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Research Article

On Some New Sequence Spaces in 2-Normed Spaces Using Ideal Convergence and an Orlicz Function

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The purpose of this paper is to introduce certain new sequence spaces using ideal convergence and an Orlicz function in 2-normed spaces and examine some of their properties.

1. Introduction

The notion of ideal convergence was introduced first by Kostyrko et al. [1] as a generalization of statistical convergence which was further studied in topological spaces [2]. More applications of ideals can be seen in [3, 4].

The concept of 2-normed space was initially introduced by Gähler [5] as an interesting nonlinear generalization of a normed linear space which was subsequently studied by many authors (see, [6, 7]). Recently, a lot of activities have started to study summability, sequence spaces and related topics in these nonlinear spaces (see, [8–10]).

Recall in [11] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is continuous, convex, nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [12] and others.

If convexity of Orlicz function, M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called Modulus function, which was presented and discussed by Ruckle [13] and Maddox [14].

Note that if M is an Orlicz function then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let $(X, \|\cdot\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if (i) $\emptyset \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$, [9, 10].

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ is a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$, [1, 3].

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent, (ii) $\|x, y\| = \|y, x\|$, (iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$, and (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space [6].

Recall that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X.

Quite recently Savaş [15] defined some sequence spaces by using Orlicz function and ideals in 2-normed spaces.

In this paper, we continue to study certain new sequence spaces by using Orlicz function and ideals in 2-normed spaces. In this context it should be noted that though sequence spaces have been studied before they have not been studied in nonlinear structures like 2-normed spaces and their ideals were not used.

2. Main Results

Let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 0$ and let I be an admissible ideal of $\mathbb{N}$, let M be an Orlicz function, and let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Further, let $p = (p_k)$ be a bounded sequence of positive real numbers. By $S(2 - X)$ we denote the space of all sequences defined over $(X, \|\cdot,\cdot\|)$. Now, we define the following sequence spaces:

$$\begin{aligned}
&W^I(\lambda, M, p, \|,\cdot,\|) \\
&\quad = \left\{ x \in S(2-X) : \forall\varepsilon > 0 \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\| \frac{x_k - L}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \right. \\
&\qquad\qquad \left. \text{for some } \rho > 0,\ L \in X \text{ and each } z \in X \right\}, \\
&W_0^I(\lambda, M, p, \|,\cdot,\|) \\
&\quad = \left\{ x \in S(2-X) : \forall\varepsilon > 0 \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\| \frac{x_k}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \right. \\
&\qquad\qquad \left. \text{for some } \rho > 0, \text{ and each } z \in X \right\},
\end{aligned}$$

$$W_{\infty}(\lambda, M, p, \|, \cdot, \|)$$
$$= \left\{ x \in S(2-X) : \exists K > 0 \text{ s.t. } \sup_{n\in\mathbb{N}} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\| \frac{x_k}{\rho}, z \right\| \right) \right]^{p_k} \le K \right.$$
$$\left. \text{for some } \rho > 0, \text{ and each } z \in X \right\},$$
$$W_{\infty}^{I}(\lambda, M, p, \|, \cdot, \|)$$
$$= \left\{ x \in S(2-X) : \exists K > 0 \text{ s.t.} \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\| \frac{x_k}{\rho}, z \right\| \right) \right]^{p_k} \ge K \right\} \in I \right.$$
$$\left. \text{for some } \rho > 0, \text{ and each } z \in X \right\}, \tag{2.1}$$

where $I_n = [n - \lambda_n + 1, n]$.

The following well-known inequality [16, page 190] will be used in the study.

$$\text{If} \quad 0 \le p_k \le \sup p_k = H, \qquad D = \max\left(1, 2^{H-1}\right) \tag{2.2}$$

then

$$|a_k + b_k|^{p_k} \le D\left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\} \tag{2.3}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

Theorem 2.1. *$W^I(\lambda, M, p, \|, \cdot, \|)$, $W_0^I(\lambda, M, p, \|, \cdot, \|)$, and $W_{\infty}^I(\lambda, M, p, \|, \cdot, \|)$ are linear spaces.*

Proof. We will prove the assertion for $W_0^I(\lambda, M, p, \|, \cdot, \|)$ only and the others can be proved similarly. Assume that $x, y \in W_0^I(\lambda, M, \|, \cdot, \|)$ and $\alpha, \beta \in \mathbb{R}$, so

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\| \frac{x_k}{\rho_1}, z \right\| \right) \right]^{p_k} \ge \varepsilon \right\} \in I \quad \text{for some } \rho_1 > 0,$$
$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_r} \left[M\left(\left\| \frac{x_k}{\rho_2}, z \right\| \right) \right]^{p_k} \ge \varepsilon \right\} \in I \quad \text{for some } \rho_2 > 0. \tag{2.4}$$

Since $\|,\cdot,\|$ is a 2-norm, and M is an Orlicz function the following inequality holds:

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{(\alpha x_k+\beta y_k)}{(|\alpha|\rho_1+|\beta|\rho_2)},z\right\|\right)\right]^{p_k} \\
&\quad\le D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k} \\
&\qquad+D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k} \\
&\quad\le DF\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k}+DF\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k},
\end{aligned} \tag{2.5}$$

where

$$F=\max\left[1,\left(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H,\left(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H\right]. \tag{2.6}$$

From the above inequality, we get

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{(\alpha x_k+\beta y_k)}{(|\alpha|\rho_1+|\beta|\rho_2)},z\right\|\right)\right]^{p_k}\ge\varepsilon\right\} \\
&\quad\subseteq\left\{n\in\mathbb{N}:DF\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k}\ge\frac{\varepsilon}{2}\right\} \\
&\quad\cup\left\{n\in\mathbb{N}:DF\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k}\ge\frac{\varepsilon}{2}\right\}.
\end{aligned} \tag{2.7}$$

Two sets on the right hand side belong to I and this completes the proof. □

It is also easy to see that the space $W_\infty(\lambda,M,p,\|,\cdot,\|)$ is also a linear space and we now have the following.

Theorem 2.2. *For any fixed $n\in\mathbb{N}$, $W_\infty(\lambda,M,p,\|,\cdot,\|)$ is paranormed space with respect to the paranorm defined by*

$$g_n(x)=\inf\left\{\rho^{p_n/H}:\rho>0\ s.t.\ \left(\sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{x_k}{\rho},z\right\|\right)\right]^{p_k}\right)^{1/H}\le 1,\ \forall z\in X\right\}. \tag{2.8}$$

Proof. That $g_n(\theta)=0$ and $g_n(-x)=g(x)$ are easy to prove. So we omit them.

(iii) Let us take $x = (x_k)$ and $y = (y_k)$ in $W_\infty(\lambda, M, p, \|,\cdot,\|)$. Let

$$A(x) = \left\{\rho > 0 : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \le 1,\ \forall z \in X\right\},$$
$$A(y) = \left\{\rho > 0 : \sup_n \frac{1}{\lambda_\mathrm{n}} \sum_{k\in I_n} \left[M\left(\left\|\frac{y_k}{\rho}, z\right\|\right)\right]^{p_k} \le 1,\ \forall z \in X\right\}. \tag{2.9}$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$, then if $\rho = \rho_1 + \rho_2$, then, we have

$$\begin{aligned}\sup_n \frac{1}{\lambda_n} \sum_{n\in I_n} M\left(\left\|\frac{(x_k + y_k)}{\rho}, z\right\|\right) &\le \frac{\rho_1}{\rho_1+\rho_2} \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\left\|\frac{x_k}{\rho_1}, z\right\|\right) \\ &\quad + \frac{\rho_2}{\rho_1+\rho_2} \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\left\|\frac{y_k}{\rho_2}, z\right\|\right).\end{aligned} \tag{2.10}$$

Thus, $\sup_n (1/\lambda_n) \sum_{n\in I_n} M(\|(x_k + y_k)/(\rho_1 + \rho_2), z\|)^{p_k} \le 1$ and

$$\begin{aligned} g_n(x+y) &\le \inf\left\{(\rho_1+\rho_2)^{p_n/H} : \rho_1 \in A(x), \rho_2 \in A(y)\right\} \\ &\le \inf\left\{\rho_1^{p_n/H} : \rho_1 \in A(x)\right\} + \inf\left\{\rho_2^{p_n/H} : \rho_2 \in A(y)\right\} \\ &= g_n(x) + g_n(y). \end{aligned} \tag{2.11}$$

(iv) Finally using the same technique of Theorem 2 of Savaş [15] it can be easily seen that scalar multiplication is continuous. This completes the proof. □

Corollary 2.3. *It should be noted that for a fixed $F \in I$ the space*

$$\begin{aligned} &W_\infty(F)(\lambda, M, p, \|,\cdot,\|) \\ &\quad = \left\{x \in S(2-X) : \exists K > 0\ s.t.\ \sup_{n\in\mathbb{N}-F} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \le K \right. \\ &\qquad\qquad \left. \textit{for some } \rho > 0, \textit{ and each } z \in X\right\}, \end{aligned} \tag{2.12}$$

which is a subspace of the space $W_\infty^I(\lambda, M, p, \|,\cdot,\|)$ is a paranormed space with the paranorms g_n for $n \notin F$ and $g^F = \inf_{n\in(\mathbb{N}-F)} g_n$.

Theorem 2.4. *Let M, M_1, M_2, be Orlicz functions. Then we have*

(i) $W_0^I(\lambda, M_1, p, \|,\cdot,\|) \subseteq W_0^I(\lambda, M \circ M_1, p, \|,\cdot,\|)$ *provided (p_k) is such that $H_0 = \inf p_k > 0$.*

(ii) $W_0^I(\lambda,\ M_1, p,\ \|,\cdot,\|) \cap W_0^I(\lambda, M_2, p, \|,\cdot,\|) \subseteq W_0^I(\lambda,\ M_1 + M_2,\ p,\ \|,\cdot,\|)$.

Proof. (i) For given $\varepsilon > 0$, first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of M choose $0 < \delta < 1$ such that $0 < t < \delta \Rightarrow M(t) < \varepsilon_0$. Let $(x_k) \in W_0(\lambda, M_1, p, \|,\cdot,\|)$. Now from the definition

$$A(\delta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{n\in I_n}\left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \geq \delta^H\right\} \in I. \tag{2.13}$$

Thus if $n \notin A(\delta)$ then

$$\frac{1}{\lambda_n}\sum_{n\in I_n}\left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < \delta^H, \tag{2.14}$$

that is,

$$\sum_{n\in I_n}\left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < \lambda_n\delta^H, \tag{2.15}$$

that is,

$$\left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < \delta^H, \quad \forall k \in I_n, \tag{2.16}$$

that is,

$$M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right) < \delta, \quad \forall k \in I_n. \tag{2.17}$$

Hence from above using the continuity of M we must have

$$M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right) < \varepsilon_0, \quad \forall k \in I_n, \tag{2.18}$$

which consequently implies that

$$\sum_{k\in I_n}\left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} < \lambda_n \max\left\{\varepsilon_0^H, \varepsilon_0^{H_0}\right\} < \lambda_n\varepsilon, \tag{2.19}$$

that is,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} < \varepsilon. \tag{2.20}$$

This shows that

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} \geq \varepsilon\right\} \subset A(\delta) \tag{2.21}$$

and so belongs to I. This proves the result.

(ii) Let $(x_k) \in W_0^I(M_1, p, \|, \cdot, \|) \cap W_0^I(M_2, p, \|, \cdot, \|)$, then the fact

$$\frac{1}{\lambda_n}\left[M_1 + M_2\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \leq D\frac{1}{\lambda_n}\left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} + D\frac{1}{\lambda_n}\left[M_2\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \tag{2.22}$$

gives us the result. □

Definition 2.5. Let X be a sequence space. Then X is called solid if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

Theorem 2.6. *The sequence spaces $W_0^I(\lambda, M, p, \|, \cdot, \|)$, $W_\infty^I(\lambda, M, p, \|, \cdot, \|)$ are solid.*

Proof. We give the proof for $W_0^I(\lambda, M, p, \|, \cdot, \|)$ only. Let $(x_k) \in W_0^I(\lambda, M, p, \|, \cdot, \|)$ and let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\left\|\frac{(\alpha_k x_k)}{\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{C}{\lambda_n} \sum_{k \in I_n} \left[M\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \tag{2.23}$$

where $C = \max_k\{1, |\alpha_k|^H\}$. Hence $(\alpha_k x_k) \in W_0^I(\lambda, M, p, \|, \cdot, \|)$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$ whenever $(x_k) \in W_0^I(\lambda, M, p, \|, \cdot, \|)$. □

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