* $\beta_2^+(s,t) = \max_{\substack{n_1+1<s<m_1-1 \\ n_2+1<t<m_2-1}}\{\beta_2(s,t), 0\}$.

Proof. Let $(x(s,t), u(s,t))$ be nontrivial real solution of system (3.1) such that $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ and $x(s,t)$ is not identically zero on $[n_1,m_1] \times [n_2,m_2]$.

Then multiplying the first equation of (3.1) by $u(s,t)$ and the second one by $x(s+1,t+1)$, adding the result, and noting $\Delta_1 x(s,t+1)\cdot\Delta_2 u(s,t)+\Delta_2 x(s+1,t)\cdot\Delta_1 u(s,t)=0$, and

$$\begin{aligned}
&\Delta_1\Delta_2[x(s,t)u(s,t)]\\
&\quad= \Delta_2((x(s+1,t)-x(s,t))u(s,t)+x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad= \Delta_2((x(s+1,t)-x(s,t))u(s,t))+\Delta_2(x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad= (x(s+1,t+1)-x(s,t+1)-(x(s+1,t)-x(s,t)))u(s,t)\\
&\qquad+(x(s+1,t+1)-x(s,t+1))(u(s,t+1)-u(s,t))\\
&\qquad+(x(s+1,t+1)-x(s+1,t))(u(s+1,t)-u(s,t))\\
&\qquad+x(s+1,t+1)(u(s+1,t+1)-u(s,t+1)-(u(s+1,t)-u(s,t)))\\
&\quad= (\Delta_1\Delta_2 x(s,t))u(s,t)+\Delta_1 x(s,t+1)\Delta_2 u(s,t)\\
&\qquad+\Delta_2 x(s+1,t)\Delta_1 u(s,t)+x(s+1,t+1)(\Delta_1\Delta_2 u(s,t)),
\end{aligned}\tag{3.3}$$

we have

$$\Delta_1\Delta_2[x(s,t)u(s,t)]=\beta_1(s,t)|u(s,t)|^{\gamma}-\beta_2(s,t)|x(s+1,t+1)|^{\beta}.\tag{3.4}$$

Summing the left side of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, we have

$$\begin{aligned}
&\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\Delta_1\Delta_2(x(s,t)u(s,t))\\
&\quad=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}(x(s+1,t+1)u(s+1,t+1)-x(s+1,t)u(s+1,t)\\
&\qquad\qquad-(x(s,t+1)u(s,t+1)-x(s,t)u(s,t)))\\
&\quad=\sum_{s=n_1}^{m_1-1}(x(s+1,m_2)u(s+1,m_2)-x(s,m_2)u(s,m_2)\\
&\qquad\quad-(x(s+1,n_2)u(s+1,n_2)-x(s,n_2)u(s,n_2)))\\
&\quad=x(m_1,m_2)u(m_1,m_2)-x(n_1,m_2)u(n_1,m_2)-x(m_1,n_2)u(m_1,n_2)\\
&\qquad+x(n_1,n_2)u(n_1,n_2).
\end{aligned}\tag{3.5}$$

Summing both sides of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, and noting $x(n_1,t)=x(m_1,t)=0$, we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=\mathrm{m}_1}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_2(s,t)|x(s+1,t+1)|^{\beta}.\tag{3.6}$$

Noticing that $x(m_1,t) = x(s,m_2) = 0$ and $\beta_2^+(s,t) = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}\{\beta_2(s,t),0\}$, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma} &= \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2(s,t)|x(s+1,t+1)|^{\beta}\\ &\le \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)|x(s+1,t+1)|^{\beta}.\end{aligned} \tag{3.7}$$

Choose $(\tau,\sigma) \in [n_1+1,m_1-1]\times[n_2+1,m_2-1]$ such that $M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}|x(s,t)|$. Hence $M = |x(\tau,\sigma)| > 0$. Summing the first equation of (3.1) over t from n_2 to $\sigma-1$ and over s from n_1 to $\tau-1$, respectively, we obtain

$$\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2 x(s,t) = \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1) + \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t). \tag{3.8}$$

Considering the left side of (3.8) and noting $x(n_1,t) = x(s,n_2) = 0$ for all $(s,t) \in [n_1,m_1]\times[n_2,m_2]$, we have

$$\begin{aligned}\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2 x(s,t) &= \sum_{s=n_1}^{\tau-1}\left(\sum_{t=n_2}^{\sigma-1}(x(s+1,t+1)-x(s+1,t)-(x(s,t+1)-x(s,t)))\right)\\ &= \sum_{s=n_1}^{\tau-1}(x(s+1,\sigma)-x(s,\sigma)-(x(s+1,n_2)-x(s,n_2)))\\ &= x(\tau,\sigma)-x(n_1,\sigma)-x(\tau,n_2)+x(n_1,n_2)\\ &= x(\tau,\sigma).\end{aligned} \tag{3.9}$$

Hence,

$$x(\tau,\sigma) = \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1) + \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t), \tag{3.10}$$

and similarly, we have

$$x(\tau,\sigma) = \sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}\alpha_1(s,t)x(s+1,t+1) + \sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t). \tag{3.11}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \leq \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}, \tag{3.12}$$

$$|x(\tau,\sigma)| \leq \sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.13}$$

Summing (3.12) and (3.13), we obtain

$$2|x(\tau,\sigma)| \leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.14}$$

On the other hand, using Hölder inequality on the second sum of the right side of (3.14) with indices α and γ, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} &= \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1} \\ &\leq \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}\right)^{1/\alpha} \\ &= \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}\right)^{1/\alpha},\end{aligned} \tag{3.15}$$

where $(1/\alpha) + (1/\gamma) = 1$. Therefore, from (3.7) and (3.10), we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} \leq \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}. \tag{3.16}$$

Substituting (3.16) to (3.14), we have

$$\begin{aligned}2|x(\tau,\sigma)| \leq{}& \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| \\ &+ \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}.\end{aligned} \tag{3.17}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}} |x(s,t)| > 0$, we get

$$2 \leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} |\alpha_1(s,t)| + M^{\beta/\alpha-1} \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s,t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s,t) \right)^{1/\alpha}. \tag{3.18}$$

This completes the proof. □

Remark 3.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (3.2) and with suitable changes, (3.2) changes to the following result:

$$2 \leq \sum_{t=n}^{m-2} |\alpha_1(t)| + M^{\beta/\alpha-1} \left(\sum_{t=n}^{m-1} \beta_1(t) \right)^{1/\gamma} \left(\sum_{t=n}^{m-2} \beta_2^+(t) \right)^{1/\alpha}. \tag{3.19}$$

This is just a new Lyapunov-type inequality which was given by Ünal et al. [2].

4. An application

Two-dimensional Emden-Fowler-type equation

$$\frac{\partial}{\partial s \partial t} \left(r(s,t) \left| \frac{\partial x(x,t)}{\partial s \partial t} \right|^{\alpha-2} \frac{\partial x(x,t)}{\partial s \partial t} \right) + q(s,t) |x(s,t)|^{\beta-2} x(s,t) = 0, \tag{4.1}$$

where $\alpha > 1$ is a constant, $r(s,t)$ and $q(s,t)$ are real functions, and $r(s,t) > 0$ for all $(s,t) \in \mathbb{R} \times \mathbb{R}$.

Consider the following special case of system (2.1), which is an equivalent system for the two-dimensional Emden-Fowler-type equation (4.1)

$$\begin{aligned} \frac{\partial x^2(s,t)}{\partial s \partial t} &= \beta_1(s,t) |u(s,t)|^{\gamma-2} u(s,t), \\ \frac{\partial u^2(s,t)}{\partial s \partial t} &= -\beta_2(s,t) |x(s,t)|^{\beta-2} x(s,t), \end{aligned} \tag{4.2}$$

where $\beta_1(s,t) = r(s,t)^{1-\gamma}$ and $\beta_2(s,t) = q(s,t)$.

Obviously Theorem 2.1 for the two-dimensional nonlinear system (2.1) with $\alpha_1(s,t) \equiv 0$ is satisfied for system (4.2). Therefore, we have

$$2 \leq M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t) dt\, ds \right)^{1/\alpha}. \tag{4.3}$$

A nontrivial solution $(x(s,t), u(s,t))$ of system (4.2) defined on $[s_0,\infty) \times [t_0,\infty)$ is said to be *proper* if and only if

$$\sup\{|x(s,t)| + |u(s,t)| : a \le s < \infty, c \le t < \infty\} > 0, \tag{4.4}$$

for any $a \ge s_0, c \ge t_0$.A proper solution $(x(s,t), u(s,t))$ of system (4.2) is called *weakly oscillatory* if and only if at least one component has a sequence of zeros tending to $+\infty$.

Theorem 4.1. *If* $|x(\tau,\sigma)| = \max\{|x(s,t)| : a < s < b, c < t < d\}$, *where* $a > s_0$, $c > t_0$ *and* $s_0, t_0, a, b, c, d \in \mathbb{R}$, $u(\tau,t)$ *is bounded on* $[t_0,\infty)$ *and* $u(s,\sigma)$ *is bounded on* $[s_0,\infty)$,

$$\int^{\infty}\int^{\infty} \beta_1(s,t)dt\,ds < \infty, \qquad \int^{\infty}\int^{\infty} \left|\beta_2(s,t)\right|dt\,ds < \infty, \tag{4.5}$$

then every weakly oscillatory proper solution of (4.2) *is bounded on* $I = [s_0,\infty) \times [t_0,\infty)$.

Proof. Let $(x(s,t), u(s,t))$ be any nontrivial weakly oscillatory proper solution of nonlinear system (4.2) on $I = [s_0,\infty) \times [t_0,\infty)$ such that $x(s,t)$ has a sequence of zeros tending to $+\infty$. Suppose to the contrary that $\limsup |x(s,t)| = \infty$; then given any positive number M_0, we can find positive numbers S_0 and T_0 such that $|x(s,t)| > M_0$ for all $s > S_0, t > T_0$. Since $x(s,t)$ is an oscillatory solution, there exist $(a,b) \times (c,d) \in \mathbb{R} \times \mathbb{R}$ with $a > S_0, c > T_0$ such that $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $|x(s,t)| > 0$ on $(a,b) \times (c,d)$. Choose (τ,σ) in $(a,b) \times (c,d)$ such that $M = |x(\tau,\sigma)| = \max\{|x(s,t)| : a < s < b, c < t < d\} > M_0$; in view of (4.5), we can choose S_0 and T_0 large enough such that for every $a \ge S_0$, $c \ge T_0$,

$$\int_a^{\infty}\int_c^{\infty} \beta_1(s,t)dt\,ds < M^{-(\beta-\alpha)/(\alpha-1)}, \qquad \int_a^{\infty}\int_c^{\infty} \left|\beta_2(s,t)\right|dt\,ds < 1. \tag{4.6}$$

Taking αth power of both sides of (4.3) and combining (4.6), we obtain

$$\begin{aligned}
2^{\alpha} &\le M^{\beta-\alpha}\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^b\int_c^d \beta_2^{+}(s,t)dt\,ds\right)\\
&\le M^{\beta-\alpha}\left(\int_a^{\infty}\int_c^{\infty} \beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^{\infty}\int_c^{\infty} \left|\beta_2(s,t)\right|dt\,ds\right)\\
&< M^{\beta-\alpha}M^{-\beta+\alpha} = 1,
\end{aligned} \tag{4.7}$$

where $\alpha > 1$ and $\beta_2^{+}(s,t) \le |\beta_2(s,t)|$.

This contradiction shows that $|x(s,t)|$ is bounded on $I = [s_0,\infty) \times [t_0,\infty)$. Therefore, there exists a positive constant K such that $|x(s,t)| \le K$ for all $(s,t) \in I$.

On the other hand, integrating the second equation of system (4.2) over t from σ to t and over s from σ to s, respectively, we obtain

$$u(s,t) - u(\tau,t) - u(s,\sigma) + u(\tau,\sigma) = \int_{\sigma}^{s}\int_{\tau}^{t} -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t)dt\,ds. \tag{4.8}$$

Notice that $u(\tau,t)$ is bounded on $[t_0,\infty)$, $u(s,\sigma)$ is bounded on $[s_0,\infty)$, and in view of triangle inequality, we have

$$\begin{aligned} |u(s,t)| &\le |u(\tau,t)+u(s,\sigma)-C| + \int_\sigma^s\int_\tau^t |\beta_2(s,t)||x(s,t)|^{\beta-1}dt\,ds \\ &\le |u(\tau,t)+u(s,\sigma)-C| + K^{\beta-1}\int_\sigma^\infty\int_\tau^\infty |\beta_2(s,t)|dt\,ds, \end{aligned} \tag{4.9}$$

where $C=u(\tau,\sigma)$ is a constant.

Equation (4.9) implies that $|u(s,t)|$ is bounded on $I = [s_0,\infty)\times[t_0,\infty)$ since $\int_\tau^\infty\int_\sigma^\infty |\beta_2(s,t)|dt\,ds<\infty$. It follows from

$$\limsup\{|x(s,t)|+|u(s,t)|\} \le \limsup|x(s,t)| + \limsup|u(s,t)| \tag{4.10}$$

that $\limsup\{|x(s,t)|+|u(s,t)|\}$ is bounden on $I=[s_0,\infty)\times[t_0,\infty)$.

This completes the proof. □

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