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Review Article

Bounds for the Ratio of Two Gamma Functions

Feng Qi

Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City 300160, China

Correspondence should be addressed to Feng Qi, qifeng618@gmail.com, qifeng618@qq.com

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By looking back at the long history of bounding the ratio $\Gamma(x+a)/\Gamma(x+b)$ for $x > -\min\{a,b\}$ and $a, b \in \mathbb{R}$, various origins of this topic are clarified, several developed courses are followed, different results are compared, useful methods are summarized, new advances are presented, some related problems are pointed out, and related references are collected.

1. Basic Definitions and Notations

In order to fluently and smoothly understand what follows in this paper, some basic concepts and notations need to be stated at first in this section.

1.1. The Gamma Function and Related Formulas

1.1.1. The Gamma Function

It is well known that the classical Euler gamma function can be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \tag{1.1}$$

the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function and denoted by $\psi(x)$, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions.

It is general knowledge that

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{1.2}$$

Taking the logarithm and differentiating on both sides of (1.2) give

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad x > 0. \tag{1.3}$$

1.1.2. Stirling's Formula

For $x > 0$, there exists $0 < \theta < 1$ such that

$$\Gamma(x+1) = \sqrt{2\pi}\ x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right). \tag{1.4}$$

See [1, page 257, 6.1.38].

1.1.3. Wallis Cosine Formula

Wallis cosine or sine formula reads [2] that

$$\begin{aligned} \int_0^{\pi/2} \cos^n x \mathrm{d}x &= \int_0^{\pi/2} \sin^n x \mathrm{d}x \\ &= \frac{\sqrt{\pi}\ \Gamma((n+1)/2)}{n\Gamma(n/2)} = \begin{cases} \dfrac{\pi}{2} \cdot \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ even}, \\ \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ odd}, \end{cases} \end{aligned} \tag{1.5}$$

where $n!!$ denotes a double factorial. Therefore,

$$\frac{(2k)!!}{(2k-1)!!} = \frac{\sqrt{\pi}\ \Gamma(k+1)}{\Gamma(k+1/2)}, \quad k \in \mathbb{N}. \tag{1.6}$$

1.1.4. Duplication Formula

For $x > 0$,

$$2^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \sqrt{\pi}\ \Gamma(x). \tag{1.7}$$

1.1.5. Binet's First Formula

Binet's first formula for $\ln\Gamma(x)$ is given by

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi}\ + \theta(x) \tag{1.8}$$

for $x > 0$, where

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-xt}}{t} \mathrm{d}t \tag{1.9}$$

for $x > 0$ is called the remainder of Binet's first formula for the logarithm of the gamma function. See [3, page 11].

1.1.6. Wendel's Limit

For real numbers a and b,

$$\lim_{x \to \infty} \left[x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} \right] = 1. \tag{1.10}$$

See [1, page 257, 6.1.46].

If $z \neq -a, -a-1, \ldots$ and $z \neq -b, -b-1, \ldots$, then

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2z} + \frac{(a-b)(a-b-1)\left[3(a+b-1)^2 - a + b - 1\right]}{24z^2} + \cdots \tag{1.11}$$

as $z \to \infty$ along any curve joining $z = 0$ and $z = \infty$. See [4, pages 118-119].

1.1.7. Legendre's Formula

For $x > 0$,

$$\psi(x) = -\gamma + \int_0^1 \frac{t^{x-1} - 1}{t - 1} \mathrm{d}t. \tag{1.12}$$

1.1.8. Gauss' Theorem

For $\mathrm{Re}(c - a - b) > 0$,

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} = {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.13}$$

See [5, page 66, Theorem 2.2].

1.2. The q-Gamma Function and Related Formulas

It is well known (see [5, pages 493–496] and [6]) that the q-gamma function, the q-analogue of the gamma function $\Gamma(x)$, is defined for $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=0}^{\infty} \frac{1-q^{i+1}}{1-q^{i+x}} \tag{1.14}$$

for $0 < q < 1$ and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\binom{x}{2}} \prod_{i=0}^{\infty} \frac{1-q^{-(i+1)}}{1-q^{-(i+x)}} \tag{1.15}$$

for $q > 1$. It has the following basic properties

$$\lim_{q \to 1^+} \Gamma_q(z) = \lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z), \quad \Gamma_q(x) = q^{\binom{x-1}{2}} \Gamma_{1/q}(x). \tag{1.16}$$

The q-psi function $\psi_q(x)$, the q-analogue of the psi function $\psi(x)$, for $0 < q < 1$ and $x > 0$ may be defined by

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} = -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k}, \tag{1.17}$$

and $\psi_q^{(k)}(x)$, the q-analogues of the polygamma functions $\psi^{(k)}(x)$, for $k \in \mathbb{N}$ are called the q-polygamma functions. The following Stieltjes integral representation for $\psi_q(x)$ is given in [7]:

$$\psi_q(x) = -\ln(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t) \tag{1.18}$$

for $0 < q < 1$ and $x > 0$, where

$$\gamma_q(t) = -\ln q \sum_{k=1}^{\infty} \delta(t + k \ln q). \tag{1.19}$$

1.3. Logarithmically Convex Functions

Definition 1.1 (see [8, 9]). For $k \in \mathbb{N}$, a positive and k-time differentiable function $f(x)$ is said to be k-log-convex on an interval I if

$$[\ln f(x)]^{(k)} \geq 0 \tag{1.20}$$

on I. If the inequality (1.20) is reversed, then f is said to be k-log-concave on I.

Remark 1.2. It is clear that a 1-log-convex function (or 1-log-concave function, resp.) is equivalent to a positive and increasing (or decreasing, resp.) function and that a 2-log-convex function is positive and convex. Conversely, a convex function may not be 2-log-convex. See [8, page 7, Remark 1.16].

1.4. Completely Monotonic Functions

Definition 1.3 ([10, Chapter XIII] and [11, Chapter IV]). A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.21}$$

for $x \in I$ and $n \geq 0$.

Remark 1.4. The famous Bernstein-Widder's Theorem [11, page 161] states that a function f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xs} \mathrm{d}\mu(s), \tag{1.22}$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral (1.22) converges for all $x > 0$. This means that a completely monotonic function f on $(0, \infty)$ is a Laplace transform of the measure μ.

Remark 1.5. A result of [12, page 98] asserts that for a completely monotonic function f on (a, ∞), inequalities in (1.21) strictly hold unless $f(x)$ is constant. This assertion can also be found in [13].

Definition 1.6 (see [14]). If $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of the kth order on an interval I.

1.5. Logarithmically Completely Monotonic Functions

Definition 1.7 (see [14, 15]). A positive function f is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k \left[\ln f(x)\right]^{(k)} \geq 0 \tag{1.23}$$

for $k \in \mathbb{N}$ on I.

Remark 1.8. In [15–19], it was recovered that any logarithmically completely monotonic function f on I must be completely monotonic on I, but not conversely. However, it was discovered in [20, Section 5] that every completely monotonic function on $(0,\infty)$ is logarithmically convex.

Remark 1.9. The following conclusions may be useful: A logarithmically convex function is also convex. If f is nonnegative and concave, then it is logarithmically concave. The sum of finite logarithmically convex functions is also a logarithmically convex function. But, the sum of two logarithmically concave functions may not be logarithmically concave. See [20, Section 3].

Remark 1.10. In [16, Theorem 1.1] and [13, 21] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [22, Theorem 4.4] and that the set of all Stieltjes transforms is a subset of the set of logarithmically completely monotonic functions on $(0,\infty)$.

Remark 1.11. For more information on characterizations, applications and history of the class of logarithmically completely monotonic functions, please refer to [13–17] and related references therein.

Definition 1.12 (see [23, 24]). Let f be a positive function which has derivatives of all orders on an interval I. If $[\ln f(x)]^{(k)}$ for some nonnegative integer k is completely monotonic on I, but $[\ln f(x)]^{(k-1)}$ is not completely monotonic on I, then f is said to be a logarithmically completely monotonic function of the kth order on I.

Definition 1.13 (see [11, 25]). A function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and

$$f^{(k-1)}(t) \geq 0 \tag{1.24}$$

for $t \in I$ and $k \in \mathbb{N}$.

Definition 1.14 (see [23, 24]). Let f be a positive function which has derivatives of all orders on an interval I. If $[\ln f(x)]^{(k)}$ for some nonnegative integer k is absolutely monotonic on I, but $[\ln f(x)]^{(k-1)}$ is not absolutely monotonic on I, then f is said to be a logarithmically absolutely monotonic function of the kth order on I.

Definition 1.15 (see [23, 24]). A positive function f which has derivatives of all orders on an interval I is said to be logarithmically absolutely convex on I if

$$\left[\ln f(x)\right]^{(2k)} \geq 0 \tag{1.25}$$

on I for $k \in \mathbb{N}$.

1.6. Some Useful Formulas and Inequalities

1.6.1. Jensen's Inequality

If ϕ is a convex function on $[a,b]$, then

$$\phi\left(\sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k \phi(x_k), \tag{1.26}$$

where $n \in \mathbb{N}$, $x_k \in [a,b]$, and $p_k \geq 0$ for $1 \leq k \leq n$ satisfying $\sum_{k=1}^{n} p_k = 1$.

1.6.2. Hölder's Inequality for Integrals

Let p and q be positive numbers satisfying $1/p + 1/q = 1$. If f and g are absolutely integrable on $(0,\infty)$, then

$$\int_0^\infty |f(t)g(t)|\mathrm{d}t \leq \left[\int_0^\infty |f(t)|^p \mathrm{d}t\right]^{1/p} \left[\int_0^\infty |g(t)|^q \mathrm{d}t\right]^{1/q}, \tag{1.27}$$

with equality when $|g(x)| = c|f(x)|^{p-1}$.

1.6.3. Convolution Theorem of Laplace Transform (See [26])

Let $f_i(t)$ for $i = 1,2$ be piecewise continuous in arbitrary finite intervals included on $(0,\infty)$. If there exist some constants $M_i > 0$ and $c_i \geq 0$ such that $|f_i(t)| \leq M_i e^{c_i t}$ for $i = 1,2$, then

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right] e^{-st}\mathrm{d}t = \int_0^\infty f_1(u)e^{-su}\mathrm{d}u \int_0^\infty f_2(v)e^{-sv}\mathrm{d}v. \tag{1.28}$$

1.6.4. Mean Values

The generalized logarithmic mean $L_p(a,b)$ of order $p \in \mathbb{R}$ for positive numbers a and b with $a \neq b$ is defined in [27, page 385] by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & p \neq -1, 0, \\ \dfrac{b-a}{\ln b - \ln a} & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & p = 0. \end{cases} \tag{1.29}$$

Note that

$$L_1(a,b) = \frac{a+b}{2} = A(a,b), \quad L_{-1}(a,b) = L(a,b), \quad L_0(a,b) = I(a,b) \tag{1.30}$$

are called, respectively, the arithmetic mean, the logarithmic mean, and the identric or exponential mean in the literature. Since the generalized logarithmic mean $L_p(a,b)$ is increasing in p for $a \neq b$, see [27, pages 386-387, Theorem 3], inequalities

$$L(a,b) < I(a,b) < A(a,b) \tag{1.31}$$

are valid for $a > 0$ and $b > 0$ with $a \neq b$. See also [25, 28, 29] and related references therein.

1.6.5. Bernoulli Numbers

Bernoulli numbers B_n for $n \geq 0$ can be defined as

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi. \tag{1.32}$$

The first six Bernoulli numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}. \tag{1.33}$$

1.6.6. A Completely Monotonic Function

For any real number α, let

$$\Theta_\alpha(x) = x^\alpha \left[\ln x - \psi(x)\right], \quad x \in (0,\infty). \tag{1.34}$$

The function $\Theta_1(x)$ was proved in [30, Theorem 3.1] to be decreasing and convex on $(0,\infty)$.

By using Binet's first formula (1.9) and complicated calculating techniques for proper integrals, a general result was presented in [31, pages 374-375, Theorem 1]. For real number α, the function $\Theta_\alpha(x)$ is completely monotonic on $(0,\infty)$ if and only if $\alpha \leq 1$.

Recently the completely monotonic property of $\Theta_\alpha(x)$ was also proved by different approaches in [32–34].

1.7. Properties of A Function Involving the Exponential Function

For $t \in \mathbb{R}$ and real numbers α and β satisfying $\alpha \neq \beta$ and $(\alpha, \beta) \notin \{(0,1),(1,0)\}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \tag{1.35}$$

In [9, 35–40], sufficient and necessary conditions that the function $q_{\alpha,\beta}(x)$ is monotonic, logarithmically convex, and logarithmically concave on $(0,\infty)$ were discovered step by step.

1.7.1. Monotonic Properties of $q_{\alpha,\beta}(x)$

The earliest complete conclusions on monotonic properties of $q_{\alpha,\beta}(x)$ were discussed in the paper [36] little by little but thoroughly.

Theorem 1.16 (see [35, 36]). *Let α and β satisfying $\alpha \neq \beta$ and $(\alpha,\beta) \notin \{(0,1),(1,0)\}$ be real numbers and $t \in \mathbb{R}$.*

(1) *The function $q_{\alpha,\beta}(t)$ is increasing on $(0,\infty)$ if and only if*

$$(\beta - \alpha)(1 - \alpha - \beta) \geq 0, \qquad (\beta - \alpha)(|\alpha - \beta| - \alpha - \beta) \geq 0. \tag{1.36}$$

(2) *The function $q_{\alpha,\beta}(t)$ is decreasing on $(0,\infty)$ if and only if*

$$(\beta - \alpha)(1 - \alpha - \beta) \leq 0, \qquad (\beta - \alpha)(|\alpha - \beta| - \alpha - \beta) \leq 0. \tag{1.37}$$

(3) *The function $q_{\alpha,\beta}(t)$ is increasing on $(-\infty,0)$ if and only if*

$$(\beta - \alpha)(1 - \alpha - \beta) \geq 0, \qquad (\beta - \alpha)(2 - |\alpha - \beta| - \alpha - \beta) \geq 0. \tag{1.38}$$

(4) *The function $q_{\alpha,\beta}(t)$ is decreasing on $(-\infty,0)$ if and only if*

$$(\beta - \alpha)(1 - \alpha - \beta) \leq 0, \qquad (\beta - \alpha)(2 - |\alpha - \beta| - \alpha - \beta) \leq 0. \tag{1.39}$$

(5) *The function $q_{\alpha,\beta}(t)$ is increasing on $(-\infty,\infty)$ if and only if*

$$(\beta - \alpha)(|\alpha - \beta| - \alpha - \beta) \geq 0, \qquad (\beta - \alpha)(2 - |\alpha - \beta| - \alpha - \beta) \geq 0. \tag{1.40}$$

(6) *The function $q_{\alpha,\beta}(t)$ is decreasing on $(-\infty,\infty)$ if and only if*

$$(\beta - \alpha)(|\alpha - \beta| - \alpha - \beta) \leq 0, \qquad (\beta - \alpha)(2 - |\alpha - \beta| - \alpha - \beta) \leq 0. \tag{1.41}$$

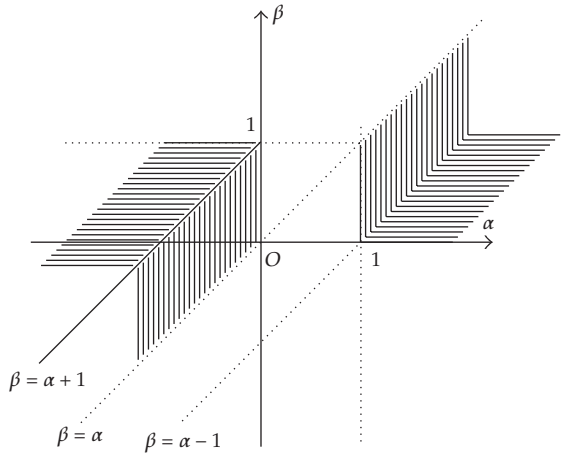

Figure 1: (α, β)-domain where $q_{\alpha,\beta}(t)$ increases on $(-\infty, \infty)$.

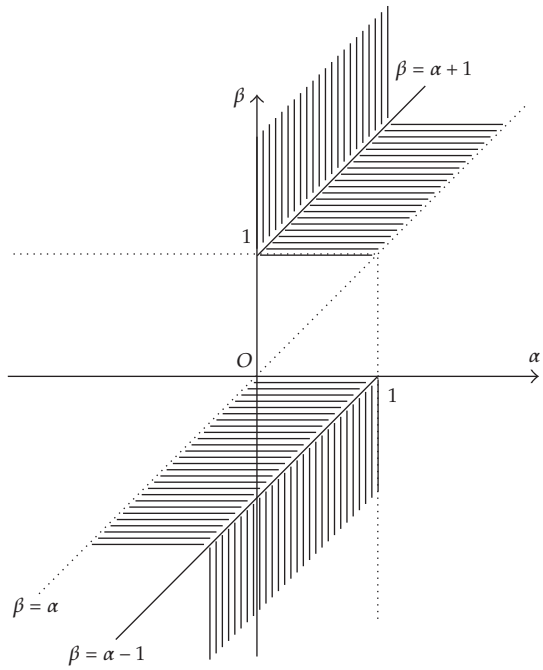

Figure 2: (α, β)-domain where $q_{\alpha,\beta}(t)$ decreases on $(-\infty, \infty)$.

Remark 1.17. The (α, β)-domains from (1.36) to (1.41) can be described, respectively, by Figures 1, 2, 3, 4, 5, and 6.

Remark 1.18. Theorem 1.16 and Figure 1 to Figure 4 correct several minor errors in [9, 36].

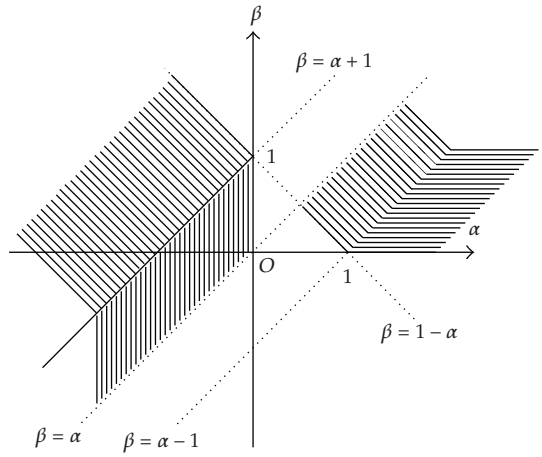

Figure 3: (α, β)-domain where $q_{\alpha,\beta}(t)$ is increasing on $(0, \infty)$.

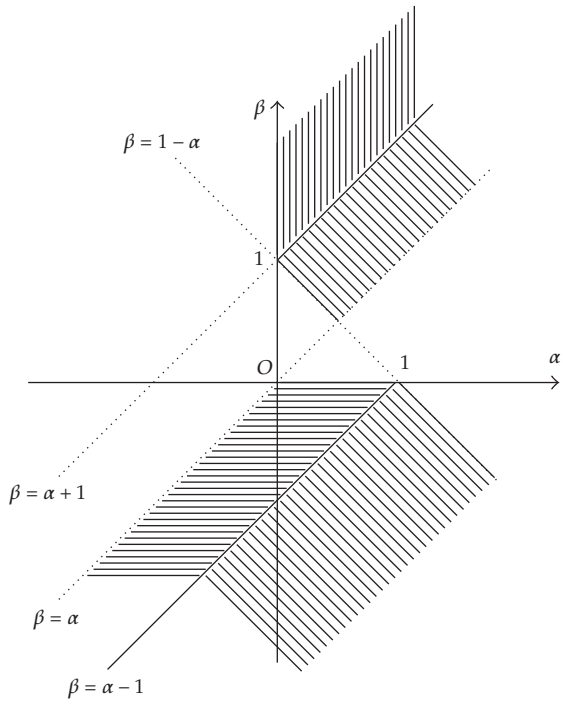

Figure 4: (α, β)-domain where $q_{\alpha,\beta}(t)$ is decreasing on $(0, \infty)$.

1.7.2. Logarithmically Convex Properties of $q_{\alpha,\beta}(t)$

These results were founded at first in [39, Lemma 1] and [40, Lemma 1] earlier than monotonic properties of $q_{\alpha,\beta}(t)$.

Theorem 1.19 (see [35–40]). *The function $q_{\alpha,\beta}(t)$ on $(-\infty, \infty)$ is logarithmically convex if $\beta - \alpha > 1$ and logarithmically concave if $0 < \beta - \alpha < 1$.*

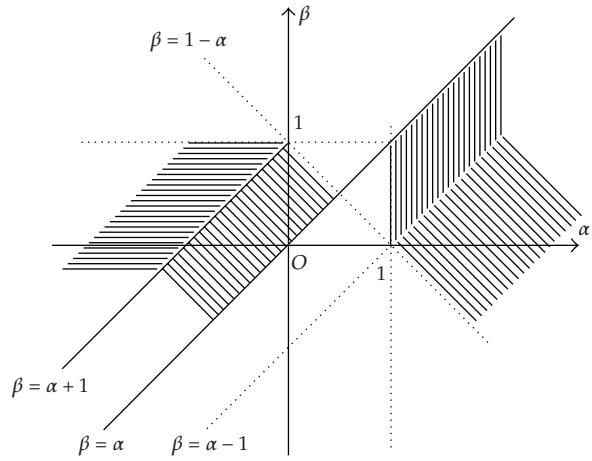

Figure 5: (α, β)-domain where $q_{\alpha,\beta}(t)$ is increasing on $(-\infty, 0)$.

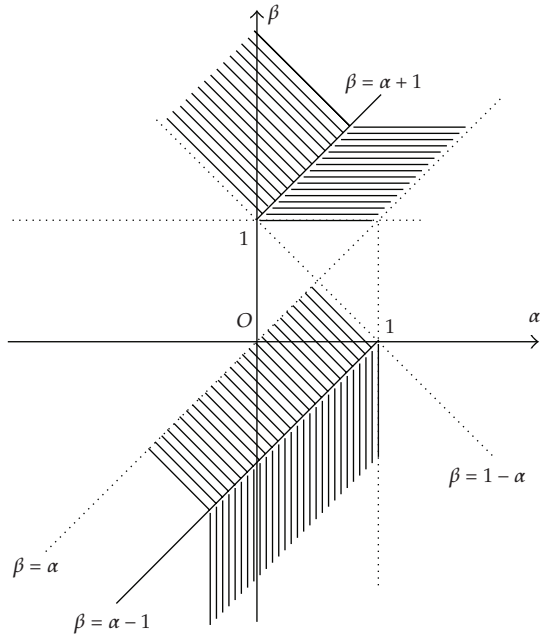

Figure 6: (α, β)-domain where $q_{\alpha,\beta}(t)$ is decreasing on $(-\infty, 0)$.

Remark 1.20. This theorem tells us that the logarithmic convexity and logarithmic concavity of $q_{\alpha,\beta}(t)$ on the interval $(-\infty, 0)$, showed in [39, Lemma 1] and [40, Lemma 1] , are wrong. However, this does not affect the correctness of any other results established in [39, 40], since the wrong conclusions about $q_{\alpha,\beta}(t)$ on the interval $(-\infty, 0)$ are idle there luckily.

1.7.3. Three-Log-Convex Properties of $q_{\alpha,\beta}(t)$

Theorem 1.21 (see [9]). *If $1 > \beta - \alpha > 0$, then $q_{\alpha,\beta}(t)$ is 3-log-convex on $(0,\infty)$ and 3-log-concave on $(-\infty,0)$; if $\beta - \alpha > 1$, then $q_{\alpha,\beta}(t)$ is 3-log-concave on $(0,\infty)$ and 3-log-convex on $(-\infty,0)$.*

Remark 1.22. So far no application of 3-log-convex properties of $q_{\alpha,\beta}(t)$ is disclosed, unlike monotonic and logarithmically convex properties of $q_{\alpha,\beta}(t)$ already having applications in [35, 37–41], respectively.

Remark 1.23. One of the key steps proving Theorems 1.16 to 1.21 is to rewrite the function $q_{\alpha,\beta}(t)$ as

$$q_{\alpha,\beta}(t) = \frac{\sinh(\beta-\alpha)t/2}{\sinh(t/2)} \exp\frac{(1-\alpha-\beta)t}{2}. \tag{1.42}$$

Remark 1.24. The monotonic and convex properties of $q_{\alpha,\beta}(t)$ have important applications to investigations of the gamma and q-gamma functions.

2. The History and Origins

In the history of this topic, there are several independent origins and different motivations of bounding the ratio of two gamma functions, no matter their appearances were early or late.

2.1. Wendel's Double Inequality and Proof

As early as in 1948, in order to establish the classical asymptotic relation

$$\lim_{x\to\infty} \frac{\Gamma(x+s)}{x^s\Gamma(x)} = 1 \tag{2.1}$$

for real s and x, using Hölder's inequality (1.27), Wendel proved in [42] the double inequality

$$\left(\frac{x}{x+s}\right)^{1-s} \le \frac{\Gamma(x+s)}{x^s\Gamma(x)} \le 1 \tag{2.2}$$

for $0 < s < 1$ and $x > 0$.

Wendel's Proof for (2.1) *and* (2.2). Let

$$\begin{aligned} &0 < s < 1, \qquad p = \frac{1}{s}, \qquad q = \frac{p}{p-1} = \frac{1}{1-s}, \\ &f(t) = e^{-st}t^{sx}, \qquad g(t) = e^{-(1-s)t}t^{(1-s)x+s-1}, \end{aligned} \tag{2.3}$$

and apply Hölder's inequality (1.27) and the recurrent formula (1.2) to obtain

$$\Gamma(x+s)=\int_0^\infty e^{-t}t^{x+s-1}\mathrm{d}t\le\left(\int_0^\infty e^{-t}t^{x}\mathrm{d}t\right)^{s}\left(\int_0^\infty e^{-t}t^{x-1}\mathrm{d}t\right)^{1-s}=[\Gamma(x+1)]^{s}[\Gamma(x)]^{1-s}=x^{s}\Gamma(x). \tag{2.4}$$

Replacing s by $1-s$ in (2.4), we get

$$\Gamma(x+1-s)\le x^{1-s}\Gamma(x), \tag{2.5}$$

from which we obtain

$$\Gamma(x+1)\le(x+s)^{1-s}\Gamma(x+s), \tag{2.6}$$

by substituting $x+s$ for x.

Combining (2.4) and (2.6), we get

$$\frac{x}{(x+s)^{1-s}}\Gamma(x)\le\Gamma(x+s)\le x^{s}\Gamma(x). \tag{2.7}$$

Therefore, the inequality (2.2) follows.

Letting x tend to infinity in (2.2) yields (2.1) for $0<s<1$. The extension to all real s is immediate on repeated application of (1.2). □

Remark 2.1. The inequality (2.2) can be rewritten for $0<s<1$ and $x>0$ as

$$x^{1-s}\le\frac{\Gamma(x+1)}{\Gamma(x+s)}\le(x+s)^{1-s}. \tag{2.8}$$

Remark 2.2. The limits (1.10) and (2.1) are equivalent to each other, since

$$x^{t-s}\frac{\Gamma(x+s)}{\Gamma(x+t)}=\frac{\Gamma(x+s)}{x^{s}\Gamma(x)}\cdot\frac{x^{t}\Gamma(x)}{\Gamma(x+t)}. \tag{2.9}$$

Hence, the limit (1.10) is called Wendel's limit in the literature of this paper.

Remark 2.3. The double inequality (2.2) or (2.8) is more meaningful than the limit (2.1), since the former implies the latter, but not conversely.

Remark 2.4. Due to unknown reasons, Wendel's paper [42] and inequalities (2.2) or (2.8) were possibly neglected by nearly all mathematicians for about more than fifty years, until 1999 in [43] and later in [20, 44–49], to the best of my knowledge.

2.2. Gurland's Upper Bound

In 1956, by a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance, Gurland in [50, page 645] established a closer approximation to π

$$\frac{4k+3}{(2k+1)^2}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2 < \pi < \frac{4}{4k+1}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2, \quad k \in \mathbb{N} \tag{2.10}$$

through presenting

$$\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^2 < \frac{n^2}{2n+1}, \quad n \in \mathbb{N}. \tag{2.11}$$

Remark 2.5. The double inequality (2.10) may be rearranged as

$$\frac{\sqrt{4k+3}}{\sqrt{\pi}\ (2k+1)} < \frac{(2k-1)!!}{(2k)!!} < \frac{2}{\sqrt{\pi(4k+1)}}, \quad k \in \mathbb{N}. \tag{2.12}$$

Remark 2.6. The inequality (2.11) is better than the right-hand side inequality in (2.8) for $x = (n-1)/2$ and $s = 1/2$.

Remark 2.7. Taking, respectively, $n = 2k$ and $n = 2k-1$ for $k \in \mathbb{N}$ in (2.11) leads to

$$\sqrt{k+\frac{1}{4}}\ < \frac{\Gamma(k+1)}{\Gamma(k+1/2)} < \frac{2k}{\sqrt{4k-1}}\ , \quad k \in \mathbb{N}. \tag{2.13}$$

This is better than (2.8) for $x = k$ and $s = 1/2$. We will see that it is also better than (2.23) for $s = 1/2$ and it is the same as (2.31).

Remark 2.8. It is astonishing that inequalities in (2.11) or (2.12) were recovered in [51] by a different but elementary approach. In other words, the inequality (2.11) and the right-hand side inequality in (2.30) are the same. See Section 2.5.

Remark 2.9. Just like the paper [42], Gurland's paper [50] was also neglected until 1966 in [52] and 1985 in [53]. The famous monograph [54] recorded neither of the papers [42, 50]. It is a pity, since inequalities in (2.10) and (2.11) are very sharp, as discussed in Remark 2.12 below.

Remark 2.10. For more information on new developments of bounding Wallis' formula (1.5), please refer to Section 7.4.

2.3. Kazarinoff's Bounds for Wallis' Formula

In 1956, starting from one form of the celebrated formula of John Wallis,

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n}}, \quad n \in \mathbb{N}, \tag{2.14}$$

which had been quoted for more than a century before 1950s by writers of textbooks, it was proved in [55] that the sequence $\theta(n)$ defined by

$$\frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi(n+\theta(n))}} \tag{2.15}$$

satisfies $1/4 < \theta(n) < 1/2$ for $n \in \mathbb{N}$. This implies

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \tag{2.16}$$

It was said in [55] that it is unquestionable that inequalities similar to (2.16) can be improved indefinitely but at a sacrifice of simplicity, which is why they have survived so long.

The proof of (2.16) is based upon the property

$$\left[\ln \phi(t)\right]'' - \left\{\left[\ln \phi(t)\right]'\right\}^2 > 0 \tag{2.17}$$

of the function

$$\phi(t) = \int_0^{\pi/2} \sin^t x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)} \tag{2.18}$$

for $-1 < t < \infty$. The inequality (2.17) was proved by making use of (1.12) and estimating the integrals

$$\int_0^1 \frac{x^t}{1+x} dx, \qquad \int_0^1 \frac{x^t \ln x}{1+x} dx. \tag{2.19}$$

Since (2.17) is equivalent to the statement that the reciprocal of $\phi(t)$ has an everywhere negative second derivative, therefore, for any positive t, $\phi(t)$ is less than the harmonic mean of $\phi(t-1)$ and $\phi(t+1)$; simplifying this leads to the fact that

$$\frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)} < \frac{2}{\sqrt{2t+1}}, \quad t > 0. \tag{2.20}$$

As a subcase of this result, the right-hand side inequality in (2.16) is established.

Remark 2.11. Replacing t by $2t$ for $t > 0$ in (2.20) leads to

$$\frac{\Gamma(t+1/2)}{\Gamma(t+1)} < \frac{1}{\sqrt{t+1/4}} \tag{2.21}$$

for $t > 0$, which is better than the left-hand side inequality in (2.8) for $s = 1/2$ and extends the left-hand side inequality in (2.13).

Remark 2.12. The right-hand side inequality in (2.12) is the same as the corresponding one in (2.16), and the left-hand side inequality in (2.12) is better than the corresponding one in (2.16) and (3.6) for $n \geq 2$. Therefore, Gurland's inequality (2.11) is much sharp.

Remark 2.13. A double inequality bounding the quantity $(2k-1)!!/(2k)!!$ can be reduced to an upper or a lower bound for the ratio $\Gamma((n+1)/2)/\Gamma(n/2)$. Conversely, either the upper or the lower bound for the ratio $\Gamma((n+1)/2)/\Gamma(n/2)$ can be used to derive a double inequality bounding the quotient $(2k-1)!!/(2k)!!$.

Remark 2.14. The idea and spirit of Kazarinoff in [55] would be developed by Watson in [56]. See Section 3.1.

2.4. Gautschi's Double Inequalities

In 1959, among other things, by a different motivation from Wendel in [42], Gautschi established independently in [57] two double inequalities for $n \in \mathbb{N}$ and $0 \leq s \leq 1$:

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq \exp\big((1-s)\psi(n+1)\big), \tag{2.22}$$

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s}. \tag{2.23}$$

Remark 2.15. It is clear that the upper bound and the domain in the inequality (2.23) are not better and more extensive than the corresponding ones in (2.8).

Remark 2.16. The upper bounds in (2.8), (2.22), and (2.23) have the following relationships:

$$\exp\big((1-s)\psi(n+1)\big) \leq (n+1)^{1-s} \tag{2.24}$$

for $0 < s \leq 1$ and $n \in \mathbb{N}$,

$$(n+s)^{1-s} \leq \exp\big((1-s)\psi(n+1)\big) \tag{2.25}$$

for $0 \leq s \leq 1/2$ and $n \in \mathbb{N}$, and the inequality (2.25) reverses for $s > e^{1-\gamma} - 1 = 0.52620\cdots$, since the function

$$Q(x) = e^{\psi(x+1)} - x \tag{2.26}$$

was proved in [58, Theorem 2] to be strictly decreasing on $(-1,\infty)$, with

$$\lim_{x\to\infty} Q(x) = \frac{1}{2}. \tag{2.27}$$

This means that Wendel's double inequality (2.8) and Gautschi's first double inequality (2.22) are not included in each other but they all contain Gautschi's second double inequality (2.23).

Remark 2.17. By the convex property of $\ln\Gamma(x)$, Merkle recovered in [20, 59–63] inequalities in (2.22) and (2.23) once again. See Section 4.

Remark 2.18. The monotonic and convex properties of the function (2.27) are also derived in [64]. See Section 3.20.1 and Remark 3.56 to Remark 3.58.

Remark 2.19. The Mathematical Reviews' comments MR0103289 on the paper [57] are cited as follows. The author gives lower and upper bounds of the form $c[(x^p + (1/c))^{1/p} - x]$ for $e^{x^p}\int_x^\infty e^{-t^p}\mathrm{d}t$ in the range $p > 1$ and $0 \le x < \infty$; the respective values of c are 2 and $[\Gamma(1 + (1/p))]^{p/(p-1)}$. As it stands, the proof is only valid if p is an integer, but, in a correction, the author has indicated a modification which validates it for all $p > 1$.

Remark 2.20. There is no word commenting on inequalities in (2.22) and (2.23) by the Mathematical Reviews' reviewer of the paper [57]. However, these two double inequalities later became a major source of a series of research on bounding the ratio of two gamma functions.

Remark 2.21. The function $e^{x^p}\int_x^\infty e^{-t^p}\mathrm{d}t$ was further investigated in [65–72] and related references therein.

2.5. Chu's Double Inequality

In 1962, by discussing that $b_{n+1}(c) \gtreqless b_n(c)$ if and only if $(1-4c)n + 1 - 3c \gtreqless 0$, where

$$b_n(c) = \frac{(2n-1)!!}{(2n)!!}\sqrt{n+c}\,, \tag{2.28}$$

it was demonstrated in [51, Theorem 1] that

$$\frac{1}{\sqrt{\pi[n+(n+1)/(4n+3)]}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n\in\mathbb{N}. \tag{2.29}$$

As an application of (2.29), by using $\Gamma(1/2) = \sqrt{\pi}$ and (1.2), the following double inequality

$$\sqrt{\frac{2n-3}{4}} < \frac{\Gamma(n/2)}{\Gamma(n/2-1/2)} \le \sqrt{\frac{(n-1)^2}{2n-1}} \tag{2.30}$$

for positive integers $n \ge 2$ was given in [51, Theorem 2].

Remark 2.22. After letting $x = (n-1)/2$ the inequality (2.30) becomes

$$\sqrt{x-\frac{1}{4}} < \frac{\Gamma(x+1/2)}{\Gamma(x)} < \frac{x}{\sqrt{x+1/4}}, \tag{2.31}$$

which is the same as (2.13).

Remark 2.23. When n is large enough, the lower bound in (2.29) is better than the one in (3.6).

Remark 2.24. Any one of the bounds in (2.31) may be derived from the other one by Boyd's method in [73] (see Section 3.4), by Shanbhag's technique in [74] (see Section 3.5), by Raja Rao's technique in [75] (see Section 3.10), by Slavić's method in [76] (see Section 3.10), or by the β-transform in Section 4.1. This implies that the double inequality (2.30) is equivalent to the inequality (2.11).

Remark 2.25. The double inequality (2.29) and the right-hand side inequality in (2.30) are a recovery of (2.12) and (2.11), respectively. Notice that the reasoning directions in the two papers [50, 51] are opposite:

$$\frac{(2n-1)!!}{(2n)!!} \underset{[50]}{\overset{[51]}{\rightleftharpoons}} \frac{\Gamma(n/2)}{\Gamma(n/2-1/2)}. \tag{2.32}$$

This confirms again what is said in Remark 2.13.

Remark 2.26. The idea of Chu's proof in [51, Theorem 1] has the same spirit as Kershaw's in [77]. See Section 3.12.

2.6. Zimering's Inequality

In 1962, Zimering obtained in [78, page 88] that

$$\frac{\Gamma(n+r)}{n!} \leq \frac{n^r-(n-1)^r}{r} \tag{2.33}$$

for $0 < r < 1$ and $n \in \mathbb{N}$.

Remark 2.27. From (1.2) it is easy to see that $n! = \Gamma(n+1)$. Hence, the inequality (2.33) can be rearranged as

$$\frac{\Gamma(n+1)}{\Gamma(n+r)} \geq \frac{r}{n^r-(n-1)^r} \tag{2.34}$$

for $0 < r < 1$ and $n \in \mathbb{N}$. Although the inequality (2.33) or (2.34) is not better than the left-hand side inequality in (2.22) or (2.23), since its motivation is particular, it is believed that it was obtained independently, and so the paper [78] can also be regarded as an origin of this topic.

2.7. Further Remarks

Remark 2.28. To the best of our knowledge and understanding, two evidences that there was no cross-citation between them and that their motivations are different convince us to believe that the above origins are independent. Actually, the very real origin(s) may not be found out forever.

Remark 2.29. Except Wendel's result, all inequalities above take values on $\mathbb{N}$, the set of positive integers. In other words, only Wendel's double inequality (2.8), the earliest result on this topic, takes values on $(0, \infty)$, the set of real numbers.

Remark 2.30. As one will see, in the history of this topic, the works by Wendel, Gurland, and Zimering did not become a source of bounding the ratio of two gamma functions.

Remark 2.31. In [53], some of the extensive previous background of the papers [55, 56] was outlined.

Remark 2.32. Currently, we may conclude that the very origins of bounding the ratio of two gamma functions are asymptotic analysis, estimation of Wallis' cosine formula, estimation of π, and mathematical statistics.

Remark 2.33. The good bounds for the ratio of two gamma functions should satisfy one or several of the following criteria.

(1) The bounds should be easily computed by hand or by computers.

(2) Sharper the bounds are, better the bounds are.

(3) The bounds should be simple in form.

(4) The bounds should be beautiful in form.

(5) The bounds should be expressed by elementary functions or any other easily calculated functions.

(6) The bounds are of some recurrent or symmetric properties.

(7) The bounds should have origin(s) and background(s).

(8) The bounds should have application(s) in mathematics or mathematical sciences.

Maybe these standards are also suitable for judging any other inequalities and estimates in mathematics.

3. Refinements and Extensions

In this section, the refinements and extensions of bounds for the ratio of two gamma functions from 1959 will be collected, to the best of our ability.

3.1. Watson's Monotonicity Result

In 1959, motivated by the result in [55], mentioned in Section 2.3, and basing on Gauss' Theorem (1.13), Watson observed in [56] that

$$\begin{aligned}\frac{[\Gamma(x+1)]^2}{x[\Gamma(x+1/2)]^2} &= {}_2F_1\left(-\frac{1}{2},-\frac{1}{2};x;1\right)\\ &= 1+\frac{1}{4x}+\frac{1}{32x(x+1)}+\sum_{r=3}^{\infty}\frac{[(-1/2)\cdot(1/2)\cdot(3/2)\cdot(r-3/2)]^2}{r!x(x+1)\cdots(x+r-1)}\end{aligned} \tag{3.1}$$

for $x > -1/2$, which implies that the much general function

$$\theta(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right]^2 - x, \tag{3.2}$$

ever discussed in [55] or Section 2.3 as a special case $\theta(n)$ for $n \in \mathbb{N}$, for $x > -1/2$ is decreasing and with

$$\lim_{x\to\infty}\theta(x) = \frac{1}{4}, \qquad \lim_{x\to(-1/2)^+}\theta(x) = \frac{1}{2}. \tag{3.3}$$

This apparently implies the sharp inequalities

$$\frac{1}{4} < \theta(x) < \frac{1}{2} \tag{3.4}$$

for $x > -1/2$,

$$\sqrt{x+\frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \sqrt{x+\frac{1}{4}+\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^2} = \sqrt{x+0.36423\cdots} \tag{3.5}$$

for $x \ge -1/4$, and, by (1.5),

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \tag{3.6}$$

In [56], an alternative proof of the double inequality (3.4) was provided as follows. Let

$$f(x) = \frac{2}{\sqrt{\pi}}\int_0^{\pi/2}\cos^{2x}t\,dt = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-xt^2\right)\frac{t\exp\left(-t^2/2\right)}{\sqrt{1-\exp(-t^2)}}dt \tag{3.7}$$

for $x > 1/2$. By using the fairly obvious inequalities

$$\begin{aligned} \sqrt{1-\exp(-t^2)} &\le t \\ \frac{t\exp(-t^2/4)}{\sqrt{1-\exp(-t^2)}} &= \frac{t}{\sqrt{2\sinh(t^2/2)}} \le 1, \end{aligned} \tag{3.8}$$

we have, for $x > -1/4$,

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(-\left(x+\frac{1}{2}\right)t^2\right)\mathrm{d}t < f(x) < \frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(-\left(x+\frac{1}{4}\right)t^2\right)\mathrm{d}t, \tag{3.9}$$

that is to say

$$\frac{1}{\sqrt{x+1/2}} < f(x) < \frac{1}{\sqrt{x+1/4}}. \tag{3.10}$$

Remark 3.1. In [56, page 8], the following interesting relation was provided:

$$x+\theta(x) = \frac{x^2}{x-1/2+\theta(x-1/2)} \tag{3.11}$$

for appropriate ranges of values of x.

Remark 3.2. The formula (3.1) would be used in [73] to obtain the inequality (3.23).

Remark 3.3. The function $\theta(x)$ defined by (3.2) was extended and studied in [37–40, 64, 79–84] later.

Remark 3.4. It is easy to see that the inequality (3.5) extends and improves (2.8) if $s = 1/2$, say nothing of (2.22) and (2.23) if $s = 1/2$.

Remark 3.5. The left-hand side inequality in (3.6) is better than the corresponding one in (2.16) but worse than the corresponding one in (2.12) for $n \ge 2$.

Remark 3.6. The double inequality (3.6) for bounding Wallis' formula (1.5) was recovered, refined, or generalized recently in [70, 85–94] and related references therein. For more information on bounds for Wallis' formula (1.5), please refer to Sections 2.2, 2.3, and 7.4 of this paper.

Remark 3.7. It is easy to see that

$$\frac{\theta(x)}{x}+1 = \frac{\Gamma(x)\Gamma(x+1)}{[\Gamma(x+1/2)]^2} \tag{3.12}$$

which is a special case of Gurland's ratio

$$T(x,y)=\frac{\Gamma(x)\Gamma(y)}{\left[\Gamma\left((x+y)/2\right)\right]^2} \tag{3.13}$$

defined first in [95] for positive numbers x and y.

The formula (3.12) reveals that bounds for Gurland's ratio $T(x,y)$ can be reduced to bounds for $\Gamma(x)/\Gamma(x+1/2)$.

For more information on bounding Gurland's ratio, please refer to [20, 44, 96, 97] and related references therein. However, there does not exist a general identity similar to (3.12) between Gurland's ratio and the ratio of two gamma functions. As a result, considering the limitation of length of this paper, new developments on Gurland's ratio (3.13) will not be involved in detail.

3.2. Erber's Inequality

Gurland proved in [95] that

$$\frac{[\Gamma(\delta+\alpha)]^2}{\Gamma(\delta)\Gamma(\delta+2\alpha)}\le\frac{\delta}{\delta+\alpha^2}, \tag{3.14}$$

where $\alpha\neq 0$, $\alpha+2\delta>0$, and $\delta>0$. In [97], the following results were derived from the right-hand side inequality in (2.23) and (3.14).

(1) Taking in (3.14) $\delta=n\in\mathbb{N}$ and $\alpha=(s+1)/2$ for $s\in(0,1)$ and rearranging lead to

$$\frac{\Gamma(n+1)}{\Gamma(n+s)}<\frac{4(n+s)}{4n+(s+1)^2}\left[\frac{\Gamma(n+1)}{\Gamma(n+(1+s)/2)}\right]^2,\quad 0<s<1,\ n\in\mathbb{N}. \tag{3.15}$$

Since $0<(1+s)/2<1$, applying the right-hand side inequality in (2.23) to the ratio in the bracket yields a strengthened upper bound of (2.23)

$$\frac{\Gamma(n+1)}{\Gamma(n+s)}<\frac{4(n+s)}{4n+(s+1)^2}(n+1)^{1-s},\quad 0<s<1,\ n\in\mathbb{N}. \tag{3.16}$$

(2) Letting $\delta=n$ and $0<s=\alpha<1$ in (3.14) and using the right-hand side inequality in (2.23), then

$$\frac{\Gamma(n+s)}{\Gamma(n+2s)}<\frac{(n+1)^{1-s}}{n+s^2},\quad 0<s<1,\ n\in\mathbb{N}. \tag{3.17}$$

(3) After $k+1$ iterations of the above process, it was obtained that

$$\frac{\Gamma(n+s)}{\Gamma(n+2s)} < \frac{(n+1)^{1-s}}{(n+s^2)R(n,s,k)}, \qquad \frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{(n+1)^{1-s}}{R(n,s,k)}, \tag{3.18}$$

where

$$R(n,s,k) = \prod_{i=0}^{k} \left\{ \frac{n + \left[\left(s + 2^{i+1} - 1\right)/2^{i+1}\right]^2}{n + \left(s + 2^i - 1\right)/2^i} \right\}^{2^i} \tag{3.19}$$

for $n, k \in \mathbb{N}$ and $0 < s < 1$.

In the final of [97], it was pointed out that it is ready to verify that the limit $\lim_{k\to\infty} R(n,s,k)$ exists and that it would be interesting to know the value of this infinite product in closed form.

Remark 3.8. It is easy to observe that bounds for Gurland's ratio provide a method to refine bounds for ratio of two gamma functions. Conversely, it is also done.

3.3. Uppuluri's Bounds

If X is a random variable defined on a probability space and E denotes the expectation operator, then $\{E|X|^r\}^{1/r}$ is a nondecreasing function of $r > 0$. See [98, page 156]. Using this conclusion, the double inequality (2.8) was recovered in [99] for $x > 0$ and $0 \le s \le 1$, which sharpens the inequality (2.23) given in [57].

Following the same lines as in [97] or Section 3.2, after $k+1$ iterations, Rao Uppuluri further obtained in [99] that

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{\left[x + \left(s - 1 + 2^{k+1}\right)/2^{k+1}\right]^{1-s}}{R(x,s,k)}, \qquad \frac{\Gamma(x+s)}{\Gamma(x+2s)} < \frac{\left[x + \left(s - 1 + 2^{k+1}\right)/2^{k+1}\right]^{1-s}}{(x+s^2)R(x,s,k)} \tag{3.20}$$

for $x > 0$, $0 \le s \le 1$, and $k \in \mathbb{N}$, which improve inequalities (3.18).

Remark 3.9. In [74, page 48], Shanbhag pointed out that the discussion concerning (3.20) and (3.21) in [99] is misleading.

3.4. Uppuluri-Boyd's Double Inequality

Motivated by (2.14) and the results in [50] (see Section 2.2), the following double inequality for $m \in \mathbb{N}$ was obtained in [52]:

$$\left(m+\frac{1}{4}+\frac{9}{48m+32}\right)^{1/2} < \frac{\Gamma(m+1)}{\Gamma(m+1/2)} < \left[\frac{(m+1/2)^2}{m+3/4+9/(48m+56)}\right]^{1/2}. \tag{3.21}$$

In [73], the left-hand side inequality in (3.21) was pointed out to be false and it was further demonstrated by using the formula (3.1) that the inequality

$$\frac{\Gamma(m+1)}{\Gamma(m+1/2)} > \left(m+\frac{1}{4}+\frac{1}{am+b}\right)^{1/2} \tag{3.22}$$

is impossible to be true for all positive integers m if $a < 32$. Moreover, the following double inequality for $m \in \mathbb{N}$ was listed in [73]:

$$\sqrt{m+\frac{1}{4}+\frac{1}{32(m+1)}} < \frac{\Gamma(m+1)}{\Gamma(m+1/2)} < \frac{m+1/2}{\sqrt{m+3/4+1/(32m+48)}} \tag{3.23}$$

by considering

$$\frac{\Gamma(m+1)}{\Gamma(m+1/2)} = \frac{m+1/2}{\Gamma(m+3/2)/\Gamma(m+1)}. \tag{3.24}$$

Reminded by [73], the author of [52] went through the computations in finding the Bhattacharya bounds in [52] and made corrections in [100]. The double inequality (3.23) was recovered in [100].

Remark 3.10. The technique used in (3.24) was employed once again in [76], see also Section 3.10, and summarized in [20] as the so-called β-transform and π_n-transform, see also Section 4.

Remark 3.11. It is obvious that the lower bound in (3.23) is better than the corresponding ones in (2.8), (2.11) and (2.13), (2.22) and (2.23), (2.30) and (2.31), (2.33) and (2.34), and (3.5).

3.5. Shanbhag's Inequalities

Motivated by [99], it was first pointed out in [74] that the right-hand side inequality in (2.8) may be deduced from the left-hand side inequality in (2.8) by observing

$$\frac{\Gamma((x+s)+1)}{\Gamma((x+s)+(1-s))} \geq (x+s)^{1-(1-s)} \Longleftrightarrow \frac{(x+s)\Gamma(x+s)}{\Gamma(x+1)} \geq (x+s)^s. \tag{3.25}$$

Then, by (2.8) and the technique stated in (3.25), a more general double inequality was established:

$$\alpha_0(x,s) < \alpha_1(x,s) < \cdots < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \cdots < \beta_1(x,s) < \beta_0(x,s), \tag{3.26}$$

where

$$\begin{aligned} \alpha_k(x,s) &= \frac{(x+k)^{1-s}(x+s+k-1)^{(k)}}{(x+k)^{(k)}}, \\ \beta_k(x,s) &= \frac{(x+k+s)^{1-s}(x+s+k-1)^{(k)}}{(x+k)^{(k)}} \end{aligned} \tag{3.27}$$

for $k \geq 0$, and

$$y^{(m)} = \begin{cases} 1, & m = 0, \\ y(y-1)\cdots(y-m+1), & m \geq 1. \end{cases} \tag{3.28}$$

From the inequality (3.26), the following corollaries were deduced in [74].

(1) If $x \notin \mathbb{N}$, then

$$\theta_0(x) < \theta_1(x) < \cdots < \gamma(x) < \cdots < \xi_1(x) < \xi_0(x), \tag{3.29}$$

where

$$\begin{aligned} \theta_k(x) &= \frac{(x+k)^{x-[x]}([x]+k)!}{(x+k)^{(k+1)}}, \\ \xi_k(x) &= \frac{([x]+k+1)^{x-[x]}([x]+k)!}{(x+k)^{(k+1)}} \end{aligned} \tag{3.30}$$

for all nonnegative integer k and with $[x]$ being the largest integer less than x.

(2) If $0 < s \leq 1$, then

$$s + \frac{1}{s} - 1 < \Gamma(s) < \frac{1}{s}. \tag{3.31}$$

(3) If $x > 0$, $0 < s < 1$, and k is a nonnegative integer, then

$$\eta_0(x,s) < \eta_1(x,s) < \cdots < \frac{\Gamma(x+s)}{\Gamma(x+2s)} < \cdots < \rho_1(x,s) < \rho_0(x,s), \tag{3.32}$$

where

$$\eta_k(x,s) = \frac{(x+s+k)^{1-s}(x+2s+k-1)^{(k)}}{(x+s+k)^{(k+1)}},$$
$$\rho_k(x,s) = \frac{(x+2s+k)^{1-s}(x+2s+k-1)^{(k)}}{(x+s+k)^{(k+1)}}. \tag{3.33}$$

It was also proved in [74] that

$$\beta_0(x,s) < T_0(x,s) < T_1(x,s) < \cdots,$$
$$\rho_0(x,s) < P_0(x,s) < P_1(x,s) < \cdots, \tag{3.34}$$

where

$$T_k(x,s) = \frac{\left[x + \left(s - 1 + 2^{k+1}\right)/2^{k+1}\right]^{1-s}}{R(x,s,k)},$$
$$P_k(x,s) = \frac{T_k(x,s)}{x+s^2} \tag{3.35}$$

for $x > 0$, $0 < s < 1$, and k is a nonnegative integer, hence Shanbhag pointed out in [74, page 48] that the discussion concerning (3.20) and (3.21) in [99] is misleading.

Remark 3.12. The method used in [74] is the same as the technique utilized in (3.24) which has been summarized as the π_n-transform II (x,β,n) in Section 4.2.

3.6. Raja Rao's Results

Based on [57, 74, 99] and by using Liapounoff's inequality and probability distribution functions, the double inequalities (2.23) and (3.26) were recovered in [75].

It was also showed in [75] that

$$\beta_k(x,s) = \frac{x+s}{\alpha_k(x+s,1-s)}, \tag{3.36}$$

so the inequality (3.26) can be written as

$$\alpha_k(x,s) \le \frac{\Gamma(x+1)}{\Gamma(x+s)} \le \frac{x+s}{\alpha_k(x+s,1-s)}. \tag{3.37}$$

Moreover, the following double inequalities on the hypergeometric functions were also obtained in [75]:

$$\begin{aligned}\frac{\Gamma(x+1)}{\Gamma(x+s)}(x+s+k)^{s-1} &\le {}_2F_1(-k,1-s;x+1;1) \le \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+k)^{s-1},\\ \frac{x+k}{x+k+1} &\le \left[\frac{{}_2F_1(-k-1,1-s;x+1;1)}{{}_2F_1(-k,1-s;x+1;1)}\right]^{1/(1-s)} \le \frac{x+s+k}{x+s+k+1},\end{aligned} \tag{3.38}$$

where $x > 0$, $0 \le s \le 1$, $k = 0,1,2,\ldots$, and ${}_2F_1(a,b;c;1)$ is the hypergeometric function defined by (1.13).

In [101–104], Raja Rao established some generalized inequalities and analogues for incomplete gamma functions, beta functions and hypergeometric functions, similar to the double inequality (2.8).

3.7. Kečkić-Vasić's Double Inequality

In 1971, by considering monotonic properties of

$$x + \ln\Gamma(x) - x\ln x + \alpha\ln x \tag{3.39}$$

on $(1,\infty)$ for $\alpha = 1/2$ or 1, respectively, among other things, Kečkić and Vasić gave in [105, Theorem 1] the following double inequality for $b > a > 1$:

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b}. \tag{3.40}$$

Remark 3.13. Taking $b = x+1$ and $b = x+s$ in Kečkić and Vasić's double inequality (3.40) gives

$$\frac{(x+1)^x}{(x+s)^{x+s-1}}e^{-(1-s)} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{(x+1)^{x+1/2}}{(x+s)^{x+s-1/2}}e^{-(1-s)} \tag{3.41}$$

for $0 < s < 1$ and $x > 0$.

Remark 3.14. In [105], inequalities in (3.40) were compared with those in (2.23), (2.30), and (2.33). For example, if taking $b = n/2$ and $a = (n-1)/2$ and letting n be large enough, then the double inequality (3.40) is not sharper than (2.30), say nothing of the inequality (3.23).

Remark 3.15. For more information on extensions and refinements of the inequality (3.40), please refer to Remark 3.43 and Section 5.

3.8. Amos' Sharp Upper Bound

In 1973, in an appendix of the paper [106], starting with the asymptotic expansion

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + R \tag{3.42}$$

for $z > 0$ and estimate R by the next term $|R| \le 1/1260z^5$, see [1, page 257, 6.1.42], the following inequality was established in [106, pages 425–427]:

$$\frac{[\Gamma(x+1)]^2}{[\Gamma(x+1/2)]^2} < x\left(1 + \frac{1}{4x} + \frac{1}{32x^2} - \frac{1}{128x^3} + \frac{6}{5x^4}\right) \tag{3.43}$$

for $x \ge 2$. This expression is asymptotically correct in all terms except the last.

Remark 3.16. In virtue of the techniques used in [73–75], a lower bound for (3.43) can be procured from its upper bound.

3.9. Lazarević-Lupaş's Convexity

In 1974, among other things, the function

$$\theta_\alpha(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+\alpha)}\right]^{1/(1-\alpha)} - x \tag{3.44}$$

on $(0,\infty)$ for $\alpha \in (0,1)$ was claimed in [80, Theorem 2] to be decreasing and convex, and so

$$\frac{\alpha}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+\alpha)}\right]^{1/(1-\alpha)} - x \le [\Gamma(\alpha)]^{1/(1-\alpha)}. \tag{3.45}$$

Remark 3.17. Although Lazarević-Lupaş's proof given in [80] on monotonic and convex properties of $\theta_\alpha(x)$ is wrong, as commented in [64, page 240], these properties are correct, as we know now.

Remark 3.18. Taking $\alpha = 1/2$ in (3.44) leads to Watson's monotonicity result in Section 3.1, but the range of x here is slightly smaller. Note that Watson did not discuss in [56] the convex property of the function $\theta(x)$ defined by (3.2).

Remark 3.19. The function $\theta_\alpha(x)$ would be extended and the same properties would be verified in [37–40, 64, 79]. See Sections 3.20.1 and 6.1.

Remark 3.20. It seems that the problem discussed in [80, Theorem 1] on characterization of the gamma function was further carried out by Merkle in [20, 59] and Lorch in [107] and related references therein.

3.10. Slavić's Double Inequalities

In 1975, by virtue of (1.2), the following implications were pointed out in [76, page 19]:

$$f(x) \le \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \Longrightarrow \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \frac{x+1/2}{f(x+1/2)}, \tag{3.46}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le g(x) \Longrightarrow \frac{x}{g(x-1/2)} \le \frac{\Gamma(x+1)}{\Gamma(x+1/2)}. \tag{3.47}$$

In particular, adopting

$$g(x) = \sqrt{x + \frac{1}{4} + \frac{1}{32x+8}} \tag{3.48}$$

in (3.47) leads to

$$\sqrt{x + \frac{1}{4} + \frac{1}{32x+8+36/(4x-1)}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x + \frac{1}{4} + \frac{1}{32x+8}}\,. \tag{3.49}$$

On basis of Duplication formula (1.7) and Binet's first formula (1.9), the following integral representation was also given in [76]:

$$\begin{aligned} &\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \\ &\quad = \sqrt{x}\ \exp\left\{\sum_{k=1}^{n} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}} \times \int_0^\infty \left[\frac{\tanh t}{2t} - \sum_{k=1}^{n} \frac{2^{2k}(2^{2k}-1)B_{2k}}{2\cdot(2k)!} t^{2k-2}\right] e^{-4tx}\,\mathrm{d}t\right\}, \end{aligned} \tag{3.50}$$

from which, a more accurate double inequality was procured:

$$\sqrt{x}\ \exp\left(\sum_{k=1}^{2m} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right) < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x}\ \exp\left(\sum_{k=1}^{2\ell-1} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right) \tag{3.51}$$

for $x > 0$, where m and ℓ are natural numbers and B_{2k} for $k \in \mathbb{N}$ are Bernoulli numbers.

Remark 3.21. Why can the function $g(x)$ in (3.47) be taken as (3.48)? There was not any clue to it in [76], but the double inequality (3.49) is surely sound.

Remark 3.22. What are the ranges of x in the double inequalities (3.46), (3.47) and (3.49)? These were not provided explicitly in [76]. As we know now, the double inequality (3.49) is valid for $x > -1/4$.

Remark 3.23. It was claimed in [76] that inequalities in (3.49) are sharper than those in (3.23) and many other inequalities mentioned above. In fact, it is true.

Remark 3.24. It was also claimed in [76] that inequalities in (3.51) are sharper than those in (3.49), but there was no proof supplied in it.

Remark 3.25. We conjecture that the constants 32 and 8 in the upper bound of (3.49) are the best possible.

Remark 3.26. The lower bound in (3.49) would be refined by the corresponding one in (7.24) obtained in [108, Theorem 1].

Remark 3.27. The method showed by (3.46) and (3.47) had been used in [73–75] when proving the double inequality (3.23) and it was summarized in [20] as the β-transform in Section 4.1.

3.11. Imoru's Refinements of Inequalities by Uppuluri-Boyd and Slavić

In [109], it was obtained that

$$\sqrt{m+\theta(m)} < \frac{\Gamma(m+1)}{\Gamma(m+1/2)} < \frac{m+1/2}{\sqrt{m+1/2+\theta(m+1/2)}} \tag{3.52}$$

for

$$\frac{1}{4} < \theta(m) \le \frac{1}{2}. \tag{3.53}$$

In particular,

(1) taking in (3.52)

$$\theta(m) = \frac{1}{4} + \frac{1}{32(m+1)} \tag{3.54}$$

for $m \in \mathbb{N}$ leads to

$$\sqrt{m+\frac{1}{4}+\frac{1}{32(m+1)}} < \frac{\Gamma(m+1)}{\Gamma(m+1/2)} < \frac{m+1/2}{\sqrt{m+3/4+1/32(m+1)}}, \tag{3.55}$$

an improved version of (3.23).

(2) Putting in (3.52)

$$\theta(m) = \frac{1}{4} + \frac{1}{32m+8+36/(4m-1)} \tag{3.56}$$

for $m \in \mathbb{N}$ yields (3.49).

(3) Letting in (3.52)

$$\theta(m) = \frac{1}{4} + \frac{1}{32m+8+36/(4m+5)} \tag{3.57}$$

for $m \in \mathbb{N}$ gives

$$\sqrt{m+\frac{1}{4}+\frac{1}{32m+8+36/(4m+5)}} < \frac{\Gamma(m+1)}{\Gamma(m+1/2)} < \frac{m+1/2}{\sqrt{m+3/4+1/4[4m+6+9/(4m+7)]}}. \tag{3.58}$$

3.12. Kershaw's Double Inequalities and Proofs

In 1983, motivated by the inequality (2.22) in [57], Kershaw presented in [77] the following two double inequalities for $0 < s < 1$ and $x > 0$:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}, \tag{3.59}$$

$$\exp\left[(1-s)\psi\left(x+\sqrt{s}\,\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{3.60}$$

They are called in the literature Kershaw's first and second double inequalities, respectively, although the order of these two inequalities (3.59) and (3.60) reverses the original order in [77].

Kershaw's Proof for (3.59) *and* (3.60). Define the functions f_α and g_β by

$$f_\alpha(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\big((s-1)\psi(x+\alpha)\big), \qquad g_\beta(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+\beta)^{s-1} \tag{3.61}$$

for $x > 0$ and $0 < s < 1$, where the parameters α and β are to be determined.

It is not difficult to show, with the aid of Stirling's formula (1.4), that

$$\lim_{x\to\infty} f_\alpha(x) = \lim_{x\to\infty} g_\beta(x) = 1. \tag{3.62}$$

Now let

$$F(x) = \frac{f_\alpha(x)}{f_\alpha(x+1)} = \frac{x+s}{x+1}\exp\frac{1-s}{x+\alpha}. \tag{3.63}$$

Then

$$\frac{F'(x)}{F(x)} = (1-s)\frac{(\alpha^2-s)+(2\alpha-s-1)x}{(x+1)(x+s)(x+\alpha)^2}. \tag{3.64}$$

It is easy to show that

(1) if $\alpha = s^{1/2}$, then $F'(x) < 0$ for $x > 0$,

(2) if $\alpha = (s+1)/2$, then $F'(x) > 0$ for $x > 0$.

Consequently if $\alpha = s^{1/2}$ then F strictly decreases, and since $F(x) \to 1$ as $x \to \infty$ it follows that $F(x) > 1$ for $x > 0$. But, from (3.62), this implies that $f_\alpha(x) > f_\alpha(x+1)$ for $x > 0$ and so $f_\alpha(x) > f_\alpha(x+n)$. Take the limit as $n \to \infty$ to give the result that $f_\alpha(x) > 1$, which can be rewritten as the left-hand side inequality in (3.60). The corresponding upper bound can be verified by a similar argument when $\alpha = (s+1)/2$, the only difference is that in this case f_α strictly increases to unity.

To prove the double inequality (3.59) define

$$G(x) = \frac{g_\beta(x)}{g_\beta(x+1)} = \frac{x+s}{x+1}\left(\frac{x+\beta+1}{x+\beta}\right)^{1-s}, \tag{3.65}$$

from which it follows that

$$\frac{G'(x)}{G(x)} = \frac{(1-s)\left[\left(\beta^2+\beta-s\right)+\left(2\beta-s\right)x\right]}{(x+1)(x+s)\left(x+\beta\right)\left(x+\beta+1\right)}. \tag{3.66}$$

This will leads to

(1) if $\beta = s/2$, then $G'(x) < 0$ for $x > 0$,

(2) if $\beta = -1/2 + (s+1/4)^{1/2}$, then $G'(x) > 0$ for $x > 0$.

The same arguments which were used on F can now be used on G to give the double inequality (3.59). □

Remark 3.28. The limits in (3.62) can also be derived by using (1.10).

Remark 3.29. Since the limits in (3.62) hold, the left-hand side inequality in (3.59) and the right-hand side inequality in (3.60) are immediate consequences of the fact that $f_{(s+1)/2}$ and $g_{s/2}$ are decreasing on $(0,\infty)$.

Remark 3.30. The spirit of Kershaw's proof is similar to Chu's in [51, Theorem 1]. See also Section 2.5.

Remark 3.31. The method used by Kershaw in [77] to prove (3.59) and (3.60) was utilized to construct many similar inequalities in several papers such as [107, 110, 111]. See Remark 3.39.

Remark 3.32. It is easy to see that the inequality (3.59) refines and extends the inequality (2.8), say nothing of (2.23).

Remark 3.33. Since the function $Q(x)$ defined by (2.26) was proved in [58, Theorem 2] to be strictly decreasing on $(-1,\infty)$, the functions

$$h_{1;s}(x) = e^{\psi(x+\sqrt{s}\,)} - \left(x+\frac{s}{2}\right) = e^{\psi(x+\sqrt{s}\,)} - \left(x+\sqrt{s}\,-1\right) - \frac{s}{2} + \sqrt{s}\,-1 \tag{3.67}$$

for $x > -\sqrt{s}$ and

$$\begin{aligned} h_{2;s}(x) &= e^{\psi(x+(s+1)/2)} - \left[x - \frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}\right] \\ &= e^{\psi(x+(s+1)/2)} - \left(x + \frac{s+1}{2} - 1\right) - \left(s + \frac{1}{4}\right)^{1/2} + \frac{s}{2} \end{aligned} \tag{3.68}$$

for $x > -(s+1)/2$, where $0 < s < 1$, are both strictly deceasing. From (2.27), it follows that

$$\lim_{x\to\infty} h_{1;s}(x) = \sqrt{s} \ - \frac{s+1}{2} < 0, \qquad \lim_{x\to\infty} h_{2;s}(x) = \frac{s+1}{2} - \left(s + \frac{1}{4}\right)^{1/2} < 0 \tag{3.69}$$

for $0 < s < 1$. It is apparent that

$$\begin{aligned} h_{1;s}(0) &= e^{\psi(\sqrt{s}\,)} - \frac{s}{2} \triangleq h_1(s), \\ h_{2;s}(0) &= e^{\psi(s+1/2)} + \frac{1}{2} - \left(s + \frac{1}{4}\right)^{1/2} \triangleq h_2(s) \end{aligned} \tag{3.70}$$

for $0 < s < 1$. Direct computation gives

$$\lim_{s\to 1^-} h_1(s) = e^{-\gamma} - \frac{1}{2} > 0, \qquad \lim_{s\to 0^+} h_2(s) = e^{\psi(1/2)} > 0. \tag{3.71}$$

These calculations show that neither (3.59) nor (3.60) is the outright winner. When x is large enough, the lower bound in (3.60) is not better than the one in (3.59), but the upper bound in (3.60) is better than the one in (3.59).

Remark 3.34. Kershaw proved in [77] that if $2x + s \geq 1$ and $0 < s < 1$ then the lower bound in Kershaw's first double inequality (3.59) is an improvement over the lower bound in (3.41).

Remark 3.35. In [77], Kershaw compared his upper bounds with Erber's inequality (3.16), but it is sure that there may be something wrong with his arguments.

3.13. Lorch's Double Inequality

In 1984, by initially unaware utilization of Kershaw's method in [77], see also Section 3.12, Lorch gave in [107] the following results. For nonnegative integers $k \geq 0$, the upper bound in the inequality

$$\left(k + \frac{s}{2}\right)^{s-1} < \frac{\Gamma(k+s)}{\Gamma(k+1)} < (k+s)^{s-1} \tag{3.72}$$

is valid for all $s > 1$, the lower bound in (3.72) is valid for $1 < s < 2$, the left-hand side inequality in (3.72) reverses for $s > 2$, and the double inequality (3.72) reverses for $0 < s < 1$.

Remark 3.36. For $0 < s < 1$, the double inequality (3.72) is not better than (3.59) for $0 < s < 1$, but the range of the parameter s was extended.

Remark 3.37. In the special case in which $s = 1/2$, the inequalities in (3.72) had been established first by Kazarinoff in [55] and then by Watson in [56]. From Watson's monotonicity result in Section 3.1, the upper bounds in (3.59) for $s = 1/2$ and (3.72) may be derived.

Remark 3.38. The motivation of Lorch in [107] was to refine an inequality for ultraspherical polynomials. Inequalities in (3.72) were used in [107] to obtain a very interesting sharpened inequality for ultraspherical polynomials:

$$(\sin\theta)^{\lambda}\left|P_n^{(\lambda)}(\cos\theta)\right| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}(n+\lambda)^{\lambda-1}, \tag{3.73}$$

where $\theta \in [0,\pi]$ and

$$P_n^{(\lambda)}(x) = \sum_{k=0}^{[n/2]}(-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)\Gamma(k+1)\Gamma(n-2k+1)}(2x)^{n-2k} \tag{3.74}$$

for $n \geq 0$ being an integer and $\lambda > 0$ being a real number. The inequality (3.73) refines the Bernstein inequality

$$(\sin\theta)^{\lambda}\left|P_n^{(\lambda)}(\cos\theta)\right| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}n^{\lambda-1} \tag{3.75}$$

for $n \geq 0$, $0 < \lambda < 1$, and $0 \leq \theta \leq \pi$. Earlier in 1975, Durand generalized in [112] the Bernstein inequality (3.75) and, as a consequence of (23) in [112], the following inequality may be derived:

$$(\sin\theta)^{\lambda}\left|P_n^{(\lambda)}(\cos\theta)\right| < \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)} \tag{3.76}$$

for $n \geq 0$, $0 < \lambda < 1$ and $0 \leq \theta \leq \pi$. For more information on further refinements of the Bernstein inequality (3.75), please refer to [113, pages 388-389] and the related references therein.

3.14. Laforgia's Inequalities

In 1984, starting from [56, 57, 77, 107] and employing more carefully Kershaw's and Lorch's methods in [77, 107], by discussing the monotonicity of the function $G(x)$ defined by (3.65) more delicately, Laforgia constructed in [111] a number of inequalities of the type

$$(x+\alpha)^{s-1} < \frac{\Gamma(x+s)}{\Gamma(x+1)} < (x+\beta)^{s-1} \tag{3.77}$$

for $s > 0$ and real number $x \geq 0$.

Remark 3.39. As in [111], thorough analyses of Kershaw's and Lorch's methods had been carried out continuously in [110, 114, 115] and [113, pages 389-390], respectively.

Remark 3.40. By discussing (3.63) subtly, the double inequality (3.60) was extended on s in [110, Section 5].

3.15. Dutka's Double Inequalities

In 1985, some of the extensive previous background of the papers [55, 56] associated with bounding Wallis's cosine formula or Wallis' product formula was outlined in [53].

On the other hand, by using continued fraction expansions for the quotient of beta functions, several bounds for the sequence $\theta(n)$ defined by (2.15), or the function $\theta(x)$ defined by (3.2), or the function $\theta_\alpha(x)$ defined by (3.44) were established in [53]. For $n \in \mathbb{N}$,

$$\left(1+\frac{1}{2n}\right)^{1/2} < \frac{\theta(n)}{n} + 1 < \left(1-\frac{1}{2n}\right)^{-1/2}, \tag{3.78}$$

$$\frac{4n+3}{2(8n+5)} < \theta(n) < \frac{2n}{8n-1}, \tag{3.79}$$

$$\frac{8n^2+13n+6}{32n^2+48n+19} < \theta(n) < \frac{8n+3}{8(4n+1)}, \tag{3.80}$$

$$\frac{1}{4} < \theta(n) < \frac{n+1}{4n+3}, \tag{3.81}$$

and inequalities in (2.10) and (2.29) were recovered.

Remark 3.41. The left-hand side inequalities in (3.4) and (3.79) are better than the one in (3.78), the right-hand side inequalities in (3.78) and (3.79) are better than the corresponding one in (3.4), and the right-hand side inequality in (3.79) is better than the corresponding one in (3.78).

3.16. Ismail-Lorch-Muldoon's Monotonicity Results

In 1986, the logarithmically complete monotonicity of three functions related to the gamma function or its ratio were obtained in [116, Theorems 2.1, 2.4, and 2.6].

(1) The function

$$x^\alpha \Gamma(x)\left(\frac{e}{x}\right)^x \tag{3.82}$$

is logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \le 1/2$, so is the reciprocal of the function (3.82) if and only if $\alpha \ge 1$.

(2) The function

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{3.83}$$

for $a > b \geq 0$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $a+b \geq 1$.

(3) Let

$$\frac{t^{x^\delta}}{\Gamma\left(x^\delta+1\right)} = e^{-h(x)} \tag{3.84}$$

for $0 < t \leq e^{-\gamma}$.

(a) For $0 < \delta \leq 1/2$, the function $h(x)$ is positive and $h'(x)$ is completely monotonic on $(0,\infty)$.

(b) For $\delta = 1$, the functions $h(x)$ and $h'(x)$ are both positive and $h''(x)$ is completely monotonic on $(0,\infty)$.

It was conjectured in [116, page 8] that $h''(x)$ remains completely monotonic for at least some values of $\delta > 1$.

Remark 3.42. The logarithmically complete monotonicity of the function (3.82) was also proved in [117, Theorem 2.1] early in 1978. See also [7, Theorem 2.1].

Remark 3.43. It is clear that the logarithm of the function (3.82) for $\alpha = 1/2$ or 1 equals the function (3.39). Therefore, the monotonic properties of the function (3.82) may be used to derive the double inequality (3.40) for $b > a > 0$ and to show the best possibilities of the constants $1/2$ and 1 in (3.40).

Remark 3.44. Since the limit (1.10) is valid, from the decreasingly monotonic property of the function (3.83), it follows that

$$x^{a-b} < \frac{\Gamma(x+a)}{\Gamma(x+b)} < \frac{x^{a-b}}{x_0^{a-b}}\cdot\frac{\Gamma(x_0+a)}{\Gamma(x_0+b)} \tag{3.85}$$

for $a > b \geq 0$ and $a+b \geq 1$ holds on $[x_0,\infty)$ for any $x_0 > 0$. It is obvious that this extends the left-hand side inequalities in (2.8), (2.22), and (2.23).

Remark 3.45. The conclusions in [116], mentioned above, were not stated using the terminology "logarithmically completely monotonic function", since the authors were not aware of the paper [14] and related papers such as [13, 15–19, 118] have not been published then.

3.17. Bustoz-Ismail's Monotonicity Results

In 1986, it was revealed in [96] that

(1) the function

$$f(x) = \frac{1}{(x+c)^{1/2}} \cdot \frac{\Gamma(x+1)}{\Gamma(x+1/2)}, \quad x > \max\left\{-\frac{1}{2}, -c\right\} \tag{3.86}$$

is logarithmically completely monotonic on $(-c, \infty)$ if $c \le 1/4$, so is the reciprocal of (3.86) on $[-1/2, \infty)$ if $c \ge 1/2$;

(2) the function

$$(x+c)^{a-b} \frac{\Gamma(x+b)}{\Gamma(x+a)} \tag{3.87}$$

for $1 \ge b - a > 0$ is logarithmically completely monotonic on the interval $(\max\{-a, -c\}, \infty)$ if $c \le (a+b)-1/2$, so is the reciprocal of (3.87) on $(\max\{-b, -c\}, \infty)$ if $c \ge a$;

(3) the functions

$$\frac{\Gamma(x+s)}{\Gamma(x+1)} \exp\left[(1-s)\psi\left(x + \frac{s+1}{2}\right)\right], \tag{3.88}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \left(x + \frac{s}{2}\right)^{s-1} \tag{3.89}$$

for $0 < s < 1$ are logarithmically completely monotonic on $(0, \infty)$;

(4) the functions

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \exp\left((s-1)\psi\left(x + \sqrt{s}\ \right)\right), \tag{3.90}$$

$$\left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\ \right)^{1-s} \frac{\Gamma(x+s)}{\Gamma(x+1)} \tag{3.91}$$

for $0 < s < 1$ are strictly decreasing on $(0, \infty)$.

Remark 3.46. The monotonic properties of the function (3.86) implies inequalities (3.4) and (3.79).

Remark 3.47. These monotonicity results generalize, extend, and refine inequalities (2.8), (2.22), (2.23), (3.59), (3.60), monotonic properties of the function (3.83), and so on.

3.18. Alzer's Monotonicity Result

In 1993, it was obtained in [119, Theorem 1] that

$$\frac{\Gamma(x+s)}{\Gamma(x+1)} \frac{(x+1)^{x+1/2}}{(x+s)^{x+s-1/2}} \exp\left(s - 1 + \frac{\psi'(x+1+\alpha) - \psi'(x+s+\alpha)}{12}\right) \tag{3.92}$$

for $\alpha > 0$ and $s \in (0,1)$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \geq 1/2$, so is the reciprocal of (3.92) for $\alpha \geq 0$ and $s \in (0,1)$ if and only if $\alpha = 0$.

In [120, Theorem 3], a slight extension of [119, Theorem 1] was presented. The function

$$\frac{\Gamma(x+s)}{\Gamma(x+t)}\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\exp\left(s-t+\frac{\psi'(x+t+\alpha)-\psi'(x+s+\alpha)}{12}\right) \tag{3.93}$$

for $0 < s < t$ and $x \in (0,\infty)$ is logarithmically completely monotonic if and only if $\alpha \geq 1/2$, so is the reciprocal of (3.93) if and only if $\alpha = 0$.

The decreasingly monotonic properties of (3.93) and its reciprocal imply that

$$\begin{aligned}\exp\left(t-s+\frac{\psi'(x+s+\beta)-\psi'(x+t+\beta)}{12}\right) &\leq \frac{\Gamma(x+s)}{\Gamma(x+t)}\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\\ &\leq \exp\left(t-s+\frac{\psi'(x+s+\alpha)-\psi'(x+t+\alpha)}{12}\right)\end{aligned} \tag{3.94}$$

for $\alpha > \beta \geq 0$ are valid for $0 < s < t$ and $x \in (0,\infty)$ if and only if $\beta = 0$ and $\alpha \geq 1/2$.

Remark 3.48. The inequality (3.94) is a slight extension of the double inequality (2.6) in [119, Corollary 2].

Remark 3.49. In [119, Theorem 4], Kečkić-Vasić's double inequality (3.40) for $b > a > 1$ was refined and sharpened. For detailed information, see Section 5.2.

3.19. Ismail-Muldoon's Monotonicity Result

In 1994, it was obtained in [7, Corollary 2.4] that, for $a > 0$, the function

$$\frac{x^{\alpha-x}\Gamma(x)}{(x+a)^{\alpha-x-a}\Gamma(x+a)} \tag{3.95}$$

is logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \leq 1/2$, so is the reciprocal of (3.95) if and only if $\alpha \geq 1$.

3.20. Elezović-Giordano-Pečarić's Results

3.20.1. The First Result

A standard argument shows that inequality (3.59) can be rearranged as

$$\frac{s}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} - x < \sqrt{s+\frac{1}{4}} - \frac{1}{2}. \tag{3.96}$$

Therefore, monotonic and convex properties of the general function

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{3.97}$$

for $x \in (-\alpha, \infty)$, where s and t are two real numbers and $\alpha = \min\{s, t\}$ was considered in [64, Theorem 1] and obtained by the following theorem.

Theorem 3.50. *The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$.*

As consequences of Theorem 3.50, the following useful conclusions are derived.

(1) The function

$$e^{\psi(x+t)} - x \tag{3.98}$$

for all $t > 0$ is decreasing and convex from $(0, \infty)$ onto $(e^{\psi(t)}, t - (1/2))$.

(2) For all $x > 0$,

$$\psi'(x)e^{\psi(x)} < 1. \tag{3.99}$$

(3) For all $x > 0$ and $t > 0$,

$$\ln\left(x + t - \frac{1}{2}\right) < \psi(x+t) < \ln\left(x + e^{\psi(t)}\right). \tag{3.100}$$

(4) For $x > -\alpha$, the inequality

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} < \frac{t-s}{\psi(x+t) - \psi(x+s)} \tag{3.101}$$

holds if $|t-s| < 1$ and reverses if $|t-s| > 1$.

Remark 3.51. Direct computation yields

$$\frac{z''_{s,t}(x)}{z_{s,t}(x) + x} = \left[\frac{\psi(x+t) - \psi(x+s)}{t-s}\right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s}. \tag{3.102}$$

To prove the positivity of the function (3.102), the following formula and inequality are used as basic tools in the proof of the following [64, Theorem 1].

(1) For $x > -1$,

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right). \tag{3.103}$$

(2) If $a \le b < c \le d$, then

$$\frac{1}{ab} + \frac{1}{cd} > \frac{1}{ac} + \frac{1}{bd}. \tag{3.104}$$

Remark 3.52. In [39, 40], a new proof for [64, Theorem 1] was supplied by making use of (1.28) and Theorem 1.19 on the logarithmically convex properties of $q_{\alpha,\beta}(t)$.

Note that a similar proof to [39, 40] for [64, Theorem 1] in the case of $|t - s| < 1$ was also given in [79].

Remark 3.53. Actually, the function (3.102) is completely monotonic under some conditions about s and t. This was verified in [81–84], and so several new proofs for [64, Theorem 1] were supplied again; see Section 6.2.

Remark 3.54. Inequality (3.99) was recovered in [121, Lemma 1.2].

Remark 3.55. It is easy to see that Elezović-Giordano-Pečarić's first main result generalizes Watson's monotonicity result in [56] and Lazarević-Lupas's convexity result in [80]; see Sections 3.1 and 3.9.

Remark 3.56. In fact, function (3.98) is deceasing and convex on $(-t, \infty)$ for all $t \in \mathbb{R}$; see [58, Theorem 2].

Remark 3.57. It is clear that the double inequality (3.100) can be deduced directly from the decreasingly monotonic property of (3.98). Furthermore, from the decreasingly monotonic and convex properties of (3.98) on $(-t, \infty)$, inequality (3.99) and

$$\psi''(x) + \left[\psi'(x)\right]^2 > 0 \tag{3.105}$$

on $(0, \infty)$ can be derived straightforwardly.

Remark 3.58. Inequalities (3.99) and (3.105) were recovered [122, page 208] and [121, Lemma 1.1]. Inequality (3.105) has been generalized to the q-analogues in [123, Lemma 4.6] and to the complete monotonicity of divided differences of $\psi(x)$ and $\psi'(x)$ in [81–84]; see Section 6.2.

3.20.2. The Second Result

It is easy to see that inequality (3.60) can be rewritten for $s \in (0, 1)$ and $x \ge 1$ as

$$\exp\left[\psi\left(x + \sqrt{s}\,\right)\right] < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} < \exp\left[\psi\left(x + \frac{s+1}{2}\right)\right]. \tag{3.106}$$

Now it is natural to ask: What are the best constants $\delta_1(s,t)$ and $\delta_2(s,t)$ such that

$$\exp\left[\psi(x+\delta_1(s,t))\right] \le \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \le \exp\left[\psi(x+\delta_2(s,t))\right] \tag{3.107}$$

holds for $x > -\min\{s,t,\delta_1(s,t),\delta_2(s,t)\}$? where s and t are two real numbers.

Elezović-Giordano-Pečarić's answer is the following [64, Theorem 4]. If the integral ψ-mean of s and t is denoted by

$$I_\psi = I_\psi(s,t) = \psi^{-1}\left(\frac{1}{t-s}\int_s^t \psi(u)\mathrm{d}u\right), \tag{3.108}$$

then the inequality

$$\psi\left(x+I_\psi(s,t)\right) < \frac{1}{t-s}\int_s^t \psi(x+u)\mathrm{d}u < \psi\left(x+\frac{s+t}{2}\right) \tag{3.109}$$

is valid for every $x \ge 0$ and positive numbers s and t.

Remark 3.59. It is clear that Elezović-Giordano-Pečarić's second main result [64, Theorem 4] is not the outright winner surely, since the ranges of s and t in (3.109) are restricted to be positive and the lower bound in (3.109) cannot be calculated easily.

Remark 3.60. The question (3.107) was also investigated in [124–127] and has been generalized in [128–133]; see Section 6.4.1 and Section 6.4.2.

3.20.3. The Third Result

The function (3.88) and its monotonicity were generalized in [64, Theorem 5] and [39, Proposition 5] and [40, Proposition 5] as follows. The function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)} \exp\left(\psi\left(x+\frac{s+t}{2}\right)\right) \tag{3.110}$$

is logarithmically completely monotonic for $x \in (-\alpha,\infty)$, where s and t are two real numbers and $\alpha = \min\{s,t\}$.

Remark 3.61. In [39, Proposition 5] and [40, Proposition 5], as a consequence of the logarithmically complete property of the function (3.110), the right-hand side inequality in (3.106) was extended as

$$\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] > \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \tag{3.111}$$

3.21. Some Results from the Viewpoint of Means

It may be worthwhile mentioning the paper [134, 135] in which some monotonicity and inequalities for the gamma and incomplete gamma functions were constructed by using properties of extended mean values $E(r,s;x,y)$ or generalized weighted mean values $M_{f,g}(r,s;x,y)$. For example, the inequality

$$e^{\gamma x} < \Gamma(x+1) < e^{x\psi(x+1)} \tag{3.112}$$

is valid for $x > 0$ and the functions

$$\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}, \qquad \left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)} \tag{3.113}$$

are increasing in $r > 0$, $s > 0$ and $x > 0$, where $\Gamma(s,x)$ and $\gamma(s,x)$ denote the incomplete gamma functions with usual notation.

Remark 3.62. The right-hand side inequality in (3.112) is valid for $x > -1$ and takes an equality at $x = 0$. Moreover, it can be rearranged by (1.2) and (1.3) as

$$x\Gamma(x) \le e^{x\psi(x)+1}, \quad x > -1. \tag{3.114}$$

But this inequality is not better than those in [136] for bounding the gamma function $\Gamma(x)$.

3.22. Batir's Double Inequality

It is clear that the double inequality (3.60) can be rearranged as

$$\psi\left(x+\sqrt{s}\,\right) < \frac{\ln\Gamma(x+1)-\ln\Gamma(x+s)}{1-s} < \psi\left(x+\frac{s+1}{2}\right) \tag{3.115}$$

for $0 < s < 1$ and $x > 1$. The middle term in (3.115) can be regarded as a divided difference of the function $\ln\Gamma(t)$ on the interval $(x+s, x+1)$. Motivated by this, Batir generalized and extended in [137, Theorem 2.7] the double inequality (3.115) as

$$-\left|\psi^{(n+1)}\left(L_{-(n+2)}(x,y)\right)\right| < \frac{\left|\psi^{(n)}(x)\right|-\left|\psi^{(n)}(y)\right|}{x-y} < -\left|\psi^{(n+1)}\left(A(x,y)\right)\right| \tag{3.116}$$

which bounds the divided differences of the polygamma functions, where x and y are positive numbers, n a positive integer, $A(x,y)$ and $L_p(a,b)$ are defined by (1.29).

Remark 3.63. In [138, Theorem 2.4], the following incorrect double inequality was obtained:

$$e^{(x-y)\psi(L(x+1,y+1)-1)} \le \frac{\Gamma(x)}{\Gamma(y)} \le e^{(x-y)\psi(A(x,y))}, \tag{3.117}$$

where x and y are positive real numbers, and $L(x,y)$ and $A(x,y)$ are mean values defined in Section 1.6.4.

Remark 3.64. Inequalities in (3.116) and (3.117) have been corrected and refined in [128–133], respectively. See Section 6.4.2.

3.23. Further Remarks

Remark 3.65. In [139, 140], by using a method of the geometric convexity for functions, the authors presented some known and new results on the ratio of two gamma functions, including a refinement of (3.6).

Remark 3.66. In [141], the authors investigated some general cases seemingly related with the ratio of two gamma functions but essentially similar to the f-means in [142] and some results appeared in [121, 137, 138].

4. Merkle's Methods and Inequalities

It is known that Merkle did many researches on bounding the ratio of two gamma functions and has his own particular methods, approaches, and notations; therefore, this section is devoted to summarize his results and to present his methods on this topic.

Merkle himself said in [20] that his method is founded on certain general convexity results, as well as on integral representations of error terms in some classical and related inequalities.

4.1. The β-Transform

This transform has been known since [73–76]; see inequality (3.23) in Section 3.4 and inequalities (3.46) and (3.47) in Section 3.10.

The inequality

$$A(x,\beta) \le \frac{\Gamma(x+\beta)}{\Gamma(x)} \tag{4.1}$$

implies, replacing x by $x+\beta$ and β by $1-\beta$,

$$A(x+\beta,1-\beta) \le \frac{\Gamma(x+1)}{\Gamma(x+\beta)}, \tag{4.2}$$

and therefore

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le \frac{x}{A(x+\beta,1-\beta)}, \tag{4.3}$$

so, only the lower bound (4.1) is enough, or vice versa. It is said that inequality (4.3) is derived from (4.1) by a β-transform.

4.2. The π_n-Transform

This transform was firstly applied in [74, 75], see also Sections 3.5 and 3.6. It works for inequalities of both the ratio of two gamma functions and Gurland's ratio.

For $n \in \mathbb{N}$, let

$$\Pi(x,\beta,n) = \frac{x(x+1)\cdots(x+n-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)}. \tag{4.4}$$

Start from the inequality

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le B(x,\beta), \tag{4.5}$$

write it for $x+n$ and β and then apply the recurrence (1.2), to obtain

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le B(x+n,\beta)\Pi(x,\beta,n). \tag{4.6}$$

Similarly, for an inequality

$$\frac{\Gamma(x)\Gamma(y)}{\left[\Gamma\left((x+y)/2\right)\right]^2} \le B(x,y) \tag{4.7}$$

one obtains

$$\frac{\Gamma(x)\Gamma(y)}{\left[\Gamma\left((x+y)/2\right)\right]^2} \le B(x+n,y+n)\rho(x,y,n), \tag{4.8}$$

where

$$\rho(x,y,n) = \frac{(x+y)^2(x+y+2)^2\cdots(x+y+2n-2)^2}{2^{2n}x(x+1)\cdots(x+n-1)y(y+1)\cdots(y+n-1)}. \tag{4.9}$$

Remark 4.1. The β-transform and the π_n-transform are connected closely. The techniques among these transforms are essentially the same, that is, using formula (1.2) iteratively.

4.3. Convexities Used by Merkle

The following texts are the main conclusions that Merkle used in his papers bounding the ratio of two gamma functions.

4.3.1. Convexities of a Divided Difference

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ have a continuous derivative f' and let

$$F(x,y) = \frac{f(y) - f(x)}{y - x} \tag{4.10}$$

on I^2 for $x \neq y$, and $F(x,x) = f'(x)$. Then it was obtained in [59, pages 273-274] that the following conclusions are equivalent to each other:

(1) f' is convex on I,

(2) $f'((x+y)/2) \leq F(x,y)$ for all x, y of I,

(3) $F(x,y) \leq (f'(x) + f'(y))/2$ for all x, y of I,

(4) F is convex on I^2,

(5) F is Schur-convex on I^2.

4.3.2. Convexity of the Error Term for Asymptotic Expansion of the Gamma Function

The function

$$\ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \tag{4.11}$$

is convex on $(0,\infty)$ if n is even and it is concave if n is odd. See [62, Theorem 1].

Remark 4.2. The complete monotonicity of the function (4.11) was proved in [31, page 383, Theorem 8] and [143].

4.3.3. Convexity of a Function Involving the Gamma and Tri-Gamma Functions

The following convexity were proved by [61, Theorem 1]. If

$$F_a(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x - \frac{1}{12}\psi'(x+a), \tag{4.12}$$

then the function $F_0(x)$ is strictly concave and the function $F_a(x)$ for $a \geq 1/2$ is strictly convex on $(0,\infty)$.

4.4. Discrete Inequalities Produced by Convexities

By utilizing the above convexity or their corresponding concavity, the following discrete inequalities for bounding the ratio of two gamma functions were demonstrated in term of Merkle's own expressions, notations, and style.

4.4.1. The First Kind Bounds for the Ratio of Two Gamma Functions

In 1996, the following recurrent conclusions were obtained by using the convexity of the function (4.11) in [62, Theorem 2]. Let B_{2k} be Bernoulli numbers, $L_0(x) = 0$,

$$L(x) = L_{2n}(x) = -\sum_{k=1}^{2n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \quad n \in \mathbb{N}, \tag{4.13}$$

$$R(x) = R_{2n+1}(x) = -\sum_{k=1}^{2n+1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \quad n \in \mathbb{N} \cup \{0\}, \tag{4.14}$$

$$U(x,\beta,S) = S(x) - \beta S(x-1+\beta) - (1-\beta)S(s+\beta), \tag{4.15}$$

$$V(x,\beta,S) = -U(x+\beta,1-\beta,S) = (1-\beta)S(x) + \beta S(x+1) - S(x+\beta), \tag{4.16}$$

$$A(x,\beta) = \frac{\left[(x-1+\beta)^{\beta}(x+\beta)^{1-\beta}\right]^{x-1/2+\beta}}{x^{x-1/2}}, \quad x > 1-\beta, \tag{4.17}$$

$$B(x,\beta) = \frac{x}{A(x+\beta,1-\beta)} = \frac{(x+\beta)^{x+\beta-1/2}x^{\beta}}{x^{(1-\beta)(x-1/2)}(x+1)^{\beta(x+1/2)}}, \quad x > 0. \tag{4.18}$$

Then for $\beta \in [0,1]$ and $x > 1-\beta$, we have

$$\begin{aligned} e^{U(x,\beta,L(x))}A(x,\beta) &\le \frac{\Gamma(x+\beta)}{\Gamma(x)} \le e^{V(x,\beta,L(x))}B(x,\beta), \\ e^{V(x,\beta,R(x))}B(x,\beta) &\le \frac{\Gamma(x+\beta)}{\Gamma(x)} \le e^{U(x,\beta,R(x))}A(x,\beta). \end{aligned} \tag{4.19}$$

With equalities if and only if $\beta = 0$ and $\beta = 1$. As $x \to \infty$, the absolute and relative error in all four inequalities tends to zero.

In [62, Theorem 3], it was obtained that, for $x \ge (1-\beta)/2$ and $\beta \in [0,1]$,

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge \left(x - \frac{1-\beta}{2} + \frac{1-\beta^2}{24x+12}\right)^{\beta}, \tag{4.20}$$

with equality if and only if $\beta = 0$ and $\beta = 1$.

4.4.2. The Second Kind Bounds for the Ratio of Two Gamma Functions

In 1998, by the convexity of $\ln\Gamma(x)$ and convex properties mentioned in Section 4.3.1, the following inequalities were obtained in [59].

(1) For positive numbers x and y,

$$\frac{\psi(x)+\psi(y)}{2} \le \frac{\ln\Gamma(y) - \ln\Gamma(x)}{y-x} = \frac{1}{y-x}\int_x^y \psi(u)\mathrm{d}u \le \psi\left(\frac{x+y}{2}\right). \tag{4.21}$$

If letting $y = x + \beta$ for $\beta > 0$ in (4.21), then

$$\exp\left(\frac{\beta[\psi(x)+\psi(x+\beta)]}{2}\right) \le \frac{\Gamma(x+\beta)}{\Gamma(x)} \le \exp\left(\beta\psi\left(x+\frac{\beta}{2}\right)\right). \tag{4.22}$$

(2) For $x > 0$ and $0 \le \beta \le 1$,

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge \frac{x^{(1+\beta)(2-\beta)/2}(x+1)^{\beta(1+\beta)/2}}{x+\beta}. \tag{4.23}$$

The equalities in (4.23) hold for $\beta = 0$ and $\beta = 1$.
(3) For $x > 0$ and $0 \le \beta \le 1$,

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge x^{\beta^2} e^{\beta(1-\beta)\psi(x)}. \tag{4.24}$$

(4) For $x > 0$ and $\beta < 1/2$,

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le x^{-\beta^2/(1-2\beta)} \exp\left(\frac{\beta(1-\beta)}{1-2\beta}\psi(x+\beta)\right). \tag{4.25}$$

(5) For $x > 0$ and $\beta > 0$,

$$\frac{\Gamma(x+3\beta)}{\Gamma(x)} \le \left(\frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)}\right)^2. \tag{4.26}$$

(6) For $x > 0$,

$$\ln\left(x-\frac{1}{2}\right) \le \psi \le \ln x - \frac{1}{2x}. \tag{4.27}$$

Remark 4.3. The lower bound in (4.24) is closer than the one in (4.22). The upper bound in (4.22) was also obtained in [57]. The lower bound in (4.22) is closer than a lower bound in [57].

4.4.3. The Third Kind Bounds for the Ratio of Two Gamma Functions

The inequality in the right-hand side of (2.22), the double inequalities (3.60) and (3.94) for $t = 1$ are rewritten in [61] as

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} < \exp(\beta\psi(x+\beta)), \quad 0 < \beta < 1,\ x > 0, \tag{4.28}$$

$$\exp\left(\beta\psi\left(x+\beta-1+\sqrt{1-\beta}\right)\right) < \frac{\Gamma(x+\beta)}{\Gamma(x)} < \exp\left(\beta\psi\left(x+\frac{\beta}{2}\right)\right) \tag{4.29}$$

for $0 < \beta < 1$ and $x - 1 + \beta > 0$;

$$\frac{(x+\beta)^{x+\beta-1/2}}{x^{x-1/2}} \exp\left(-\beta + \frac{\psi'(x+\beta) - \psi'(x)}{12}\right) < \frac{\Gamma(x+\beta)}{\Gamma(x)} < \frac{(x+\beta)^{x+\beta-1/2}}{x^{x-1/2}} \exp\left(-\beta + \frac{\psi'(x+\beta+a) - \psi'(x+a)}{12}\right) \tag{4.30}$$

for $\beta \in (0,1)$, $x - 1 + \beta > 0$, and $a \geq 1/2$.

By using the convex properties of $F_a(x)$ defined by (4.12) and the mean value theorem for derivative, the double inequality (4.30) was extended in [61, Corollary 1] to $x > 0$, $\beta > 0$ and $a \geq 1/2$.

By making use of the convex properties of $F_a(x)$ defined by (4.12) and Jensen's inequality for convex functions, the following double inequalities were presented in [61, Corollary 2].

(1) If $\beta \in (0,1)$ and $s \geq 1/2$, then

$$A(x,\beta) \exp\left(\frac{\beta\psi'(x+a-1+\beta) + (1-\beta)\psi'(x+a+\beta) - \psi'(x+a)}{12}\right) < \frac{\Gamma(x+\beta)}{\Gamma(x)} < A(x,\beta) \exp\left(\frac{\beta\psi'(x-1+\beta) + (1-\beta)\psi'(x+\beta) - \psi'(x)}{12}\right) \tag{4.31}$$

holds for $x > 1 - \beta$, where $A(x,\beta)$ is defined by (4.17).

(2) If $\beta \in (0,1)$ and $s \geq 1/2$, then

$$B(x,\beta) \exp\left(\frac{\psi'(x+\beta) - (1-\beta)\psi'(x) - \beta\psi'(x+1)}{12}\right) < \frac{\Gamma(x+\beta)}{\Gamma(x)} < B(x,\beta) \exp\left(\frac{\psi'(x+a+\beta) - (1-\beta)\psi'(x+a) - \beta\psi'(x+a+1)}{12}\right) \tag{4.32}$$

holds for $x > 0$, where $B(x,\beta)$ is defined by (4.18).

(3) If $\beta > 1$, then the reversed inequalities of (4.31) and (4.32) hold for $x > 0$.

In [61, Theorem 1], it was proved that the double inequality (4.32) for $\beta \in (0,1)$ and $x > 0$ is sharper than (4.30).

It is pointed out that an advantage of inequalities based on convexity is that they can be infinitely sharpened. As an example, the following inequality was shown in [61, Theorem 3].

For any $x > 0$ and $\beta \in (0,1)$, the inequality

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} < \frac{x(x+1)\cdots(x+n-1)A(x+n,\beta)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)} \times \exp\left(\frac{\beta\psi'(x+n-1+\beta)+(1-\beta)\psi'(x+n+\beta)-\psi'(x+n)}{12}\right) \tag{4.33}$$

holds for $n \in \mathbb{N}$.

By proving the fact that the function (4.10) applied to $f(x) = \ln\Gamma(x)$ is Schur-convex for $x > 0$ and $y > 0$, inequality (4.22) was recovered in [61, Corollary 3] and was proved in [61, Theorem 5] to be sharper than (4.29).

4.4.4. The Fourth Kind Bounds for the Ratio of Two Gamma Functions

In 1999, Merkle used in [43, Section 4] only the convex property of $\ln\Gamma(x)$ to discover the left-hand side of Wendel's double inequality (2.8) and

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le (x+n)^{\beta}\mathrm{II}(x,\beta,n), \quad x > 0,\ \beta \in [0,1], \tag{4.34}$$

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge (x-1+\beta)^{\beta}, \quad x > 0,\ \beta \in [0,1], \tag{4.35}$$

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge (x+n-1+\beta)^{\beta}\mathrm{II}(x,\beta,n), \quad x > 0,\ \beta \in [0,1], \tag{4.36}$$

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge \left[x(x-1+\beta)\right]^{\beta/2}, \quad x > 0,\ \beta \in [0,1], \tag{4.37}$$

$$\frac{\Gamma(y)}{\Gamma(x)} \ge \left[(x-1)y\right]^{(y-x)/2}, \quad 0 < x < y. \tag{4.38}$$

It is easy to see that (4.37) is sharper than (4.35) and (4.38); however, (4.38) is not restricted to $y - x \le 1$.

4.4.5. The Fifth Kind Bounds For the Ratio of Two Gamma Functions

In [60], the author rediscovered by the same method as in [59, 61, 62] inequalities (4.27), (4.22), (4.26) and gave by using the β-transform the following inequalities:

$$\mathrm{II}(x,n,\beta)\exp\left(\frac{\beta[\psi(x+n)+\psi(x+n+\beta)]}{2}\right) \le \frac{\Gamma(x+\beta)}{\Gamma(x)} \le \mathrm{II}(x,n,\beta)\exp\left(\beta\psi\left(x+n+\frac{\beta}{2}\right)\right),$$
$$\ln\left(x+n-\frac{1}{2}\right)-\sum_{k=0}^{n-1}\frac{1}{x+k} \le \psi(x) \le \ln(x+n)-\frac{1}{2(x+n)}-\sum_{k=0}^{n-1}\frac{1}{x+k}. \tag{4.39}$$

4.5. Further Remarks

Remark 4.4. Actually, a wonderful survey of Merkle's methods, tools, and techniques which can be used to produce inequalities for the gamma function has been given in [20]. Therefore, the readers should refer to [20] for the very real ideas and spirits of Professor Merkle.

5. Refinements and Extensions of Kečkić-Vasić's Inequality

Due to the importance of Kečkić-Vasić's double inequality (3.40) and the enrichment of results on it, now we devote this special section to this topic.

5.1. An Extension and the Sharpness

In Section 3.7, Kečkić-Vasić's double inequality (3.40) obtained in [105] was introduced. An extension and the sharpness of (3.40), followed by the decreasingly monotonic property of the function (3.82) considered in [116], has been claimed in Remark 3.43.

5.2. Alzer's Refinement and Sharpness

In 1993, Kečkić-Vasić's double inequality (3.40) was rearranged in [119, page 342] as

$$\left(\frac{b}{a}\right)^{1/2}\frac{\Gamma(b)}{\Gamma(a)} < [I(a,b)]^{b-a} < \frac{b}{a}\cdot\frac{\Gamma(b)}{\Gamma(a)} \tag{5.1}$$

for $b > a \geq 1$, where

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{5.2}$$

for $a > 0$ and $b > 0$ with $a \neq b$ is the so-called identric or exponential mean; see [25, 27–29]. This provides a relationship between the identric and the gamma function.

Alzer in [119, Theorem 4] sharpened the inequality (5.1) as

$$\left(\frac{b}{a}\right)^{\alpha}\frac{\Gamma(b)}{\Gamma(a)} < [I(a,b)]^{b-a} < \left(\frac{b}{a}\right)^{\beta}\frac{\Gamma(b)}{\Gamma(a)} \tag{5.3}$$

for $b > a \geq 1$ if and only if $\alpha \leq 1/2$ and $\beta \geq \gamma$.

If setting $b = x + 1$ and $a = x + s$, then inequality (5.3) with $\alpha = 1/2$ and $\beta = \gamma$ yields

$$\frac{(x+1)^{x+1-\gamma}}{(x+s)^{x+s-\gamma}}e^{s-1} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{(x+1)^{x+1/2}}{(x+s)^{x+s-1/2}}e^{s-1} \tag{5.4}$$

for all real numbers x and s satisfying $s < 1$ and $x + s \geq 1$.

Remark 5.1. The lower bound in (5.4) improves the corresponding one in (3.41).

Remark 5.2. The double inequality (5.4) is not better than (3.94).

5.3. ABRVV's Monotonic and Log-Convex Functions

In 1995, by using the monotonicity of $\Theta_1(x)$ defined by (1.34), it was proved with no any application in [30, Theorem 3.2] that the function $x^{1/2-x}e^x\Gamma(x)$ is decreasing and logarithmically convex from $(0,\infty)$ onto $(\sqrt{2\pi},\infty)$ and that the function $x^{1-x}e^x\Gamma(x)$ is increasing and logarithmically concave from $(0,\infty)$ onto $(1,\infty)$.

It is remarked in [30, page 1720] that a result similar to the above monotonic and log-convex properties appears in [144, page 17].

Remark 5.3. These conclusions extend the corresponding results in [105], are particular cases of properties of the function (3.79) in [116], and may be used to establish the double inequality (3.40) for $b > a > 0$.

Remark 5.4. The convex properties of these two functions can be applied to obtain bounds for Gurland's ratio (3.13), as done in [145] and mentioned in Remark 5.7 below.

5.4. Alzer's Necessary and Sufficient Conditions

In 1997, the monotonic and convex properties of the function (3.82) were slightly extended in [31, page 376, Theorem 2]. For $c \geq 0$, function (3.82) is decreasing on (c,∞) if and only if $\alpha \leq 1/2$ and increasing on (c,∞) if and only if

$$\alpha \geq \begin{cases} \Theta_1(c) & \text{if c} > 0, \\ 1 & \text{if } c = 0. \end{cases} \tag{5.5}$$

Remark 5.5. From these monotonic properties, Kečkić-Vasić's double inequality (3.40) may be generalized as follows. If $b > a > c \geq 0$, then the double inequality (5.3) holds if and only if $\alpha \leq 1/2$ and $\beta \geq \Theta_1(c)$.

5.5. Chen-Qi's Necessary and Sufficient Conditions

In 2006, the following necessary and sufficient conditions were demonstrated in [146]. For $\alpha \in \mathbb{R}$ and $\beta \geq 0$, the function

$$f_{\alpha,\beta}(x) = \frac{e^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}} \tag{5.6}$$

is logarithmically completely monotonic on $(0,\infty)$ if $2\alpha \leq 1 \leq \beta$; the function $f_{\alpha,1}(x)$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $2\alpha \leq 1$; so is the function $[f_{\alpha,1}(x)]^{-1}$ on $(0,\infty)$ if and only if $\alpha \geq 1$.

From monotonic properties of $f_{1/2,1}(x)$ and $[f_{1,1}(x)]^{-1}$, Kečkić-Vasić's double inequality (3.40) obtained in [105] was extended in [146] from $b > a > 1$ to $b > a > 0$.

Remark 5.6. The functions defined by (3.39) and (3.82) and discussed in [30, Theorem 3.2] are particular cases of the function $f_{\alpha,\beta}(x)$ defined by (5.6).

Remark 5.7. From convex properties of $f_{1/2,1}(x)$ and $[f_{1,1}(x)]^{-1}$ and Jensen's inequality (1.26), bounds for Gurland's ratio (3.13) were established in [145]. For positive numbers x and y,

$$\frac{x^{x-1/2}y^{y-1/2}}{\left[(x+y)/2\right]^{x+y-1}} \le T(x,y) \le \frac{x^{x-1}y^{y-1}}{\left[(x+y)/2\right]^{x+y-2}}. \tag{5.7}$$

The left-hand side inequality in (5.7) is same as the corresponding one in [44, Theorem 1], but their upper bounds do not include each other.

5.6. Guo-Qi-Srivastava's Double Inequality

For $\beta \in \mathbb{R}$, let

$$g_\beta(x) = \frac{e^x\Gamma(x+1)}{(x+\beta)^{x+\beta}} \tag{5.8}$$

on the interval $(\max\{0,-\beta\},\infty)$. In [147, 148], it was showed that the function $g_\beta(x)$ is logarithmically completely monotonic if and only if $\beta \ge 1$ and that the function $[g_{\alpha,\beta}(x)]^{-1}$ is logarithmically completely monotonic if and only if $\beta \le 1/2$.

As consequences of the monotonicity results of the function $g_\beta(x)$, the following two-sided inequality was derived in [148]

$$\frac{(x+1)^{x+1}}{(y+1)^{y+1}}e^{y-x} < \frac{\Gamma(x+1)}{\Gamma(y+1)} < \frac{(x+1/2)^{x+1/2}}{(y+1/2)^{y+1/2}}e^{y-x} \tag{5.9}$$

or, equivalently,

$$I\left(x+\frac{1}{2},y+\frac{1}{2}\right) < \left[\frac{\Gamma(x+1)}{\Gamma(y+1)}\right]^{1/(x-y)} < I(x+1,y+1) \tag{5.10}$$

for $y > x > 0$, where the constants 1 and $1/2$ in the very left and the very right sides of the two-sided inequalities (5.9) and (5.10) cannot be replaced, respectively, by smaller and larger ones.

Remark 5.8. By convex properties of $g_\beta(x)$ and Jensen's inequality (1.26), a new double inequality for a generalization of Gurland's ratio (3.13) may be deduced. For $n \in \mathbb{N}$, $x_k > 0$ for $1 \le k \le n$, and $p_k \ge 0$ satisfying $\sum_{k=1}^n p_k = 1$, the inequality

$$\begin{aligned}\frac{\left(\sum_{k=1}^n p_k x_k\right)\prod_{k=1}^n (x_k+b)^{p_k(x_k+b)}}{\left[\sum_{k=1}^n p_k(x_k+b)\right]^{\sum_{k=1}^n p_k(x_k+b)}\prod_{k=1}^n x_k^{p_k}} &< \frac{\prod_{k=1}^n [\Gamma(x_k)]^{p_k}}{\Gamma\left(\sum_{k=1}^n p_k x_k\right)} \\ &< \frac{\left(\sum_{k=1}^n p_k x_k\right)\prod_{k=1}^n (x_k+a)^{p_k(x_k+a)}}{\left[\sum_{k=1}^n p_k(x_k+a)\right]^{\sum_{k=1}^n p_k(x_k+a)}\prod_{k=1}^n x_k^{p_k}}\end{aligned} \tag{5.11}$$

holds if and only if $a \le 1/2$ and $b \ge 1$. In particular, Gurland's ratio (3.13) can be bounded by

$$\begin{aligned}\frac{2^{x+y+2b-2}(x+y)^2(x+b)^{x+b}(y+b)^{y+b}}{xy(x+y+2b)^{x+y+2b}} &< T(x,y) \\ &< \frac{2^{x+y+2a-2}(x+y)^2(x+a)^{x+a}(y+a)^{y+a}}{xy(x+y+2a)^{x+y+2a}}\end{aligned} \tag{5.12}$$

if and only if $a \le 1/2$ and $b \ge 1$, where x and y are positive numbers.

5.7. Some More Kečkić-Vasić Type Inequalities

5.7.1. Guo-Qi-Srivastava's One-Sided Inequality

In 2008, some new conclusions of the function $f_{\alpha,\beta}(x)$ were procured in [149].

(1) If $\beta \in (0,\infty)$ and $\alpha \le 0$, then $f_{\alpha,\beta}$ is logarithmically completely monotonic on $(0,\infty)$.

(2) If $\beta \in (0,\infty)$ and $f_{\alpha,\beta}$ is a logarithmically completely monotonic function on $(0,\infty)$, then $\alpha \le \min\{\beta, 1/2\}$.

(3) If $\beta \ge 1$, then $f_{\alpha,\beta}$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \le 1/2$.

As direct consequences of monotonic properties above, a Kečkić-Vasić type inequality is deduced immediately. If a and b are positive numbers with $a \ne b$, then

(1) the inequality

$$I(a,b) > \left[\left(\frac{a}{b}\right)^{\alpha-\beta}\frac{\Gamma(a+\beta)}{\Gamma(b+\beta)}\right]^{1/(a-b)} \tag{5.13}$$

holds true for $\beta \ge 1$ if and only if $\alpha \le 1/2$;

(2) inequality (5.13) holds true also for $\beta \in (0,\infty)$ if $\alpha \le 0$.

5.7.2. More Inequalities by Guo and Qi

Subsequently, a necessary and sufficient condition and a necessary condition for $[f_{\alpha,\beta}(x)]^{-1}$ to be logarithmically completely monotonic on $(0,\infty)$ were established in [150] as follows.

Theorem 5.9 (see [150]). *If the function $[f_{\alpha,\beta}(x)]^{-1}$ is logarithmically completely monotonic on $(0,\infty)$, then either $\beta > 0$ and $\alpha \geq \max\{\beta, 1/2\}$ or $\beta = 0$ and $\alpha \geq 1$.*

Theorem 5.10 (see [150]). *If $\beta \geq 1/2$, the necessary and sufficient condition for the function $[f_{\alpha,\beta}(x)]^{-1}$ to be logarithmically completely monotonic on $(0,\infty)$ is $\alpha \geq \beta$.*

As the first application of Theorem 5.10, the following inequalities are derived by using logarithmically complete monotonicity of the functions $[f_{\alpha,\beta}(x)]^{\pm 1}$ on $(0,\infty)$.

Theorem 5.11 (see [150]). *Let β be a positive number.*

(1) *For $k \in \mathbb{N}$, double inequalities*

$$\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x - \frac{1}{2x}, \tag{5.14}$$

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \leq (-1)^{k+1}\psi^{(k)}(x) \leq \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{5.15}$$

hold on $(0,\infty)$.

(2) *When $\beta > 0$, inequalities*

$$\begin{aligned} \psi(x+\beta) &\leq \ln x + \frac{\beta}{x}, \\ (-1)^k\psi^{(k-1)}(x+\beta) &\geq \frac{(k-2)!}{x^{k-1}} - \frac{\beta(k-1)!}{x^k} \end{aligned} \tag{5.16}$$

hold on $(0,\infty)$ for $k \geq 2$.

(3) *When $\beta \geq 1/2$, inequalities*

$$\psi(x+\beta) \geq \ln x, \qquad (-1)^k\psi^{(k-1)}(x+\beta) \leq \frac{(k-2)!}{x^{k-1}} \tag{5.17}$$

hold on $(0,\infty)$ for $k \geq 2$.

(4) *When $\beta \geq 1$, inequalities*

$$\begin{aligned} \psi(x+\beta) &\leq \ln x + \frac{\beta - 1/2}{x}, \\ (-1)^k\psi^{(k-1)}(x+\beta) &\geq \frac{(k-2)!}{x^{k-1}} - \frac{(\beta-1/2)(k-1)!}{x^k} \end{aligned} \tag{5.18}$$

holds on $(0,\infty)$ for $k \geq 2$.

As the second application of Theorem 5.10, the following inequalities are derived by using logarithmically convex properties of the function $[f_{\alpha,\beta}(x)]^{\pm 1}$ on $(0,\infty)$.

Theorem 5.12 (see [150]). *Let $n \in \mathbb{N}$, $x_k > 0$ for $1 \le k \le n$, $p_k \ge 0$ satisfying $\sum_{k=1}^{n} p_k = 1$. If either $\beta > 0$ and $\alpha \le 0$ or $\beta \ge 1$ and $\alpha \le 1/2$, then*

$$\frac{\prod_{k=1}^{n}[\Gamma(x_k+\beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k+\beta\right)} \ge \frac{\prod_{k=1}^{n} x_k^{p_k(x_k+\beta-\alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k+\beta-\alpha}}. \tag{5.19}$$

If $\alpha \ge \beta \ge 1/2$, then inequality (5.19) reverses.

As the final application of Theorem 5.10, the following inequality may be derived by using the decreasingly monotonic property of the function $[f_{\alpha,\beta}(x)]^{-1}$ on $(0,\infty)$.

Theorem 5.13 (see [150]). *If $\alpha \ge \beta \ge 1/2$, then*

$$I(x,y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta}\frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{5.20}$$

holds true for $x, y \in (0,\infty)$ with $x \ne y$.

Remark 5.14. The double inequality (5.15) and its sharpness can also be deduced from [45, Thorem 1.3] and [46, Thorem 3].

5.8. Guo-Zhang-Qi's Inequality

In 2008, by using the monotonicity and convexity of the function $\Theta_1(x)$ defined by (1.34), it was showed in [136] that the function

$$h(x) = \frac{e^x\Gamma(x)}{x^{x-\Theta_1(x)}} \tag{5.21}$$

on $(0,\infty)$ has a unique maximum e at $x = 1$, with the following two limits:

$$\lim_{x\to 0^+} h(x) = 1, \qquad \lim_{x\to\infty} h(x) = \sqrt{2\pi}. \tag{5.22}$$

As consequences of the monotonicity of the function $h(x)$, it was concluded in [136] that the following inequality:

$$I(x,y) < \left[\frac{x^{\Theta_1(x)}\Gamma(x)}{y^{\Theta_1(y)}\Gamma(y)}\right]^{1/(x-y)} \tag{5.23}$$

holds true for $x \ge 1$ and $y \ge 1$ with $x \ne y$. If $0 < x \le 1$ and $0 < y \le 1$ with $x \ne y$, the inequality (5.23) is reversed.

5.9. Further Remarks

Remark 5.15. The above discussins show that Kečkić-Vasić type inequalities and bounds for Gurland's ratio (3.13) may be established, respectively, from the monotonic and convex properties of the same functions. Kečkić-Vasić type inequalities follow from the monotonic properties while bounds for Gurland's ratio (3.13) do from the convex properties.

6. Qi and His Coauthors' Results

Beginning from 2005, a large part of my academic attention was concentrated on bounding the ratio of two gamma functions.

6.1. Proofs of Elezović-Giordano-Pečarić's Monotonicity Result

In [39, 40], by making use of the convolution theorem (1.28) for Laplace transform and Theorem 1.19 on logarithmically convex properties of the function $q_{\alpha,\beta}(x)$ defined by (1.35), a new proof of [64, Theorem 1], that is, Theorem 3.50 in Section 3.20.1, was given.

By a similar approach to the one in [39, 40], a concise proof for [64, Theorem 1] was presented in [37, 38] recently.

6.2. Complete Monotonicity of Divided Differences

After giving a new proof for [64, Theorem 1], complete monotonicity of the function (3.102), which involves divided differences of the digamma and trigamma functions, was considered in [81–84] by employing one of the main results in [151, 152].

Our main results can be stated as follows.

Theorem 6.1. *Let s and t be two real numbers and $\alpha = \min\{s,t\}$. Define*

$$\delta_{s,t}(x) = \begin{cases} \dfrac{\psi(x+t)-\psi(x+s)}{t-s} - \dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s \neq t, \\ \psi'(x+s) - \dfrac{1}{x+s} - \dfrac{1}{2(x+s)^2}, & s = t, \end{cases} \tag{6.1}$$

on $x \in (-\alpha, \infty)$. Then the functions $\delta_{s,t}(x)$ for $|t-s| < 1$ and $-\delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic on $x \in (-\alpha, \infty)$.

Theorem 6.2. *Let s and t be two real numbers and $\alpha = \min\{s,t\}$. Define*

$$\Delta_{s,t}(x) = \begin{cases} \left[\dfrac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2 + \dfrac{\psi'(x+t)-\psi'(x+s)}{t-s}, & s \neq t, \\ \left[\psi'(x+s)\right]^2 + \psi''(x+s), & s = t, \end{cases} \tag{6.2}$$

on $x \in (-\alpha, \infty)$. Then the functions $\Delta_{s,t}(x)$ for $|t-s| < 1$ and $-\Delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic on $x \in (-\alpha, \infty)$.

Theorem 6.3. *Let k be a nonnegative integer and $\theta > 0$ a constant.*

If $a > 0$ and $b > 0$, then

$$\sum_{i=0}^{k} \frac{1}{(a+\theta)^{i+1}(b+\theta)^{k-i+1}} + \sum_{i=0}^{k} \frac{1}{a^{i+1}b^{k-i+1}} > 2\sum_{i=0}^{k} \frac{1}{(a+\theta)^{i+1}b^{k-i+1}} \tag{6.3}$$

holds for $b - a > -\theta$ and reveres for $b - a < -\theta$.

If $a < -\theta$ and $b < -\theta$, then inequalities

$$\sum_{i=0}^{2k} \frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+1}} + \sum_{i=0}^{2k} \frac{1}{a^{i+1}b^{2k-i+1}} > 2\sum_{i=0}^{2k} \frac{1}{(a+\theta)^{i+1}b^{2k-i+1}}, \tag{6.4}$$

$$\sum_{i=0}^{2k+1} \frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+2}} + \sum_{i=0}^{2k+1} \frac{1}{a^{i+1}b^{2k-i+2}} < 2\sum_{i=0}^{2k+1} \frac{1}{(a+\theta)^{i+1}b^{2k-i+2}} \tag{6.5}$$

hold for $b - a > -\theta$ and reverse for $b - a < -\theta$.

If $-\theta < a < 0$ and $-\theta < b < 0$, then inequality (6.4) *holds and inequality* (6.5) *is valid for $a + b + \theta > 0$ and is reversed for $a + b + \theta < 0$.*

If $a < -\theta$ and $b > 0$, then inequality (6.4) *holds and inequality* (6.5) *is valid for $a + b + \theta > 0$ and is reversed for $a + b + \theta < 0$.*

If $a > 0$ and $b < -\theta$, then inequality (6.4) *is reversed and inequality* (6.5) *holds for $a+b+\theta < 0$ and reverses for $a + b + \theta > 0$.*

If $b = a - \theta$, then inequalities (6.3), (6.4), *and* (6.5) *become equalities.*

Theorem 6.4. *Inequality* (6.3) *for positive numbers a and b is equivalent to Theorem 6.1.*

Remark 6.5. One of the key tools or ideas used in the proofs of Theorems 6.1 and 6.2 is that if $f(x)$ is a function defined on an infinite interval I and satisfies $\lim_{x\to\infty} f(x) = \delta$ and $f(x) - f(x+\varepsilon) > 0$ for $x \in I$ and some fixed number $\varepsilon > 0$, then $f(x) > \delta$. This lemma has been validated in [81–83, 151, 152] and related references therein to be especially successful in proving the monotonicity, the complete monotonicity, and the logarithmically complete monotonicity of functions involving the gamma, psi, and polygamma functions.

Remark 6.6. It is clear that the complete monotonicity of the functions $\delta_{s,t}(x)$ and $\Delta_{s,t}(x)$ in Theorems 6.1 and 6.2 generalize the positivity and complete monotonicity of the function

$$\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} \tag{6.6}$$

on $(0, \infty)$, presented in [153, 154], [45, Thorem 1.3], and [46, Thorem 3], and the inequality (3.105), obtained initially and, respectively, in [122, page 208] and [121, Lemma 1.1].

Remark 6.7. Along another direction, results on the convex and completely monotonic properties for the generalization of the function (6.6) were procured in [31, page 383, Theorem 8] and [62, page 370, Theorem 1] by different approaches, respectively. Recently, a new proof for it was published in [143].

6.3. Necessary and Sufficient Conditions

For real numbers a, b, and c, denote $\rho = \min\{a, b, c\}$ and let

$$H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{6.7}$$

for $x \in (-\rho, \infty)$.

Making use of monotonicity of $q_{\alpha,\beta}(t)$ on $(0,\infty)$, see Theorem 1.16 procured in [35, 36] and Section 1.7 in this paper, sufficient and necessary conditions such that the function $H_{a,b,c}(x)$ are logarithmically completely monotonic on $(-\rho,\infty)$, which extend or generalize [116, Theorem 2.4], [155, Theorem 1], and [156, Theorem 1], and other known results mentioned in Section 3.17 are presented as follows.

Theorem 6.8 (see [41]). *Let a, b and c be real numbers, and $\rho = \min\{a, b, c\}$. Then*

(1) *$H_{a,b,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if and only if*

$$\begin{aligned}(a,b,c) \in D_1(a,b,c) \triangleq {} & \{(a,b,c) : (b-a)(1-a-b+2c) \ge 0\} \\ & \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \ge 0\} \\ & \setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\},\end{aligned} \tag{6.8}$$

(2) *$H_{b,a,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if and only if*

$$\begin{aligned}(a,b,c) \in D_2(a,b,c) \triangleq {} & \{(a,b,c) : (b-a)(1-a-b+2c) \le 0\} \\ & \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \le 0\} \\ & \setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}.\end{aligned} \tag{6.9}$$

From Theorem 6.8, the following double inequality for divided differences of the psi and polygamma functions can be deduced readily.

Theorem 6.9 (see [157]). *Let $b > a \ge 0$ and $k \in \mathbb{N}$. Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k} \le \frac{(-1)^{k-1}\left[\psi^{(k-1)}(x+b) - \psi^{(k-1)}(x+a)\right]}{b-a} \le \frac{(k-1)!}{(x+\beta)^k} \tag{6.10}$$

for $x \in (-\rho,\infty)$ holds if $\alpha \ge \max\{a, (a+b-1)/2\}$ and $0 \le \beta \le \min\{a, (a+b-1)/2\}$.

Remark 6.10. It is unthought of that taking $b - a = 1$ in (6.10) leads to

$$\psi^{(k-1)}(x+a+1) - \psi^{(k-1)}(x+a) = (-1)^{k-1}\frac{(k-1)!}{(x+a)^k} \tag{6.11}$$

for $a \geq 0$, $x > 0$ and $k \in \mathbb{N}$, which is equivalent to the recurrence formula

$$\psi^{(n)}(z+1) - \psi^{(n)}(z) = (-1)^n n! z^{-n-1} \tag{6.12}$$

for $z > 0$ and $n \geq 0$, listed in [1, pages 260, 6.4.6]. This has been showed in [158, 159] by a different manner from here.

Remark 6.11. The logarithmically complete monotonicity of the function $H_{a,0,0}(x)$ was demonstrated in [48, Theorem 1.3].

6.4. Some Results on Gautschi-Kershaw's Second Double Inequality

Up to now, results on refinements or generalizations of the right-hand side inequality in (2.22) and the double inequality (3.60) are the logarithmically complete monotonicity of the functions (3.88), (3.90), and (3.110) and inequalities in (3.109), (3.111), and (3.116).

6.4.1. Monotonicity Results on Gautschi-Kershaw's Second Double Inequality

Motivated by problem (3.107), the following results on logarithmically complete monotonicity of functions related to the double inequality (3.107) were demonstrated.

Theorem 6.12 (see [124, Theorem 1] and [125, Theorem 1]). *Let a, b, c be real numbers and $\rho = \min\{a, b, c\}$. Define*

$$F_{a,b,c}(x) = \begin{cases} \left[\dfrac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)} \exp\left[\psi(x+c)\right], & a \neq b, \\ \exp\left[\psi(x+c) - \psi(x+a)\right], & a = b \neq c \end{cases} \tag{6.13}$$

for $x \in (-\rho, \infty)$. Furthermore, let $\theta(t)$ be an implicit function defined by equation

$$e^t - t = e^{\theta(t)} - \theta(t) \tag{6.14}$$

on $(-\infty, \infty)$. Then $\theta(t)$ is decreasing and $t\theta(t) < 0$ for $\theta(t) \neq t$, and

(1) *$F_{a,b,c}(x)$ is logarithmically completely monotonic on $(-\rho, \infty)$ if*

$$\begin{aligned}(a,b,c) \in {} & \{c \geq a, c \geq b\} \cup \{c \geq a, 0 \geq c-b \geq \theta(c-a)\} \\ & \cup \{c \leq a, c-b \geq \theta(c-a)\} \setminus \{a = b = c\};\end{aligned} \tag{6.15}$$

(2) *$[F_{a,b,c}(x)]^{-1}$ is logarithmically completely monotonic on $(-\rho, \infty)$ if*

$$\begin{aligned}(a,b,c) \in {} & \{c \leq a, c \leq b\} \cup \{c \geq a, c-b \leq \theta(c-a)\} \\ & \cup \{c \leq a, 0 \leq c-b \leq \theta(c-a)\} \setminus \{a = b = c\}.\end{aligned} \tag{6.16}$$

Theorem 6.13 (see [126, Theorem 1] and [127, Theorem 1]). *For real numbers s and t with $s \neq t$ and $\theta(s,t)$ a constant depending on s and t, define*

$$v_{s,t}(x) = \frac{1}{\exp[\psi(x+\theta(s,t))]}\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \tag{6.17}$$

(1) *If $\theta(s,t) \leq \min\{s,t\}$, then $v_{s,t}(x)$ is logarithmically completely monotonic on $(-\theta(s,t),\infty)$.*

(2) *The function $[v_{s,t}(x)]^{-1}$ is logarithmically completely monotonic on the interval $(-\min\{s,t\},\infty)$ if and only if $\theta(s,t) \geq (s+t)/2$.*

6.4.2. Refinements of Gautschi-Kershaw's Second Double Inequality

Stimulated by the left-hand side inequality in (3.117), although it is not correct, several refinements and generalizations related to Gautschi-Kershaw's second double inequality were established.

Theorem 6.14 (see [128, Theorem 1] and [129, Theorem 1]). *For positive numbers a and b with $a \neq b$, the inequality*

$$e^{\psi(L(a,b))} < \left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{(a-b)} < e^{\psi(A(a,b))} \tag{6.18}$$

is valid, where L and A are, respectively, defined in (1.30).

Theorem 6.15 (see [128, Theorem 2] and [129, Theorem 2]). *For $s,t \in \mathbb{R}$ with $s \neq t$, the function*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} \frac{1}{e^{\psi(L(s,t;x))}} \tag{6.19}$$

is decreasing and

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)} e^{\psi(A(s,t;x))} \tag{6.20}$$

is logarithmically completely monotonic in $x > -\min\{s,t\}$, where

$$L(s,t;x) = L(x+s,x+t), \qquad A(s,t;x) = A(x+s,x+t). \tag{6.21}$$

Theorem 6.16 (see [131, 132]). *The inequalities*

$$\left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)} \le e^{\psi(I(a,b))}, \tag{6.22}$$

$$\frac{(-1)^n\left[\psi^{(n-1)}(a)-\psi^{(n-1)}(b)\right]}{a-b} \le (-1)^n\psi^{(n)}(I(a,b)) \tag{6.23}$$

hold true for $a>0$ and $b>0$, where I is defined in (1.30).

Theorem 6.17 (see [130, Theorem 1] and [133, Theorem 1]). *For real numbers $s>0$ and $t>0$ with $s\neq t$ and an integer $i\ge 0$, the inequality*

$$(-1)^i\psi^{(i)}\left(L_p(s,t)\right) \le \frac{(-1)^i}{t-s}\int_s^t \psi^{(i)}(u)\mathrm{d}u \le (-1)^i\psi^{(i)}\left(L_q(s,t)\right) \tag{6.24}$$

holds if $p\le -i-1$ and $q\ge -i$.

Theorem 6.18 (see [130, Theorem 2] and [133, Theorem 2]). *The double inequality*

$$e^{\psi(L_p(a,b))} < \left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)} < e^{\psi(L_q(a,b))} \tag{6.25}$$

for $a>0$ and $b>0$ holds if $p\le -1$ and $q\ge 0$.

Theorem 6.19 (see [130, Theorem 3] and [133, Theorem 3]). *For $i\ge 0$ being an integer and $s,t\in\mathbb{R}$ with $s\neq t$ and $x>-\min\{s,t\}$, the function*

$$(-1)^i\left[\psi^{(i)}\left(L_p(s,t;x)\right)-\frac{1}{t-s}\int_s^t\psi^{(i)}(x+u)\mathrm{d}u\right] \tag{6.26}$$

is increasing in x if either $p\le -(i+2)$ or $p=-(i+1)$ and decreasing in x if $p\ge 1$, where

$$L_p(s,t;x)=L_p(x+s,x+t). \tag{6.27}$$

Remark 6.20. The proofs of Theorems 6.16 and 6.17 need to use the main results in [160–162] on monotonic properties of some functions involving the psi and polygamma functions.

Remark 6.21. In [163], the authors recovered the left-hand side inequality in (6.18) and obtained

$$\frac{\ln\Gamma(b)-\ln\Gamma(a)}{b-a} < \psi(L(a,b))+\ln\frac{I(a,b)}{L(a,b)} \tag{6.28}$$

for all a and b with $a\neq b$.

Since

$$\frac{\psi(I(a,b))-\psi(L(a,b))}{\ln I(a,b)-\ln L(a,b)}=\xi\psi'(\xi)>1, \tag{6.29}$$

where $\xi\in(L(a,b),I(a,b))$, inequality (6.28) is better than (6.22).

Remark 6.22. In [164], the author generalized Theorem 6.15 as follows. The function (6.19) is logarithmically convex with respect to $x>-\min\{s,t\}$.

With the help of the main result obtained in [165, 166], it is claimed in [164] that the function

$$e^{-\psi(L(s,t;x))} \tag{6.30}$$

is completely monotonic with respect to $x>-\min\{s,t\}$ for real numbers s and t with $s\neq t$. Actually, the proof is not correct.

Let s and t be two real numbers and $\alpha=\min\{s,t\}$. For $x\in(-\alpha,\infty)$, define

$$\begin{aligned}\Psi_{s,t}(x)&=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}, & s\neq t,\\ e^{\psi(x+s)}, & s=t,\end{cases}\\ \mathcal{D}_{s,t}(x)&=\Psi_{s,t}(x)\frac{\psi(x+t)-\psi(x+s)}{t-s},\\ Q_{s,t}(x)&=\frac{t-s}{\psi(x+t)-\psi(x+s)}-\Psi_{s,t}(x).\end{aligned} \tag{6.31}$$

The second open problem posed in [39] was denied in [164] as follows. Neither $\mathcal{D}_{s,t}(x)$ nor $Q_{s,t}(x)$ is completely monotonic on $(-\alpha,\infty)$.

6.5. Generalizations of Lew-Frauenthal-Keyfitz's Double Inequality

In 2006, in order to improve

$$2\Gamma\left(n+\frac{1}{2}\right)\le\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)\le 2^n\Gamma\left(n+\frac{1}{2}\right) \tag{6.32}$$

for $n\ge 1$, see [167, page 213] and [168], it was proved in [49] by using (2.2) that

$$\sqrt{x}\ \le\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\le\sqrt{x+\frac{1}{2}}\ . \tag{6.33}$$

It is clear that the double inequality (6.33) is weaker than (3.59) for $s=1/2$ and the right-hand side inequality in (6.33) is also weaker than an inequality obtained in [111].

Observe that inequality (6.32) can be rearranged for $n > 1$ as

$$1 \leq \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)}\right]^{1/(n-1)} \leq 2. \tag{6.34}$$

The middle term in (6.34) hints us to define

$$g(x) = \begin{cases} \left[\dfrac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)}\right]^{1/(x-1)}, & x \neq 1, \\ \exp\left[1-\gamma-\psi\left(\dfrac{3}{2}\right)\right], & x = 1 \end{cases} \tag{6.35}$$

for $x \in (-1/2, \infty)$, with $\gamma = 0.57721566\ldots$ being the Euler-Mascheroni constant, and to consider its logarithmically complete monotonicity.

Theorem 6.23 (see [47, Theorem 1.1] and [48, Theorem 1.1]). *The function* $g(x) \in \mathcal{L}[(-1/2, \infty)]$ *with*

$$\lim_{x \to -(1/2)^+} g(x) = \infty, \qquad \lim_{x \to \infty} g(x) = 1, \tag{6.36}$$

where $\mathcal{L}[I]$ *stands for the set of logarithmically completely monotonic functions on an interval* $I \subseteq \mathbb{R}$.

The left-hand side inequality in (2.2) reminds us to introduce

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} = \frac{(x+a)^{1-a}\Gamma(x+a)}{\Gamma(x+1)} \tag{6.37}$$

for $x > 0$ and $a > 0$ and to discuss its logarithmically complete monotonicity.

Theorem 6.24 (see [47, Theorem 1.2] and [48, Theorem 1.2]). *The function* $h_a(x)$ *has the following properties.*

(1) $h_a(x) \in \mathcal{L}[(0, \infty)]$ *if* $0 < a < 1$.

(2) $[h_a(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ *if* $a > 1$.

(3) *For any* $a > 0$,

$$\lim_{x \to 0^+} h_a(x) = \frac{\Gamma(a+1)}{a^a}, \qquad \lim_{x \to \infty} h_a(x) = 1. \tag{6.38}$$

As a straightforward consequence of combining Theorem 6.8 for $H_{a,0,0}(x)$ and Theorem 6.24, the following refinement of the upper bound in inequality (2.2) is established.

Theorem 6.25 (see [47, Theorem 1.4] and [48, Theorem 1.4]). *Let $x \in (0,\infty)$. If $0 < a < 1$, then*

$$\left(\frac{x}{x+a}\right)^{1-a} < \frac{\Gamma(x+a)}{x^a\Gamma(x)} < \begin{cases} \dfrac{\Gamma(a+1)}{a^a}\left(\dfrac{x}{x+a}\right)^{1-a} \le 1, & 0 < x \le \dfrac{ap(a)}{1-p(a)}, \\ 1, & \dfrac{ap(a)}{1-p(a)} < x < \infty, \end{cases} \tag{6.39}$$

where

$$p(x) = \begin{cases} \left[\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(1-x)}, & x \ne 1, \\ e^{-\gamma}, & x = 1. \end{cases} \tag{6.40}$$

If $a > 1$, the reversed inequality of (6.39) *holds.*

Now rewrite inequality (6.32) or (6.34) for $n > 1$ as

$$1 \le \left[\frac{\Gamma(1+1/2)}{\Gamma(1+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1/2)}\right]^{1/(n-1)} \le 2. \tag{6.41}$$

The definition (6.35) of $g(x)$ and inequality (6.41) motivate us to introduce a new function $h_\beta(x)$ as follows. Let s and t be two real numbers with $s \ne t$, $\alpha = \min\{s,t\}$ and $\beta > -\alpha$. For $x \in (-\alpha,\infty)$, define

$$h_\beta(x) = \begin{cases} \left[\dfrac{\Gamma(\beta+t)}{\Gamma(\beta+s)} \cdot \dfrac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(x-\beta)}, & x \ne \beta, \\ \exp\left[\psi(\beta+s) - \psi(\beta+t)\right], & x = \beta. \end{cases} \tag{6.42}$$

Theorem 6.26 (see [45, Theorem 1.1] and [46, Thorem 1]). *The following two conclusions are valid.*

(1) *If $s > t$, then $h_\beta(x) \in \mathcal{L}[(-\alpha,\infty)]$ with*

$$\lim_{x\to-\alpha} h_\beta(x) = \infty, \qquad \lim_{x\to\infty} h_\beta(x) = 1. \tag{6.43}$$

(2) *If $s < t$, then $[h_\beta(x)]^{-1} \in \mathcal{L}[(-\alpha,\infty)]$ with*

$$\lim_{x\to-\alpha} h_\beta(x) = 0, \qquad \lim_{x\to\infty} h_\beta(x) = 1. \tag{6.44}$$

In [47] and [48, Theorem 1.5], the logarithmically complete monotonicity of $p(x)$ was proved: $p(x) \in \mathcal{L}[(0,\infty)]$ with $\lim_{x\to 0^+} p(x) = 1$ and $\lim_{x\to\infty} p(x) = 1/e$. Motivated by inequality (6.41) and the definition of $h_\beta(x)$ in (6.42), a more general function than $p(x)$ can be introduced. For $x \in (0,\infty)$ and $\alpha > 0$, let

$$p_\alpha(x) = \begin{cases} \left[\dfrac{\Gamma(\alpha+1)}{\alpha^\alpha}\cdot\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(\alpha-x)}, & x \neq \alpha, \\ \dfrac{\exp\left[\psi(\alpha+1)-1\right]}{\alpha}, & x = \alpha. \end{cases} \tag{6.45}$$

Theorem 6.27 (see [45, Theorem 1.4] and [46, Thorem 4]). *For any fixed $\alpha > 0$, $p_\alpha(x) \in \mathcal{L}[(0,\infty)]$ with*

$$\lim_{x\to 0^+} p_\alpha(x) = \frac{\sqrt[\alpha]{\Gamma(\alpha+1)}}{\alpha}, \qquad \lim_{x\to\infty} p_\alpha(x) = \frac{1}{e}. \tag{6.46}$$

Remark 6.28. The functions $h(x,y) = h_y(x)$ and $p(x,y) = p_y(x)$ satisfy $h(x,y) = h(y,x)$ and $p(x,y) = p(y,x)$ and so their Schur-convex properties can be considered.

Remark 6.29. It is clear that the papers [45–48] take a different direction from [49] to generalize and refine inequalities in (6.32).

Remark 6.30. The logarithmically complete monotonicity of the function $\sqrt[x]{\Gamma(x+1)}/x$ and several similar ones have been researched and applied in [15, 169, 170].

7. Related Problems

7.1. Monotonicity Results for the Ratio of Two q-Gamma Functions

The known results obtained by many mathematicians show that most of properties of the ratio of two gamma functions may be replanted to cases of the ratio of two q-gamma functions.

Let a, b, and c be real numbers and $\rho = \min\{a,b,c\}$, define

$$H_{q;a,b;c}(x) = \left(\frac{1-q^{x+c}}{1-q}\right)^{a-b}\frac{\Gamma_q(x+b)}{\Gamma_q(x+a)} \tag{7.1}$$

for $x \in (-\rho,\infty)$, where $\Gamma_q(x)$ for $0 < q < 1$ is the q-gamma function defined by (1.14).

In [7, Theorem 2.5], the following logarithmically complete monotonicity of $H_{q;a,b;c}(x)$ was obtained: For $a < b \le a+1$, the function $H_{q;a,b;c}(x)$ is logarithmically completely monotonic on $(-c,\infty)$ if $0 \le c \le (a+b-1)/2$, so is its reciprocal on $(-a,\infty)$ if $c \ge a \ge 0$. Note that the proof in [7, Theorem 2.5] for the conclusion "Neither is completely monotonic for $(a+b-1)/2 < c < a$" is not convincible.

In virtue of monotonic properties of $q_{\alpha,\beta}(t)$ on $(0,\infty)$, it is not difficult to see that [7, Theorem 2.5] can be extended to and [116, Theorem 2.5] and [7, Theorem 2.6] can be included in the following Theorem 7.1 easily and thoroughly, which is an analogue of Theorem 6.8.

Theorem 7.1 (see [35]). *Let a, b and c be real numbers and $\rho = \min\{a, b, c\}$.*

(1) *The function $H_{q;a,b;c}(x)$ is logarithmically completely monotonic on $(-\rho, \infty)$ if and only if*

$$\begin{aligned}(a,b;c) \in D_1(a,b;c) \triangleq {} & \{(a,b;c) : (a-b)(1-a-b+2c) \geq 0\} \\ & \cap \{(a,b;c) : (a-b)(|a-b|-a-b+2c) \geq 0\} \\ & \setminus \{(a,b;c) : a = c+1 = b+1 \text{ or } b = c+1 = a+1\}.\end{aligned} \tag{7.2}$$

(2) *The function $H_{q;b,a;c}(x)$ is logarithmically completely monotonic on $(-\rho, \infty)$ if and only if*

$$\begin{aligned}(a,b;c) \in D_2(a,b;c) \triangleq {} & \{(a,b;c) : (a-b)(1-a-b+2c) \leq 0\} \\ & \cap \{(a,b;c) : (a-b)(|a-b|-a-b+2c) \leq 0\} \\ & \setminus \{(a,b;c) : b = c+1 = a+1 \text{ or } a = c+1 = b+1\}.\end{aligned} \tag{7.3}$$

Similar to Theorem 6.9, we have the following double inequality for divided differences of the q-psi function $\psi_q(x)$ for $0 < q < 1$, which may be derived from Theorem 7.1.

Theorem 7.2 (see [35]). *Let $b > a \geq 0$, $k \in \mathbb{N}$ and $0 < q < 1$. If $0 \leq c \leq \min\{a, (a+b-1)/2\}$, then*

$$\frac{(-1)^{k-1}\left[\psi_q^{(k-1)}(x+b) - \psi_q^{(k-1)}(x+a)\right]}{b-a} \leq (-1)^k \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\frac{q^{x+c}}{1-q^{x+c}}\right)\ln q \tag{7.4}$$

holds for $x \in (0, \infty)$. If $c \geq \max\{a, (a+b-1)/2\}$, inequality (7.4) reverses on $(0, \infty)$. Consequently, the identity

$$\psi_q^{(k-1)}(x+1) - \psi_q^{(k-1)}(x) = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\frac{q^x}{1-q^x}\right)\ln q \tag{7.5}$$

holds for $x \in (0, \infty)$ and $k \in \mathbb{N}$.

7.2. Monotonicity Results for the Ratio of Products of Gamma Functions

7.2.1. Bounds for Gurland's Ratio

There have been a lot of literature on bounding Gurland's ratio $T(x, y)$ defined by (3.13). Gurland's ratio and the ratio of two gamma functions are nearly a couple of companion, therefore, to find results on bounding Gurland's ratio possibly as long as to find those bounding the ratio of two gamma functions, see [20, 96, 97] and related references therein.

There are a lot of literature provided in [44] on bounding Gurland's ratio.

7.2.2. Monotonicity Results for the Ratio of Products of Gamma Functions

As a generalization of Gurland's ratio, the function

$$\frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} \tag{7.6}$$

for nonnegative numbers a and b, related with Gauss's Theorem expressed by (1.13), was proved in [96, Theorem 6] to be logarithmically completely monotonic on $(0,\infty)$.

In [31], a more general result was obtained. The function

$$\prod_{k=1}^{n}\frac{\Gamma(x+a_k)}{\Gamma(x+b_k)} \tag{7.7}$$

is logarithmically completely monotonic on $(0,\infty)$ provided that

$$\begin{aligned} &0 \le a_1 \le a_2 \le \cdots \le a_n, \qquad 0 \le b_1 \le b_2 \le \cdots \le b_n, \\ &\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i \quad \text{for } 1 \le k \le n-1, \quad \text{and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i. \end{aligned} \tag{7.8}$$

In [21, Theorem 1.2], it was presented that the functions

$$F_n(x) = \frac{\Gamma(x)\prod_{k=1}^{[n/2]}\left[\prod_{m\in P_{n,2k}}\Gamma\left(x+\sum_{j=1}^{2k}a_{m_j}\right)\right]}{\prod_{k=1}^{[(n+1)/2]}\left[\prod_{m\in P_{n,2k-1}}\Gamma\left(x+\sum_{j=1}^{2k-1}a_{m_j}\right)\right]} \tag{7.9}$$

for any $a_k > 0$ and $k \in \mathbb{N}$ are logarithmically completely monotonic on $(0,\infty)$ and that any product of functions of the type (7.9) with different parameters a_k is logarithmically completely monotonic as well, where $P_{n,k}$ for $1 \le k \le n$ is the set of all vectors $\mathbf{m} = (m_1,\ldots,m_k)$ whose components are natural numbers such that $1 \le m_\nu < m_\mu \le n$ for $1 \le \nu < \mu \le k$ and $P_{n,0}$ is the empty set.

Let a_i and b_i for $1 \le i \le n$ be real numbers and $\rho_n = \min_{1\le i\le n}\{a_i, b_i\}$. For $x \in (-\rho_n,\infty)$, define

$$h_{\mathbf{a},\mathbf{b};n}(x) = \prod_{i=1}^{n}\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}, \tag{7.10}$$

where $\mathbf{a}$ and $\mathbf{b}$ denote $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$, respectively. By recurring to monotonic properties of $q_{\alpha,\beta}(t)$ on $(0,\infty)$, the following new sufficient conditions for $h_{\mathbf{a},\mathbf{b};n}(x)$ to be logarithmically completely monotonic on $(0,\infty)$ are devised.

Theorem 7.3 (see [35]). *If*

$$(b_i-a_i)(1-a_i-b_i) \ge 0, \qquad (b_i-a_i)(|a_i-b_i|-a_i-b_i) \ge 0 \tag{7.11}$$

hold for $1 \le i \le n$ *and*

$$\sum_{i=1}^{n} b_i \ge \sum_{i=1}^{n} a_i, \tag{7.12}$$

then the function $h_{\mathbf{a},\mathbf{b};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n, \infty)$. *If inequalities in* (7.11) *and* (7.12) *are reversed, then the function* $h_{\mathbf{b},\mathbf{a};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n, \infty)$.

Remark 7.4. The beta function $B(p,q)$ is defined by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{7.13}$$

It is a ratio among three gamma functions.

The ratios among four gamma functions have hypergeometric functions (1.13) and Gurland's ratio (3.13).

7.3. Monotonicity Results for the Ratio of Products of q-Gamma Functions

In [21, Theorems 3.2 and 3.3] and [7, Theorem 4.1], logarithmically complete monotonicity for ratios of products of q-gamma functions was discussed.

The q-analogue of Theorem 7.3 is as follows.

Theorem 7.5 (see [35]). *Let* a_i *and* b_i *for* $1 \le i \le n$ *be real numbers and* $\rho_n = \min_{1\le i\le n}\{a_i, b_i\}$. *For* $x \in (-\rho_n, \infty)$, *define*

$$h_{q;\mathbf{a},\mathbf{b};n}(x) = \prod_{i=1}^{n} \frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)} \tag{7.14}$$

for $0 < q < 1$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *denote* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$, *respectively. If inequalities in* (7.11) *and* (7.12) *hold, then the function* $h_{q;\mathbf{a},\mathbf{b};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n, \infty)$. *If inequalities in* (7.11) *and* (7.12) *are reversed, then the function* $h_{q;\mathbf{b},\mathbf{a};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n, \infty)$.

7.4. Bounds for Wallis' Formula

Bounding Wallis' formula (1.5) has a long history, as mentioned in Sections 2.2, 2.3, and 2.5. For more information, please refer to related contents in [171], [54, pages 192-193 and page 287] and [90, 91].

In this section, we would like to review concisely recent developments on it.

7.4.1. Recovery and Various Proofs

As mentioned in Remark 3.6, the double inequality (3.6) and its sharpness were recovered and proved once and again in some papers such as [70, 86–92, 172–174] and [108, Theorem 2], because of either without being aware of and finding out the original version of the paper [56], or making use of various approaches and subtle techniques, or repeating some existed routines. Moreover, almost of them were not devoted to improve the bounds in (3.6).

7.4.2. Bounds for Wallis' Formula and the Probability Integral

The following theorem connects bounds for Wallis' formula with the probability integral.

Theorem 7.6 (see [85]). *For $n \in \mathbb{N}$,*

$$\frac{\sqrt{\pi}}{2\sqrt{n+9\pi/16-1}} \le \frac{(2n)!!}{(2n+1)!!} < \frac{\sqrt{\pi}}{2\sqrt{n+3/4}}. \tag{7.15}$$

The constants $9\pi/16-1$ and $3/4$ in (7.15) are the best possible.

For all natural number n,

$$\frac{\sqrt{\pi}}{\sqrt{1+(9\pi/16-1)/n}} \le \int_{-\sqrt{n}}^{\sqrt{n}} e^{-x^2}\mathrm{d}x < \frac{\sqrt{\pi}}{\sqrt{1-3/4n}}. \tag{7.16}$$

In particular, taking $n \to \infty$ in (7.16) leads to

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}. \tag{7.17}$$

7.4.3. Koumandos' Generalization

Influenced by the iterative work in [90, 91], Koumandos established in [93] the following general double inequality.

Theorem 7.7 (see [93]). *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ and*

$$d_n(\alpha) = \frac{(1-\alpha)_n}{n!} = \begin{cases} 1, & n = 0, \\ \dfrac{\Gamma(n+1-\alpha)}{n!\Gamma(1-\alpha)}, & n \ge 1. \end{cases} \tag{7.18}$$

For all natural numbers n,

$$\frac{1}{\Gamma(1-\alpha)(n+c_2)^{\alpha}} \le d_n(\alpha) < \frac{1}{\Gamma(1-\alpha)(n+c_1)^{\alpha}}, \tag{7.19}$$

where the constants

$$c_1 = c_1(\alpha) = \frac{1-\alpha}{2}, \qquad c_2 = c_2(\alpha) = \frac{1}{[\Gamma(2-\alpha)]^{1/\alpha}} - 1 \tag{7.20}$$

are the best possible.

Remark 7.8. The double inequality (7.19) for $\alpha = 1/2$ coincides with (3.6).

7.4.4. Zhao's Refinement

In [175], the double inequality (3.6) was refined as follows. For $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi n[1+1/(4n-1/2)]}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n[1+1/(4n-1/3)]}}. \tag{7.21}$$

Remark 7.9. The double inequality (7.21) is better than (2.12), say nothing of (2.16) and (3.6).

Remark 7.10. By (1.6), inequality (7.21) may be rewritten as

$$\sqrt{k+\frac{k}{4k-1/3}} < \frac{\Gamma(k+1)}{\Gamma(1+1/2)} < \sqrt{k+\frac{k}{4k-1/2}}, \quad k \in \mathbb{N}. \tag{7.22}$$

7.4.5. Zhang's Refinements

In [108, Theorem 1], the double inequality (7.21) was refined as

$$\frac{1}{\sqrt{\pi\{n+1/[4-4/(8n+3)]\}}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\{n+1/[4-1/(2n+1)]\}}}. \tag{7.23}$$

Remark 7.11. The right-hand side inequality in (7.23) is same as the corresponding one in (3.49) for $x = k$, and the left-hand side inequality in (7.23) refines the lower bound of (3.49) for $x = k$.

Remark 7.12. In virtue of (1.6), the double inequality (7.23) may be rearranged as

$$\sqrt{k+\frac{1}{4-1/(2k+1)}} < \frac{\Gamma(k+1)}{\Gamma(1+1/2)} < \sqrt{k+\frac{1}{4-4/(8k+3)}}, \quad k \in \mathbb{N}. \tag{7.24}$$

7.4.6. Zhao-Wu's Refinements and Generalizations

In [94, 176], the following refinements and generalizations of inequalities on Wallis' formula were established.

Theorem 7.13. *For $0 < z < 1$ and $n > 1$,*

$$\frac{1}{n^z[1+(1-z)/2(n-1)]^z\Gamma(1-z)} < \frac{(1-z)(2-z)\cdots(n-z)}{n!} < \frac{1}{n^z[1+(1-z)/(2n+1-z)]^z\Gamma(1-z)}, \tag{7.25}$$

For $0 < z < 1$ and $n \geq 22$,

$$\frac{(1-z)(2-z)\cdots(n-z)}{n!} < \frac{1}{n^z[1+(1-z)/2n]^z\Gamma(1-z)}, \tag{7.26}$$

For $n \geq 1$, the left-hand side inequality in (7.21) is valid. For $n \geq 1$ and $0 < \epsilon < 1/2$,

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{n\pi[1+1/(4n-1/2+\epsilon)]}} \tag{7.27}$$

holds for $n > n^*$ if n^* is the maximal root of

$$32\epsilon n^2 + 4\epsilon^2 n + 32\epsilon n - 17n + 4\epsilon^2 - 1 = 0. \tag{7.28}$$

Remark 7.14. Inequalities in (3.6) and (7.21) can be deduced from Theorem 7.13.

7.5. Inequalities for the Volume of Unit Ball in $\mathbb{R}^n$

Let

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \tag{7.29}$$

denote the volume of the unit ball in $\mathbb{R}^n$. In [177], among other things, it was demonstrated for $n \geq 1$ that

$$\sqrt{\frac{n+1/2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+\pi/2-1}{2\pi}} \tag{7.30}$$

by recovering the decreasing monotonicity of $\theta(x)$ defined by (3.2) on $[0,\infty)$. The constants both $1/2$ and $\pi/2-1$ in (7.30) are the best possible.

It is easy to see that the ratio of the volumes of two unit balls in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ for $n \in \mathbb{N}$ is related with the ratio of two gamma functions.

For more information on inequalities of volumes of unit balls in $\mathbb{R}^n$, please see [122, 177, 178], [179, Theorem 1] and [44, Theorem 5], and related references therein.

In [178], it was proved that the sequence $\Omega_n^{1/(n\ln n)}$ for $n \geq 2$ is decreasing and converges to $e^{-1/2}$ as $n \to \infty$. In [180, Theorem 2]. It was presented that the double inequality

$$\exp\left(\frac{a}{n(\ln n)^2}\right) \leq \frac{\Omega_n^{1/(n\ln n)}}{\Omega_{n+1}^{1/[(n+1)\ln(n+1)]}} < \exp\left(\frac{b}{n(\ln n)^2}\right) \tag{7.31}$$

holds for $n \geq 2$ if and only if

$$a \leq \ln 2 \ln \pi - \frac{2(\ln 2)^2 \ln(4\pi/3)}{3\ln 3}, \qquad b \geq \frac{1+\ln(2\pi)}{2}. \tag{7.32}$$

Recently, the author obtained the following stronger results than the ones in [180, Theorem 2] and [178].

Theorem 7.15. *For $n \geq 2$, the sequence $\Omega_n^{1/(n\ln n)}$ is logarithmically convex and the sequence*

$$\frac{\Omega_n^{1/(n\ln n)}}{\Omega_{n+1}^{1/[(n+1)\ln(n+1)]}} \tag{7.33}$$

is decreasing.

For more information on this topic, please see [181] and related references.

7.6. A New Ratio of Two Gamma Functions

By using a geometrical method, the following double inequality was proved in [182]:

$$\frac{1}{n!} \leq \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)} \leq 1 \tag{7.34}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$.

By analytical arguments in [183], it was presented that the function

$$f(x,y) = \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{7.35}$$

for all $y \geq 1$ is decreasing in $x \geq 0$. From this, it is deduced that

$$\frac{1}{\Gamma(1+y)} \leq \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \leq 1 \tag{7.36}$$

for all $y \geq 1$ and $x \in [0,1]$, which is a generalization of inequality (7.34).

In [184], It was showed that if f is a differentiable and logarithmically convex function on $[0,\infty)$, then the function $[f(x)]^a/f(ax)$ for $a \geq 1$ (or $0 < a \leq 1$, resp.) is decreasing (or increasing resp.) on $[0,\infty)$. As one of applications to inequalities involving gamma function, Riemann's zeta function and the complete elliptic integrals of the first kind, inequalities (7.34) and (7.36) were deduced.

In [185], an inequality involving a positive linear operator acting on the composition of two continuous functions is presented and, as applications of this inequality, some new inequalities involving the beta, gamma and Riemann's zeta functions and a large family of functions which are Mellin transforms are produced. In particular, for $\beta > \delta > 0$, $\alpha\beta > -1$ and $\alpha\delta > -1$, if either $\alpha < 0$ or $\alpha > 1$, then the inequality

$$\frac{[\Gamma(1+\delta)]^{\alpha}}{\Gamma(1+\alpha\delta)} > \frac{[\Gamma(1+\beta)]^{\alpha}}{\Gamma(1+\alpha\beta)} \tag{7.37}$$

holds true; if $0 < \alpha \leq 1$, inequality (7.37) is reversed. It is not difficult to see that the left-hand sides in inequalities (7.34) and (7.36) are special cases of the inequality (7.37).

In [186, 187], the following logarithmically complete monotonicities, as generalizations of the decreasingly monotonic property in [183], are presented.

(1) For given $y > 1$, the function $f(x,y)$ defined by (7.35) is decreasing and logarithmically concave with respect to $x \in (0,\infty)$, and $1/f(x,y)$ is a logarithmically completely monotonic function of second order in $x \in (0,\infty)$.

(2) For given $0 < y < 1$, the function $f(x,y)$ defined by (7.35) is increasing and logarithmically convex with respect to $x \in (0,\infty)$, and $f(x,y)$ is a logarithmically completely monotonic function of second order in $x \in (0,\infty)$.

(3) For given $x \in (0,\infty)$, the function $f(x,y)$ defined by (7.35) is logarithmically concave with respect to $y \in (0,\infty)$, and $1/f(x,y)$ is a logarithmically completely monotonic function of first order in $y \in (0,\infty)$.

(4) For given $x \in (0,\infty)$, let

$$F_x(y) = \frac{\Gamma(1+y)[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{7.38}$$

on $\in (0,\infty)$. If $0 < x < 1$, then $F_x(y)$ is a logarithmically completely monotonic function of second order on $(0,\infty)$; if $x > 1$, then $1/F_x(y)$ is a logarithmically completely monotonic function of second order on $(0,\infty)$.

In [188, Theorem 2.1], it was proved that the function

$$G_{s,t}(x) = \frac{[\Gamma(1+tx)]^s}{[\Gamma(1+sx)]^t} \tag{7.39}$$

is decreasing (or increasing, resp.) in $x \in [0,\infty)$ if either $s \geq t > 0$ or $0 > s \geq t$ (or both $s > 0$ and $t < 0$, resp.) such that $1 + sx > 0$ and $1 + tx > 0$. This result generalized and extended the corresponding conclusions in [182, 183, 185].

In [23, 24], the following logarithmically complete monotonicity, logarithmically absolute monotonicity and logarithmically absolute convexity of $G_{s,t}(x)$ defined by (7.39) are verified. The function $G_{s,t}(x)$ for $x, s, t \in \mathbb{R}$ such that $1 + sx > 0$ and $1 + tx > 0$ with $s \neq t$ has the following properties.

(1) For $t > s > 0$ and $x \in (0, \infty)$, $G_{s,t}(x)$ is an increasing function and a logarithmically completely monotonic function of second order in x.

(2) For $t > s > 0$ and $x \in (-1/t, 0)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x;

(3) For $s < t < 0$ and $x \in (-\infty, 0)$, $G_{s,t}(x)$ is a decreasing function and a logarithmically absolutely monotonic function of second order in x;

(4) For $s < t < 0$ and $x \in (0, -1/s)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x.

(5) For $s < 0 < t$ and $x \in (-1/t, 0)$, $G_{t,s}(x)$ is an increasing function and a logarithmically absolutely convex function in x.

(6) For $s < 0 < t$ and $x \in (0, -1/s)$, $G_{t,s}(x)$ is a decreasing function and a logarithmically absolutely convex function in x.

As generalizations of [24, Theorem 1.4] and the main results in [187], the following i-log convex properties for $i \in \mathbb{N}$ are established in [189] and its simplified version [190, Theorem 1.2].

Theorem 7.16. *Let a and b be two real numbers, $f(x)$ a positive function on an interval I, and*

$$g_{a,b}(x) = \frac{[f(bx)]^a}{[f(ax)]^b} \tag{7.40}$$

defined for $ax \in I$ and $bx \in I$. For $i \in \mathbb{N}$, the function $g_{a,b}(x)$ has the following properties.

(1) *For either $b > a > 0$ and $x > 0$ or $0 > b > a$ and $x > 0$,*

   (a) *if the function $u^{i-1}[\ln f(u)]^{(i)}$ for all $i \in \mathbb{N}$ is increasing on I, then $g_{a,b}(x)$ is i-log-convex;*

   (b) *if the function $u^{i-1}[\ln f(u)]^{(i)}$ for all $i \in \mathbb{N}$ is decreasing on I, then $g_{a,b}(x)$ is i-log-concave.*

(2) *For $b > 0 > a$ and $x > 0$,*

   (a) *if the function $u^{i-1}[\ln f(u)]^{(i)}$ for all $i \in \mathbb{N}$ is increasing on I, then $g_{a,b}(x)$ is i-log-concave;*

   (b) *if the function $u^{i-1}[\ln f(u)]^{(i)}$ for all $i \in \mathbb{N}$ is decreasing on I, then $g_{a,b}(x)$ is i-log-convex.*

(3) *For either $b > a > 0$ and $x > 0$ or $0 > b > a$ and $x < 0$,*

   (a) *if the function $u^{i-1}[\ln f(u)]^{(i)}$ for all $i \in \mathbb{N}$ is increasing on I, then $g_{a,b}(x)$ is $(2i-1)$-log-concave and $(2i)$-log-convex;*

(b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I*, then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-convex and* $(2i)$*-log-concave.*

(4) *For* $b > 0 > a$ *and* $x < 0$*,*

(a) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is increasing on* I*, then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-convex and* $(2i)$*-log-concave;*

(b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I*, then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-concave and* $(2i)$*-log-convex.*

Remark 7.17. Most results in [191–194] are simple and direct consequences of Theorem 7.16.

Remark 7.18. Some proofs in [23, 24, 189, 190] need to use the main results in [160–162] on monotonic properties of some functions involving the psi and polygamma functions.

Remark 7.19. Recently, some new results on this topic are obtained in [84, 195–200] and closely-related references therein.

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