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## Research Article

# **Upper Semicontinuity of Solution Maps for a Parametric Weak Vector Variational Inequality**

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This paper investigates the upper semicontinuity of the solution map for a parametric weak vector variational inequality associated to a *v*-hemicontinuous and weakly *C*-pseudomonotone operator.

#### 1. Introduction and Preliminaries

A vector variational inequality (VVI, in short) was first introduced by Giannessi [1] in the setting of finite-dimensional Euclidean space. Later on, it was studied and generalized to infinite-dimensional spaces. Existences of the solutions for VVI have been studied extensively in various versions; see [2–6] and references therein.

Stability of the solution map for VVI or vector equilibrium problems is an important topic in optimization theory. A great deal of papers have been denoted to study the semicontinuity and continuity of the solution maps; see [7–18] and references therein. All results of the stability of the solution map for VVI in the literature are obtained based on continuity of the operator. It is well known that v-hemicontinuity is weaker than continuity. In this paper, our aim is to investigate the upper semicontinuity of the solution map for a parametric weak vector variational inequality associated to a v-hemicontinuous and weakly C-pseudomonotone operator.

Let X, Y, and W (the spaces of parameters) be Banach spaces and let  $C \subset Y$  be a pointed closed and convex cone with nonempty interior int C. Let L(X,Y) be the space of all linear continuous operators from X to Y. The value of a linear operator  $t \in L(X,Y)$  at  $x \in X$  is denoted by  $\langle t, x \rangle$ . Consider the following weak vector variational inequality problem:

find 
$$x \in K$$
 such that  $\langle T(x), y - x \rangle \notin - \text{int } C$ ,  $\forall y \in K$ , (WVVI)

where  $K \subset X$  is a nonempty subset and  $T: X \to L(X,Y)$  is a vector-valued mapping.

When *T* is perturbed by a parameter  $\mu$ , which varies over a nonempty set  $\Lambda \subset W$ , for a given  $\mu$ , we can define the parametric weak vector variational inequality problem

find 
$$x \in K$$
 such that  $\langle T(x, \mu), y - x \rangle \notin - \text{int } C$ ,  $\forall y \in K$ , (PWVVI)

where  $K \subset X$  is a nonempty subset and  $T : X \times \Lambda \to L(X,Y)$  is a vector-valued mapping. For each  $\mu \in \Lambda$ , we denote the solution map of (PWVVI) by  $S(\mu)$ , that is,

$$S(\mu) = \{ x \in K \mid \langle T(x, \mu), y - x \rangle \notin -\inf C, \ \forall y \in K \}. \tag{1.1}$$

Throughout the paper, we always assume that  $S(\mu)$  is nonempty for all  $\mu$  in a neighborhood of  $\overline{\mu} \in \Lambda$ . Now, we recall some basic definitions and their properties.

*Definition 1.1* (see [4,6]). Let *K* be a nonempty convex subset of *X* and let *T* : *K* → L(X,Y) be an operator. *T* is said to be *v*-hemicontinuous if and only if, for every  $x, y \in K$  and  $t \in [0,1]$ , the mapping  $t \to \langle T(ty + (1-t)x), y - x \rangle$  is continuous at  $0^+$ .

*Definition* 1.2 (see [6]). Let K be a nonempty subset of X and  $T: K \to L(X, Y)$  be an operator. T is weakly C-pseudomonotone on K if, for every pair of points  $x \in K$ ,  $y \in K$ , one has that  $\langle Tx, y - x \rangle \notin -$  int C implies that  $\langle T(y), y - x \rangle \notin -$  int C.

**Proposition 1.3** (see [6, Generalized Linearization Lemma]). *Let K be a nonempty convex subset of X and let T* :  $K \rightarrow L(X,Y)$  *be an operator. Consider the following problems:* 

- (I)  $x \in K$  such that  $\langle Tx, y x \rangle \notin \text{int } C$  for all  $y \in K$ ,
- (II)  $x \in K$  such that  $\langle Ty, y x \rangle \notin -\operatorname{int} C$  for all  $y \in K$ .

Then the following are obtained.

- (i) Problem (I) implies Problem (II) if T is weakly C-pseudomonotone.
- (ii) Problem (II) implies Problem (I) if T is v-hemicontinuous.

Let  $F: \Lambda \to 2^X$  be a set-valued mapping, given that  $\overline{\lambda} \in \Lambda$ .

*Definition 1.4* (see [19, 20]). (i) F is called lower semicontinuous (l.s.c) at  $\overline{\lambda}$  if, for any open set V satisfying  $V \cap F(\overline{\lambda}) \neq \emptyset$ , there exists  $\delta > 0$  such that for every  $\lambda \in B(\overline{\lambda}, \delta)$ ,  $V \cap F(\lambda) \neq \emptyset$ .

(ii) F is called upper semicontinuous (u.s.c) at  $\overline{\lambda}$  if, for any open set V satisfying  $F(\overline{\lambda}) \subset V$ , there exists  $\delta > 0$  such that, for every  $\lambda \in B(\overline{\lambda}, \delta)$ ,  $F(\lambda) \subset V$ .

We say F is l.s.c (resp., u.s.c) on  $\Lambda$ , if it is l.s.c (resp., u.s.c) at each  $\lambda \in \Lambda$ . F is said to be continuous on  $\Lambda$  if it is both l.s.c and u.s.c on  $\Lambda$ .

**Proposition 1.5** (see [19, 21]). (i) F is l.s.c at  $\overline{\lambda}$  if and only if, for any sequence  $\{\lambda_n\} \subset \Lambda$  with  $\lambda_n \to \overline{\lambda}$  and any  $\overline{x} \in F(\overline{\lambda})$ , there exists  $x_n \in F(\lambda_n)$  such that  $x_n \to \overline{x}$ .

(ii) If F has compact values (i.e.,  $F(\lambda)$  is a compact set for each  $\lambda \in \Lambda$ ), then F is u.s.c at  $\overline{\lambda}$  if and only if, for any sequence  $\{\lambda_n\} \subset \Lambda$  with  $\lambda_n \to \overline{\lambda}$  and for any  $x_n \in F(\lambda_n)$ , there exist  $\overline{x} \in F(\overline{\lambda})$  and a subsequence  $\{x_{n_k}\}$  of  $x_n$  such that  $x_{n_k} \to \overline{x}$ .

#### 2. Main Results

In this section, we mainly discuss the upper semicontinuity of the solution map for (PWVVI).

**Lemma 2.1.** Let K be a nonempty compact convex subset of X. Suppose that, for any  $\mu \in \Lambda$ ,  $T(\cdot, \mu)$  is v-hemicontinuous and weakly C-pseudomonotone on K. Then,  $S(\cdot)$  has compact values on  $\Lambda$ , that is,  $S(\mu)$  is a compact set for each  $\mu \in \Lambda$ .

*Proof.* For any  $\mu \in \Lambda$ , take any sequence  $x_n \in S(\mu)$  with  $x_n \to x$ ; we have

$$\langle T(x_n, \mu), y - x_n \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.1)

By Proposition 1.3 and the weakly *C*-pseudomonotonicity of  $T(\cdot, \mu)$ , we get

$$\langle T(y,\mu), y - x_n \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.2)

From  $T(y,\mu) \in L(X,Y)$ , we have  $\langle T(y,\mu), y - x_n \rangle \to \langle T(y,\mu), y - x \rangle$  as  $n \to \infty$ . It follows from the closedness of  $Y \setminus -$  int C and (2.2) that

$$\langle T(y,\mu), y-x \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.3)

Moreover, by Proposition 1.3 and the *v*-hemicontinuity of  $T(\cdot, \mu)$ , we have

$$\langle T(x,\mu), y - x \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.4)

That is  $x \in S(\mu)$ . Thus,  $S(\mu)$  is a closed set. Furthermore, it follows from  $S(\mu) \subset K$  and the compactness of K that  $S(\mu)$  is a compact set. The proof is complete.

**Theorem 2.2.** Let K be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied.

- (i) For any  $\mu \in \Lambda$ ,  $T(\cdot, \mu)$  is v-hemicontinuous on K,
- (ii) For any  $\mu \in \Lambda$ ,  $T(\cdot, \mu)$  is weakly C-pseudomonotone on K,
- (iii) For any  $x \in X$ ,  $T(x, \cdot)$  is continuous on  $\Lambda$ .

*Then,*  $S(\cdot)$  *is u.s.c on*  $\Lambda$ .

*Proof.* For any  $\mu_0 \in \Lambda$ , any sequences  $\{\mu_n\} \subset \Lambda$  with  $\mu_n \to \mu_0$ , and  $x_n \in S(\mu_n)$ , we have  $x_n \in K$  and

$$\langle T(x_n, \mu_n), y - x_n \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.5)

Since K is a compact set, there are an  $x_0 \in K$  and a subsequence  $\{x_{n_k}\} \subset \{x_n\}$  such that  $x_{n_k} \to x_0$ . Particularly, from (2.5), we get

$$\langle T(x_{n_k}, \mu_{n_k}), y - x_{n_k} \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.6)

By Proposition 1.3 and (iii), we can obtain that

$$\langle T(y, \mu_{n_k}), y - x_{n_k} \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.7)

Since  $T(y, \cdot)$  is continuous and

$$\|\langle T(y,\mu_{n_{k}}), y - x_{n_{k}} \rangle - \langle T(y,\mu_{0}), y - x_{0} \rangle \|$$

$$\leq \|\langle T(y,\mu_{n_{k}}), y - x_{n_{k}} \rangle - \langle T(y,\mu_{0}), y - x_{n_{k}} \rangle \|$$

$$+ \|\langle T(y,\mu_{0}), y - x_{n_{k}} \rangle - \langle T(y,\mu_{0}), y - x_{0} \rangle \|$$

$$\leq \|T(y,\mu_{n_{k}}) - T(y,\mu_{0}) \| \|y - x_{n_{k}} \| + \|T(y,\mu_{0}) \| \|x_{n_{k}} - x_{0}\|,$$
(2.8)

we get  $\langle T(y, \mu_{n_k}), y - x_{n_k} \rangle \rightarrow \langle T(y, \mu_0), y - x_0 \rangle$ , as  $n_k \rightarrow \infty$ . It follows from the closedness of  $Y \setminus -int C$  and (2.7) that

$$\langle T(y,\mu_0), y - x_0 \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K.$$
 (2.9)

Moreover, by Proposition 1.3 and (ii), we have

$$\langle T(x_0, \mu_0), y - x_0 \rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K,$$
 (2.10)

that is  $x_0 \in S(\mu_0)$ .

Thus, for any sequence  $\mu_n \subset \Lambda$  with  $\mu_n \to \mu_0$  and for any  $x_n \in S(\mu_n)$ , there exist  $x_0 \in S(\mu_0)$  and a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \to x_0$ . By Proposition 1.5 and Lemma 2.1, we have  $S(\cdot)$  is u.s.c at  $\mu_0$ . From the arbitrariness of  $\mu_0$ , we can get  $S(\cdot)$  is u.s.c on  $\Lambda$ . The proof is complete.

Remark 2.3. In [7–10], the upper semicontinuity of the solution map for (PVVI) has been discussed based on the continuity of the operator. Note that v-hemicontinuity is weaker than continuity. Moreover, together with the assumption of weakly C-pseudomonotonicity, v-hemicontinuity may not derive the continuity of the operator. Thus, it is necessary to investigate the upper semicontinuity of the solution map for (PVVI) associated to a v-hemicontinuous and weakly C-pseudomonotone operator. Now we give an example to illustrate our result.

Example 2.4. Let  $X = R^2$ , Y = R,  $K = [0,1] \times [0,1]$ ,  $\Lambda = [0,1]$  and

$$T(x,\mu) = \begin{cases} \frac{\mu x_1^2 x_2}{x_1^2 + x_2^2}, & \text{if } x \neq (0,0)^\top, \\ 1, & \text{if } x = (0,0)^\top. \end{cases}$$
 (2.11)

Then,

$$\langle T(x,\mu), y - x \rangle = \begin{cases} \frac{\mu x_1^2 x_2}{x_1^2 + x_2^2} (y_1 - x_1, y_2 - x_2)^\top, & \text{if } x \neq (0,0)^\top, \\ 0, & \text{if } x = (0,0)^\top. \end{cases}$$
(2.12)

It is clear that conditions (ii) and (iii) of Theorem 2.2 are satisfied. For any ray  $x_2 = kx_1(0 \le k < \infty)$ ,  $T(\cdot, \mu)$  is continuous. Thus,  $T(\cdot, \mu)$  is v-hemicontinuous on K and condition (i) of Theorem 2.2 is satisfied. By Theorem 2.2, we conclude that  $S(\cdot)$  is u.s.c on  $\Lambda$ . In fact,

$$S(\mu) = \begin{cases} \{0\}, & \text{if } \mu \neq 0, \\ [0,1], & \text{if } \mu = 0. \end{cases}$$
 (2.13)

Then, by the definition of upper semicontinuity, it follows readily that the solutions map  $S(\mu)$  is u.s.c on  $\Lambda$ .

However, for  $x = (x_1, x_2)^{\top} \to x_0 = (0, 0)^{\top}$  with  $x_2 = x_1^2$ , we have  $T(x, \mu) = 1/2$ , but  $T(x_0, \mu) = 0$ . Thus, for any  $\mu \in \Lambda$ ,  $T(\cdot, \mu)$  is not continuous at  $(0, 0)^{\top}$ . Therefore, the theorems concerning the upper semicontinuity in the literatures are not applicable.

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