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Research Article

Note on q-Nasybullin's Lemma Associated with the Modified p-Adic q-Euler Measure

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We derive the modified p-adic q-measures related to q-Nasybullin's type lemma.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such a way that $|p|_p = 1/p$ (see [1–17]). For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, let $\overline{f} = [f, p]$ be the least common multiple of f and p. We set

$$\begin{aligned}
\mathbb{Z}_{\overline{f}} &= \frac{\varprojlim_n \mathbb{Z}}{\overline{f} p^n \mathbb{Z}}, \quad \text{for } n \geq 0, \\
\mathbb{Z}_{\overline{f}}^* &= \bigcup_{\substack{0<a<\overline{f}p \\ (a,p)=1}} \left(a + \overline{f}p\, \mathbb{Z}_p\right), \\
a + \overline{f}p^n \mathbb{Z}_p &= \left\{x \in \mathbb{Z}_{\overline{f}} \mid x \equiv a \left(\operatorname{mod} \overline{f}p^n\right)\right\},
\end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < \overline{f}p^n$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ (see [1–6, 18–23]). As the definition of q-number, we use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.2}$$

(see [1–23]).

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-invariant integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.3}$$

(see [2, 3]).

The q-Euler numbers, $\varepsilon_{n,q}$, can be determined inductively by

$$\varepsilon_{0,q} = 1, \quad q(q\varepsilon+1)^n + \varepsilon_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{1.4}$$

with the usual convention of replacing ε^i by $\varepsilon_{i,q}$ (see [11]). The modified q-Euler numbers $E_{n,q}$ of $\varepsilon_{n,q}$ are defined in [2] as follows:

$$E_{0,q} = \frac{[2]_q}{2}, \quad (qE+1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{1.5}$$

with the usual convention of replacing E^i by $E_{i,q}$. For any positive integer N,

$$\mu_q\left(a + \overline{f}p^N\mathbb{Z}_p\right) = \frac{(-q)^a}{\left[\overline{f}p^N\right]_{-q}} \tag{1.6}$$

is known as a measure on $\mathbb{Z}_{\overline{f}}$ (see [9]). In [2], the Witt's type formulas for $E_{n,q}$ are given by

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x}[x]_q^n\,d\mu_q(x) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}. \tag{1.7}$$

The modified q-Euler polynomials are also defined by

$$E_{n,q}(x) = \left([x]_q + q^x E\right)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} q^{lx} [x]_q^{n-l}, \tag{1.8}$$

with the usual convention of replacing E^n by $E_{n,q}$ (see [2]). Thus, we note that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-t}[x+t]_q^n d\mu_q(t) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{1+q^l}. \tag{1.9}$$

Recently Govil and Gupta [22] have introduced a new type of q-integrated Meyer-König-Zeller-Durrmeyer (q-MKZD) operators, obtained moments for these operators, and estimated the convergence of these integrated q-MKZD operators. In this paper, we consider the q-extension which is in a direction different than that of Govil and Gupta [22].

Let K be a field over $\mathbb{Q}_p$. Then we call a function μ a K-measure on $\mathbb{Z}_{\overline{f}}^*$ if μ is finitely additive function defined on open-closed subsets in $\mathbb{Z}_{\overline{f}}^*$, whose values are in the field K. Any open-closed subset in $\mathbb{Z}_{\overline{f}}^*$ is a disjoint union of some finite intervals $I_{a,n} = a + p^n \overline{f}\, \mathbb{Z}_p$ in $\mathbb{Z}_{\overline{f}}^*$, where $a \in \mathbb{Z}$is prime to $\overline{f}$, and therefore a K-measure μ is determined by its values on all intervals in $\mathbb{Z}_{\overline{f}}^*$. Let $Q^{(f)}$ denote the set of all rational numbers, whose denominator is a divisor of $\overline{f}p^n$ for some $n \geq 0$. In Section 2, we derive the modified p-adic q-measures related to q-Nasybullin's type lemma.

2. The Modified p-Adic q-Measure

Let T be a K-valued function defined on $Q^{(f)}$ with the following property.

There exist two constants $A, B \in K$ such that

$$\begin{gathered} \sum_{k=0}^{p-1} T\left(\left[\frac{x+k}{p}\right]_{q^p}\right)(-1)^k = AT\left([x]_q\right) + BT\left([px]_{q^{1/p}}\right), \\ T\left([x+1]_q\right) = T\left([x]_q\right), \end{gathered} \tag{2.1}$$

for any number $x \in Q^{(f)}$. Suppose that ρ is a root of the equation $y^2 = Ay + Bp$. Then we define

$$\mu(I_{a,n}) = \rho^{-n}(-1)^a T\left(\left[\frac{a}{p^n \overline{f}}\right]_{q^{p^n \overline{f}}}\right) + B\rho^{-(n+1)}(-1)^a T\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right), \tag{2.2}$$

for any interval $I_{a,n}$. From (2.2), we note that

$$
\begin{aligned}
&\sum_{k=0}^{p-1}\mu\left(I_{a+p^n\overline{f}k,n+1}\right)\\
&\quad=\rho^{-(n+1)}\sum_{k=0}^{p-1}T\left(\left[\frac{a+p^n\overline{f}k}{p^{n+1}\overline{f}}\right]_{q^{p^{n+1}\overline{f}}}\right)(-1)^{a+k}+B\rho^{-(n+2)}\sum_{k=0}^{p-1}T\left(\left[\frac{a+p^n\overline{f}k}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)(-1)^{a+k}\\
&\quad=\rho^{-(n+1)}(-1)^a\sum_{k=0}^{p-1}T\left(\left[\frac{k+a/p^n\overline{f}}{p}\right]_{(q^{p^n\overline{f}})^p}\right)(-1)^k+B\rho^{-(n+2)}(-1)^a\sum_{k=0}^{p-1}T\left(\left[\frac{a}{p^n\overline{f}}+k\right]_{q^{p^n\overline{f}}}\right)(-1)^k\\
&\quad=\rho^{-(n+1)}(-1)^aAT\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)+\rho^{-(n+1)}B(-1)^aT\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right)\\
&\qquad+B\rho^{-(n+2)}(-1)^apT\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)\\
&\quad=\rho^{-(n+2)}(-1)^a(\rho A+Bp)T\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)+\rho^{-(n+1)}B(-1)^aT\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right)\\
&\quad=\mu(I_{a,n}).
\end{aligned}
\tag{2.3}
$$

Thus, we have

$$
\mu(I_{a,n})=\sum_{\substack{b\,(\mathrm{mod}\ p^{n+1}\overline{f})\\ b\equiv a\,(\mathrm{mod}\ p^n\overline{f})}}\mu(I_{b,n+1}).
\tag{2.4}
$$

Therefore we obtain the following theorem.

Theorem 2.1. *For $f\in\mathbb{N}$ with $f\equiv 1\,(\mathrm{mod}\,2)$ and $\overline{f}=[p,f]$, let T be a K-valued function defined on $Q^{(f)}$ with the following properties.*

There exist two constants $A,B\in K$ such that

$$
\begin{gathered}
\sum_{k=0}^{p-1}T\left(\left[\frac{x+k}{p}\right]_{q^p}\right)(-1)^k=AT\left([x]_q\right)+BT\left([p\ x]_{q^{1/p}}\right),\\
T\left([x+1]_q\right)=T\left([x]_q\right),
\end{gathered}
\tag{2.5}
$$

for any $x \in Q^{(f)}$. Suppose that ρ is a root of the equation $y^2 = Ay + Bp$. Then there exists a $K(\rho)$-measure μ on $\mathbb{Z}_{\overline{f}}^{*}$ such that

$$\mu(I_{a,n}) = \rho^{-n}(-1)^a T\left(\left[\frac{a}{p^n \overline{f}}\right]_{q^{p^n \overline{f}}}\right) + B\rho^{-(n+1)}(-1)^a T\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right), \quad (2.6)$$

for any interval $I_{a,n}$.

From (1.9), we note that

$$E_{n,q}(x) = [p]_q^n \frac{[2]_q}{[2]_{q^p}} \sum_{a=0}^{p-1} (-1)^a E_{n,q^p}\left(\frac{x+a}{p}\right). \quad (2.7)$$

Let $E_{m,q}(x)$ be the mth q-Euler polynomials and let $P_m([x]_q)$ be the mth q-Euler functions, that is, for $0 \leq x < 1$,

$$P_m\left([x]_q\right) = E_{m,q}(x). \quad (2.8)$$

Note that $\lim_{q \to 1} P_m([x]_q) = P_m(x)$ is the Euler function. By (2.7), we see that

$$\frac{[2]_q}{[2]_{q^p}} [p]_q^m \sum_{a=0}^{p-1} (-1)^a P_m\left(\left[\frac{x+i}{p}\right]_{q^p}\right) = P_m\left([x]_q\right). \quad (2.9)$$

Thus, the q-Euler function $P_m([x]_q)$ satisfies the properties of Theorem 2.1 with constants

$$A = [p]_q^{-m} \frac{[2]_{q^p}}{[2]_q}, \qquad B = 0. \quad (2.10)$$

Then $\rho \neq 0$ is equal to $[p]_q^{-m}([2]_{q^p}/[2]_q)$, as $\rho^2 = A\rho + Bp$ reduces simply to $\rho^2 = [p]_q^{-m}([2]_{q^p}/[2]_q)\rho$. Therefore, we obtain the following theorem.

Theorem 2.2. *For $m \in \mathbb{Z}_+$, let the function $\mu_m = \mu_{m,q}$ be defined on $I_{a,n}$ as follows:*

$$\mu_m(I_{a,n}) = \left[\overline{f} p^n\right]_q^m \frac{[2]_q}{[2]_{q^{p^n \overline{f}}}} (-1)^a P_m\left(\left[\frac{a}{p^n \overline{f}}\right]_{q^{p^n \overline{f}}}\right). \quad (2.11)$$

Then μ_m is a $\mathbb{Q}_p(q)$-measure on $\mathbb{Z}_{\overline{f}}^{*}$.

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$ and $\overline{f} = [f, p]$, let χ be a primitive Dirichlet character modulo $\overline{f}$. Then the generalized q-Euler numbers are defined as follows:

$$E_{n,\chi,q} = \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n,q^{\overline{f}}}\left(\frac{a}{\overline{f}}\right). \tag{2.12}$$

From (2.12) and (2.7), we can easily derive the following Witt's formula:

$$\begin{aligned} E_{n,\chi,q} &= \int_{\mathbb{Z}_{\overline{f}}} [x]_q^n q^{-x} \chi(x) d\mu_q(x) \\ &= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \left[\frac{a}{d} + x\right]_{q^{\overline{f}}} q^{-dx} d\mu_{q^d}(x) \\ &= \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \left[\frac{a}{\overline{f}} + x\right]_{q^{\overline{f}}} q^{-\overline{f}x} d\mu_{q^{\overline{f}}}(x) \\ &= \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n,q^{\overline{f}}}\left(\frac{a}{\overline{f}}\right). \end{aligned} \tag{2.13}$$

We can compute a q-analogue of the p-adic q-l-function by the following p-adic q-Mellin Mazur transform with respect to μ_m.

Let

$$\begin{aligned} L(\mu_m, \chi) &= \int_{\mathbb{Z}^*_{\overline{f}}} \chi(a) d\mu_m(a) \\ &= \lim_{\rho \to \infty} \sum_{\substack{a \,(\mathrm{mod}\ p^\rho \overline{f}) \\ a \in \mathbb{Z},\ (a,p)=1}} \chi(a) \mu_m(I_{a,\rho}). \end{aligned} \tag{2.14}$$

Since the character χ is constant on the interval $I_{a,0}$,

$$\begin{aligned} L(\mu_m, \chi) &= \sum_{\substack{a \,(\mathrm{mod}\ \overline{f}) \\ (a,p)=1}} \chi(a) \mu_m(I_{a,0}) \\ &= \sum_{\substack{a \,(\mathrm{mod}\ \overline{f}) \\ (a,p)=1}} \chi(a) \left[\overline{f}\right]_q^m \frac{[2]_q}{[2]_{q^{\overline{f}}}} (-1)^a P_m\left(\left[\frac{a}{\overline{f}}\right]_{q^{\overline{f}}}\right) \\ &= E_{m,\chi,q} - \chi(p) \frac{[2]_q}{[2]_{q^p}} [p]_q^m E_{m,\chi,q^p}, \end{aligned} \tag{2.15}$$

where $E_{m,\chi,q}$ are the mth generalized q-Euler numbers attached to χ. For $m \in \mathbb{Z}_+$, we have

$$L(\mu_m, \chi w^{-m}) = E_{m,\chi w^{-m},q} - \chi w^{-m}(p)\frac{[2]_q}{[2]_{q^p}}[p]_q^m E_{m,\chi w^{-m},q^p} = l_{p,q}(-m,\chi). \tag{2.16}$$

Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Let w be the Teichmüller character mod p. For $x \in \mathbb{Z}_f^*$, we set $\langle x\rangle_q = [x]_q/w(x)$. Note that $|\langle x\rangle_q - 1|_p < p^{-1/(p-1)}$ and $\langle x\rangle_q^s$ are defined by $\exp(s\log_p\langle x\rangle_q)$ for $|s|_p \leq 1$. For $s \in \mathbb{Z}_p$, we define

$$l_{p,q}(s,x) = \int_{\mathbb{Z}_f^*}\langle x\rangle_q^{-s}\chi(x)d\mu_q(x). \tag{2.17}$$

For (2.14), (2.16) and (2.17), we note that

$$l_{p,q}\left(-k,\chi w^k\right) = \int_{\mathbb{Z}_f^*}[x]_q^k\chi(x)d\mu_q(x) = \int_{\mathbb{Z}_f^*}\chi(x)d\mu_k(x). \tag{2.18}$$

Since $|\langle x\rangle_q - 1|_p < p^{-1/(p-1)}$ for $x \in \mathbb{Z}_f^*$, we have $\langle x\rangle_q^{p^n} \equiv 1\ (\bmod\ p^n)$. Let $k \equiv k'(\bmod\ p^n(p-1))$. Then we have

$$l_{p,q}\left(-k,\chi w^k\right) \equiv l_{p,q}\left(-k',\chi w^{k'}\right)\left(\bmod\ p^n\right). \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $k \equiv k'(\bmod\ p^n(p-1))$, we have*

$$L(\mu_k,\chi) \equiv L(\mu_{k'},\chi)\left(\bmod\ p^n\right). \tag{2.20}$$

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