Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 504670, 8 pages
doi:10.1155/2010/504670

Research Article

Global Asymptotic Stability of Solutions to Nonlinear Marine Riser Equation

Şevket Gür

Department of Mathematics, Sakarya University, 54100 Sakarya, Turkey

Correspondence should be addressed to Şevket Gür, sgur@sakarya.edu.tr

Received 28 May 2010; Revised 25 August 2010; Accepted 14 September 2010

Academic Editor: Michel C. Chipot

This paper studies initial boundary value problem of fourth-order nonlinear marine riser equation. By using multiplier method, it is proven that the zero solution of the problem is globally asymptotically stable.

1. Introduction

The straight-line vertical position of marine risers has been investigated with respect to dynamic stability [1]. It studies the following initial boundary value problem describing the dynamics of marine riser:

$$mu_{tt} + EIu_{xxxx} - (N_{\text{eff}}u_x)_x + au_x + bu_t|u_t| = 0, \quad x \in (0,l),\ t > 0, \tag{1.1}$$

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \quad t > 0, \tag{1.2}$$

where EI is the flexural rigidity of the riser, N_{eff} is the "effective tension", a is the coefficient of the Coriolis force, b is the coefficient of the nonlinear drag force, and m is the mass line density. u represents the riser deflection.

By using the Lyapunov function technique, Köhl has shown that the zero solution of the problem is stable.

In [2], Kalantarov and Kurt have studied the initial boundary value problem for the equation

$$mu_{tt} + ku_{xxxx} - [a(x)u_x]_x + \gamma u_{tx} + bu_t|u_t|^p = 0 \tag{1.3}$$

under boundary conditions (1.2). Here p, m, k, and b are given positive numbers, γ is given real number, $a(x)$ is a $C^1[0,l]$ function, and $a(x) \geq -c_0 > 0$for all $x \in [0,l]$. It is shown that the zero solution of the problem (1.3)-(1.2) is globally asymptotically stable, that is, the zero solution is stable and all solutions of this problem are tending to zero when $t \to \infty$. Moreover the polynomial decay rate for solutions is established.

There are many articles devoted to the investigation of the asymptotic behavior of solutions of nonlinear wave equations with nonlinear dissipative terms (see, e.g. [3, 4]), where theorems on asymptotic stability of the zero solution and estimates of the zero solution and the estimates of the rate of decay of solutions to second order wave equations are obtained.

Similar results for the higher-order nonlinear wave equations are obtained in [5].

In this study, we consider the following initial boundary value problem for the multidimensional version of (1.1):

$$u_{tt} + k\Delta^2 u - a\Delta u + \sum_{i=1}^{n} \gamma_i u_{tx_i} + b|u_t|^p u_t = 0, \quad x \in \Omega, t > 0, \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.5}$$

$$u = \Delta u = 0, \quad x \in \partial\Omega, t > 0, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. k, b, and p are given positive numbers, and a, γ_i, $i = 1, \dots, n$ are given real numbers.

Following [2, 5], we prove that all solutions of the problem (1.4)–(1.6) are tending to zero with a polynomial rate as $t \to +\infty$. In this work, $\|\cdot\|$ stands for the norm in $L^2(\Omega)$.

2. Decay Estimate

Theorem 2.1. *Suppose that k, b, and p are arbitrary positive numbers, and number a satisfies*

$$a + k\lambda_1 = m_0 > 0, \tag{2.1}$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the homogeneous Dirichlet boundary condition. p is an arbitrary positive number when $n \leq 2$ and

$$p \in \left(0, \frac{4}{n-2}\right] \quad \text{when } n \geq 3. \tag{2.2}$$

Then the following estimate holds:

$$\frac{1}{2}\|u_t\|^2 + \frac{m_0}{2}\|\nabla u\|^2 \leq \begin{cases} At^{-(p+1)/(p+2)}, & p \in (0,1), \\ At^{-2/(p+2)}, & p \geq 1,\ t \in [1,\infty), \end{cases} \tag{2.3}$$

where A depends only on the initial data and the numbers a, b, p, γ_i, $(i = 1, \dots, n)$, and λ_1.

Proof. We multiply (1.4) by u_t and integrate over Ω:

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{k}{2}\|\Delta u\|^2+\frac{a}{2}\|\nabla u\|^2\right]+\sum_{i=1}^{n}\gamma_i\int_\Omega u_{tx_i}u_t dx+b\int_\Omega |u_t|^{p+2}dx=0. \tag{2.4}$$

Since

$$\sum_{i=1}^{n}\gamma_i\int_\Omega u_{tx_i}u_t dx=\sum_{i=1}^{n}\gamma_i\int_\Omega \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_t^2\right)dx=0, \tag{2.5}$$

we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{k}{2}\|\Delta u\|^2+\frac{a}{2}\|\nabla u\|^2\right]+b\int_\Omega |u_t|^{p+2}dx=0. \tag{2.6}$$

Let $\delta>0$. Multiplying (1.4) by δu, integrating over Ω and adding to (2.6), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{k}{2}\|\Delta u\|^2+\frac{a}{2}\|\nabla u\|^2+\delta(u,u_t)\right]-\delta\|u_t\|^2+k\delta\|\Delta u\|^2\\&\quad+a\delta\|\nabla u\|^2+\delta\sum_{i=1}^{n}\gamma_i\int_\Omega u_{tx_i}u dx+b\delta\int_\Omega |u_t|^p u_t u dx+b\int_\Omega |u_t|^{p+2}dx=0.\end{aligned} \tag{2.7}$$

Using the method integrating by parts, we get

$$\delta\sum_{i=1}^{n}\gamma_i\int_\Omega u_{tx_i}u dx=-\delta\sum_{i=1}^{n}\gamma_i\int_\Omega u_{x_i}u_t dx. \tag{2.8}$$

Hence we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{k}{2}\|\Delta u\|^2+\frac{a}{2}\|\nabla u\|^2+\delta(u,u_t)\right]-\delta\|u_t\|^2+k\delta\|\Delta u\|^2+a\delta\|\nabla u\|^2\\&\quad-\delta\sum_{i=1}^{n}\gamma_i\int_\Omega u_{x_i}u_t dx+b\delta\int_\Omega |u_t|^p u_t u dx+b\int_\Omega |u_t|^{p+2}dx=0.\end{aligned} \tag{2.9}$$

Let

$$E_1(t)=\frac{1}{2}\|u_t\|^2+\frac{k}{2}\|\Delta u\|^2+\frac{a}{2}\|\nabla u\|^2+\delta(u,u_t). \tag{2.10}$$

Then we have from (2.9)

$$\frac{d}{dt}E_1(t) \leqslant \delta\|u_t\|^2 - k\delta\|\Delta u\|^2 - a\delta\|\nabla u\|^2 + \delta|\gamma|\int_\Omega |\nabla u||u_t|dx + b\delta\int_\Omega |u_t|^p u_t u dx - b\int_\Omega |u_t|^{p+2}dx, \tag{2.11}$$

where $|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2}$. Using Cauchy-Schwarz and Young's inequalities, we can get the following estimate:

$$\delta|\gamma|\int_\Omega |\nabla u||u_t|dx \leq \frac{\delta^2}{2}|\gamma|^2\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2. \tag{2.12}$$

It is not difficult to see that

$$\|\nabla u\| \leq \lambda_1^{-1/2}\|\Delta u\|. \tag{2.13}$$

Using inequalities (2.12) and (2.13) in (2.11), we obtain

$$\frac{d}{dt}E_1(t) \leqslant \left(\delta + \frac{1}{2}\right)\|u_t\|^2 - \left(\delta k - \frac{\delta^2}{2\lambda_1}|\gamma|^2\right)\|\Delta u\|^2 - a\delta\|\nabla u\|^2 + b\delta\int_\Omega |u_t|^p u_t u dx - b\int_\Omega |u_t|^{p+2}dx. \tag{2.14}$$

Let

$$0 < \delta < \frac{2\lambda_1 k}{|\gamma|^2}, \tag{2.15}$$

then

$$L = \delta k - \frac{\delta^2}{2\lambda_1}|\gamma|^2 > 0. \tag{2.16}$$

From (2.14), we get

$$\frac{d}{dt}E_1(t) \leqslant (\delta + 1)\|u_t\|^2 + b\delta\int_\Omega |u_t|^{p+1}u dx - b\int_\Omega |u_t|^{p+2}dx - \left(\frac{1}{2}\|u_t\|^2 + a\delta\|\nabla u\|^2 + L\|\Delta u\|^2\right). \tag{2.17}$$

Let

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{k}{2}\|\Delta u\|^2 + \frac{a}{2}\|\nabla u\|^2, \tag{2.18}$$

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|u_t\|^2 + \frac{k\lambda_1}{2}\|\nabla u\|^2 + \frac{a}{2}\|\nabla u\|^2 \geq \frac{1}{2}\|u_t\|^2 + \frac{m_0}{2}\|\nabla u\|^2 \\ &\geq \min\left\{\frac{1}{2}, \frac{m_0}{2}\right\}\left[\|u_t\|^2 + \|\nabla u\|^2\right]. \end{aligned} \tag{2.19}$$

From (2.6), we have

$$\frac{d}{dt}E(t) = -b\int_\Omega |u_t|^{p+2}dx \leq 0. \tag{2.20}$$

Therefore $E(t)$ is a Lyapunov functional. From (2.20), we find that

$$E(t) - E(0) = -b\int_0^t\int_\Omega |u_t|^{p+2}dx\,ds. \tag{2.21}$$

Since $E(t) \geq 0$, we obtain

$$\int_0^t\int_\Omega |u_t|^{p+2}dx\,ds \leqslant \frac{E(0)}{b}. \tag{2.22}$$

If a is nonnegative, then we have

$$\frac{1}{2}\|u_t\|^2 + a\delta\|\nabla u\|^2 + L\|\Delta u\|^2 \geq \min\left\{1, 2\delta, \frac{2L}{k}\right\}E(t) \geq D_1 E(t), \tag{2.23}$$

where $D_1 = \min\{1, 2\delta, 2L/k\}$.

If a is negative, then, using (2.13), we have

$$\begin{aligned} \frac{1}{2}\|u_t\|^2 + a\delta\|\nabla u\|^2 + L\|\Delta u\|^2 &\geq \frac{1}{2}\|u_t\|^2 + \left(\frac{a\delta}{\lambda_1} + L\right)\|\Delta u\|^2 \\ &\geq \min\left\{1, \frac{2}{k}\left(\frac{a\delta}{\lambda_1} + L\right)\right\}\left[\frac{1}{2}\|u_t\|^2 + \frac{k}{2}\|\Delta u\|^2\right] \geq D_2 E(t), \end{aligned} \tag{2.24}$$

where $D_2 = \min\{1, (2/k)(a\delta/\lambda_1 + L)\}$.

Therefore if a either nonnegative or negative then it is clear that

$$\frac{1}{2}\|u_t\|^2 + a\delta\|\nabla u\|^2 + L\|\Delta u\|^2 \geq DE(t), \tag{2.25}$$

where $D = \min\{1, 2\delta, 2L/k, (2/k)(a\delta/\lambda_1 + L)\}$ and $0 < \delta < 2m_0/|\gamma|^2$. Using (2.25), we obtain from (2.17)

$$\frac{d}{dt}E_1(t) \leqslant (\delta+1)\|u_t\|^2 - DE(t) + b\delta\int_\Omega |u_t|^{p+1}|u|dx - b\int_\Omega |u_t|^{p+2}dx. \tag{2.26}$$

Integrating (2.26) with respect to t, we can get

$$\begin{aligned}
E_1(t) - E_1(0) &\leqslant (\delta+1)\int_0^t\int_\Omega |u_t|^2 dx\,ds - D\int_0^t E(s)ds \\
&\quad + b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u|dx\,ds - b\int_0^t\int_\Omega |u_t|^{p+2}dx\,ds, \\
DtE(t) &\leqslant D\int_0^t E(s)ds \leqslant [E_1(0) - E_1(t)] + (\delta+1)\int_0^t\int_\Omega |u_t|^2 dx\,ds \\
&\quad + b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u|dx\,ds - b\int_0^t\int_\Omega |u_t|^{p+2}dx\,ds,
\end{aligned} \tag{2.27}$$

$$DtE(t) \leqslant [E_1(0) - E_1(t)] + (\delta+1)\int_0^t\int_\Omega |u_t|^2 dx\,ds + b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u|dx\,ds. \tag{2.28}$$

Using Poincare's and Cauchy-Schwarz inequalities, we can estimate $E_1(t)$ from below:

$$\begin{aligned}
E_1(t) &= \frac{1}{2}\|u_t\|^2 + \frac{k}{2}\|\Delta u\|^2 + \frac{a}{2}\|\nabla u\|^2 + \delta(u, u_t) \\
&\geqslant \frac{1}{2}\|u_t\|^2 + \frac{k}{2}\|\Delta u\|^2 + \frac{a}{2}\|\nabla u\|^2 - \delta|(u, u_t)| \\
&\geqslant \frac{1}{2}\|u_t\|^2 + \frac{m_0}{2}\|\nabla u\|^2 - \frac{\delta}{2}\|u\|^2 - \frac{\delta}{2}\|u_t\|^2 \\
&\geqslant \frac{1}{2}\|u_t\|^2 + \frac{m_0}{2}\|\nabla u\|^2 - \frac{\delta}{2\lambda_1}\|\nabla u\|^2 - \frac{\delta}{2}\|u_t\|^2 \\
&\geqslant \frac{1}{2}(1-\delta)\|u_t\|^2 + \frac{1}{2}\left(m_0 - \frac{\delta}{\lambda_1}\right)\|\nabla u\|^2,
\end{aligned} \tag{2.29}$$

thus for

$$0 < \delta < \min\left\{\frac{2\lambda_1 k}{|\gamma|^2}, 1, m_0\lambda_1, \frac{2m_0}{|\gamma|^2}\right\} \tag{2.30}$$

the following estimate holds:

$$E_1(t) \geqslant d_1\left(\|u_t\|^2 + \|\nabla u\|^2\right), \tag{2.31}$$

where

$$d_1 = \frac{1}{2}\min\left\{1-\delta, m_0 - \frac{\delta}{\lambda_1}\right\}. \tag{2.32}$$

Therefore,

$$E_1(0) - E_1(t) \leqslant E_1(0), \tag{2.33}$$

$$DtE(t) \leqslant E_1(0) + (\delta+1)\int_0^t\int_\Omega |u_t|^2 dx\, ds + b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u| dx\, ds. \tag{2.34}$$

Now we can estimate the right-hand side of (2.34) from below. Due to Holder inequality and (2.22), we obtain

$$\begin{aligned}(\delta+1)\int_0^t\int_\Omega |u_t|^2 dx\, ds &\leqslant (\delta+1)\left(\int_0^t\int_\Omega |u_t|^{p+2} dx\, ds\right)^{2/(p+2)}\left(\int_0^t\int_\Omega dx\, ds\right)^{p/(p+2)} \\ &\leqslant (\delta+1)\left(\frac{E(0)}{b}\right)^{2/(p+2)}\left(\int_0^t\int_\Omega dx\, ds\right)^{p/(p+2)} \leqslant C_1 t^{p/(p+2)},\end{aligned} \tag{2.35}$$

where C_1 is a positive constant depending on the initial data and the parameters of (1.4).

Using the Holder inequality and the Sobolev imbedding $H^1 \subset L^{p+2}$, we obtain

$$\begin{aligned}b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u| dx\, ds &\leqslant b\delta\left(\int_0^t\int_\Omega |u_t|^{p+2} dx\, ds\right)^{(p+1)/(p+2)}\left(\int_0^t\int_\Omega |u|^{p+2} dx\, ds\right)^{1/(p+2)} \\ &\leqslant b\delta\left(\int_0^t\int_\Omega |u_t|^{p+2} dx\, ds\right)^{(p+1)/(p+2)}\left(\int_0^t \|u\|_{p+2}^{p+2} ds\right)^{1/(p+2)} \\ &\leqslant C_2 b\delta\left(\int_0^t\int_\Omega |u_t|^{p+2} dx\, ds\right)^{(p+1)/(p+2)}\left(\int_0^t \|\nabla u\|^{p+2} ds\right)^{1/(p+2)},\end{aligned} \tag{2.36}$$

where C_2 is a positive constant depending on Ω. Due to (2.22) and

$$\|\nabla u\|^2 \leqslant \frac{2E(0)}{a+k\lambda_1}, \tag{2.37}$$

we obtain

$$b\delta\int_0^t\int_\Omega |u_t|^{p+1}|u| dx\, ds \leqslant C_3 t^{1/(p+2)}, \tag{2.38}$$

where

$$C_3 = b\delta C_2\left(\frac{2E(0)}{a+k\lambda_1}\right)^{1/2}\left(\frac{E(0)}{b}\right)^{(p+1)/(p+2)}. \tag{2.39}$$

Therefore

$$\begin{aligned} DtE(t) &\leqslant E_1(0) + C_1 t^{p/(p+2)} + C_3 t^{1/(p+2)}, \\ E(t) &\leqslant D^{-1}\left[E_1(0)t^{-1} + C_1 t^{-2/(p+2)} + C_3 t^{-(p+1)/(p+2)}\right]. \end{aligned} \tag{2.40}$$

It follows then that for large values of t, $t \geq 1$, the following estimate is valid:

$$E(t) \leq \begin{cases} At^{-(p+1)/(p+2)}, & p \in (0,1), \\ At^{-2/(p+2)}, & p \geq 1, \end{cases} \tag{2.41}$$

where $A = (D)^{-1}[E_1(0) + C_1 + C_3]$. Hence we have from (2.19)

$$\frac{1}{2}\|u_t\|^2 + \frac{m_0}{2}\|\nabla u\|^2 \leq \begin{cases} At^{-(p+1)/(p+2)}, & p \in (0,1), \\ At^{-2/(p+2)}, & p \geq 1,\ t \in [1,\infty). \end{cases} \tag{2.42}$$

From this inequality it follows that the zero solution (1.4)–(1.6) is globally asymptotically stable. □

Aknowledgment

Special thanks to Prof. Dr. Varga Kalantarov.

References

[1] M. Köhl, "An extended Liapunov approach to the stability assessment of marine risers," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 73, no. 2, pp. 85–92, 1993.

[2] V. K. Kalantarov and A. Kurt, "The long-time behavior of solutions of a nonlinear fourth order wave equation, describing the dynamics of marine risers," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 77, no. 3, pp. 209–215, 1997.

[3] M. Nakao, "Remarks on the existence and uniqueness of global decaying solutions of the nonlinear dissipative wave equations," *Mathematische Zeitschrift*, vol. 206, no. 2, pp. 265–276, 1991.

[4] A. Haraux and E. Zuazua, "Decay estimates for some semilinear damped hyperbolic problems," *Archive for Rational Mechanics and Analysis*, vol. 100, no. 2, pp. 191–206, 1988.

[5] P. Marcati, "Decay and stability for nonlinear hyperbolic equations," *Journal of Differential Equations*, vol. 55, no. 1, pp. 30–58, 1984.