Research Article

Hyers-Ulam Stability of a Bi-Jensen Functional Equation on a Punctured Domain

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We obtain the Hyers-Ulam stability of a bi-Jensen functional equation: $2f((x+y)/2, z) - f(x, z) - f(y, z) = 0$ and simultaneously $2f(x,(y+z)/2) - f(x, y) - f(x, z) = 0$. And we get its stability on the punctured domain.

1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms: let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

(1.1)

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

(1.2)

for all $x \in G_1$? The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1949, 1950, and 1978, Bourgin [3], Aoki [4], and Rassias [5] gave a generalization of it under the conditions bounded by variables. Since then, the further generalization has been extensively investigated by a number of mathematicians, such as Găvruta, Rassias, and so forth, [6–25].
Throughout this paper, let $X$ be a normed space and $Y$ a Banach space. A mapping $g : X \to Y$ is called a Jensen mapping if $g$ satisfies the functional equation $2g((x + y)/2) = g(x) + g(y)$. For a given mapping $f : X \times X \to Y$, we define

$$J_1 f(x, y, z) := 2f\left(\frac{x + y}{2}, z\right) - f(x, z) - f(y, z),$$

$$J_2 f(x, y, z) := 2f\left(x, \frac{y + z}{2}\right) - f(x, y) - f(x, z)$$

for all $x, y, z \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if $f$ satisfies the functional equations $J_1 f = 0$ and $J_2 f = 0$.

In 2006, Bae and Park [26] obtained the generalized Hyers-Ulam stability of a bi-Jensen mapping. The following result is a special case of Theorem 6 in [26].

**Theorem A.** Let $\varepsilon > 0$ and let $f : X \times X \to Y$ be a mapping such that

$$\|J_1 f(x, y, z)\| \leq \varepsilon,$$

$$\|J_2 f(x, y, z)\| \leq \varepsilon$$

for all $x, y, z \in X$. Then there exist two bi-Jensen mappings $F, F_0 : X \times X \to Y$ such that

$$\|f(x, y) - f(0, y) - F(x, y)\| \leq \varepsilon,$$

$$\|f(x, y) - f(x, 0) - F_0(x, y)\| \leq \varepsilon$$

for all $x, y \in X$.

In Theorem A, they did not show that there exist a $k \in \mathbb{R}$ and a unique bi-Jensen mapping $F : X \times X \to Y$ such that $\|f(x, y) - F(x, y)\| \leq k\varepsilon$ for all $x, y \in X$. In 2008, Jun et al. [7, 8] improved Bae and Park’s results.

In Section 2, we show that there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that $\|f(x, y) - F(x, y)\| \leq 4\varepsilon$ for all $x, y \in X$. In Section 3, we investigate the Hyers-Ulam stability of a bi-Jensen functional equation on the punctured domain.

## 2. Stability of a Bi-Jensen Functional Equation

From Lemma 1 in [8], we get the following lemma.

**Lemma 2.1.** Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then

$$f(x, y) = \frac{1}{4^n} f(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right) (f(2^n x, 0) + f(0, 2^n y)) + \left(1 - \frac{1}{2^n}\right)^2 f(0, 0)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we will give the Hyers-Ulam stability for a bi-Jensen mapping.
Theorem 2.2. Let \( \varepsilon > 0 \) and let \( f : X \times X \to Y \) be a mapping satisfying (1.4) for all \( x, y, z \in X \). Then there exists a unique bi-Jensen mapping \( F : X \times X \to Y \) such that

\[
\|f(x, y) - F(x, y)\| \leq 4\varepsilon \tag{2.2}
\]

for all \( x, y \in X \) with \( F(0, 0) = f(0, 0) \). In particular, the mapping \( F : X \times X \to Y \) is given by

\[
F(x, y) := \lim_{j \to \infty} \left[ \frac{1}{4^j} f\left(2^j x, 2^j y\right) + \left(\frac{1}{2^j} - \frac{1}{4^j}\right) \left(f\left(2^j x, 0\right) + f\left(0, 2^j y\right)\right) \right] + f(0, 0) \tag{2.3}
\]

for all \( x, y \in X \).

Proof. Let \( f_j \) be the map defined by

\[
f_j(x, y) = \frac{f\left(2^j x, 2^j y\right)}{4^j} + \left(\frac{1}{2^j} - \frac{1}{4^j}\right) \left(f\left(2^j x, 0\right) + f\left(0, 2^j y\right)\right) + \left(1 - \frac{1}{2^{j+1}} + \frac{1}{4^j}\right)f(0, 0) \tag{2.4}
\]

for all \( x, y \in X \) and \( j \in \mathbb{N} \). By (1.4), we get

\[
\|f_j(x, y) - f_{j+1}(x, y)\| = \left\| \frac{f_1(2^{j+1} x, 0, 0)}{2^{j+1}} + \frac{f_1(2^{j+1} x, 0, 2^{j+1} y)}{2 \cdot 4^{j+1}} + \frac{f_1(2^{j+1} x, 0, 2^{j+1} y)}{4^{j+1}} - \frac{3 f_1(2^{j+1} x, 0, 0)}{2 \cdot 4^{j+1}} - \frac{f_2(0, 0, 2^{j+1} y)}{2^{j+1}} + \frac{f_2(2^{j+1} x, 0, 2^{j+1} y)}{2 \cdot 4^{j+1}} \right\|
\]

\[
\leq \left(\frac{1}{2^j} + \frac{3}{2 \cdot 4^j}\right)\varepsilon
\]

for all \( x, y \in X \) and \( j \in \mathbb{N} \). For given integers \( l, m \) with \( 0 \leq l < m \), we obtain

\[
\|f_l(x, y) - f_m(x, y)\| \leq \sum_{j=l}^{m-1} \left(\frac{1}{2^j} + \frac{3}{2 \cdot 4^j}\right)\varepsilon \tag{2.6}
\]

for all \( x, y \in X \). By the above inequality, the sequence \( \{f_j(x, y)\} \) is a Cauchy sequence for all \( x, y \in X \). Since \( Y \) is complete, the sequence \( \{f_j(x, y)\} \) converges for all \( x, y \in X \). Define \( F : X \times X \to Y \) by

\[
F(x, y) := \lim_{j \to \infty} f_j(x, y) \tag{2.7}
\]
for all \( x, y \in X \). Putting \( l = 0 \) and taking \( m \to \infty \) in (2.6), we obtain the inequality

\[
\|f(x, y) - F(x, y)\| \leq 4\varepsilon
\]  
(2.8)

for all \( x, y \in X \). By (1.4) and the definition of \( F \), we get

\[
\begin{align*}
J_1F(x, y, z) &= \lim_{j \to \infty} \frac{1}{4^j} J_1 f(2^j x, 2^j y) + \left( \frac{1}{2^j} - \frac{1}{4^j} \right) \left( J_1 f(2^j x, 0) + J_1 f(0, 2^j y) \right) = 0, \\
J_2F_3(x, y, z) &= \lim_{j \to \infty} \frac{1}{4^j} J_2 f(2^j x, 2^j y) + \left( \frac{1}{2^j} - \frac{1}{4^j} \right) \left( J_2 f(2^j x, 0) + J_2 f(0, 2^j y) \right) = 0
\end{align*}
\]  
(2.9)

for all \( x, y, z \in X \). So \( F \) is a bi-Jensen mapping satisfying (2.2). Now, let \( F' : X \times X \to Y \) be another bi-Jensen mapping satisfying (2.2) with \( F'(0, 0) = f(0, 0) \). By Lemma 2.1, we have

\[
\|F(x, y) - F'(x, y)\| \leq \left( \frac{1}{2^n} + \frac{1}{4^{n-1}} \right) \varepsilon
\]  
(2.10)

for all \( x, y \in X \) and \( n \in \mathbb{N} \). As \( n \to \infty \), we may conclude that \( F(x, y) = F'(x, y) \) for all \( x, y \in X \). Thus the bi-Jensen mapping \( F : X \times X \to Y \) is unique. \( \square \)

**Example 2.3.** Let \( f, F, F' : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the bi-Jensen mappings defined by

\[
f(x, y) := 0, \quad F(x, y) := \varepsilon, \quad F'(x, y) := -\varepsilon
\]  
(2.11)

for all \( x, y \in \mathbb{R} \). Then \( f, F, F' \) satisfy (1.4) for all \( x, y, z \in \mathbb{R} \). In addition, \( f, F \) satisfy (2.2) for all \( x, y \in \mathbb{R} \) and \( f, F' \) also satisfy (2.2) for all \( x, y \in \mathbb{R} \). But we get \( F \neq F' \). Hence the condition \( F(0, 0) = f(0, 0) \) is necessary to show that the mapping \( F \) is unique.

Let $A$ be a subset of $X$. $X \setminus A$ and $(X \times X) \setminus (A \times A)$ are punctured domain on the spaces $X$ and $(X \times X)$, respectively.

Throughout this paper, for a given mapping $f : X \times X \to Y$, let $f_1, A_1, A_2 : X \times X \to Y$ be the mappings defined by

$$f_1(x, y) := \frac{f(x, y) - f(-x, y) - f(x, -y) + f(-x, -y)}{4},$$

$$A_1(x, y) := \sum_{m=0}^{n} \sum_{n=0}^{1} (-1)^{m+1} f((-1)^m x, (-1)^n \cdot 3x, y),$$

$$A_2(x, y) := \sum_{m=0}^{n} \sum_{n=0}^{1} (-1)^{m+1} f(x, (-1)^m \cdot 3y, (-1)^n y)$$

for all $x, y \in X$.

**Lemma 3.1.** Let $A$ be a subset of $X$ satisfying the following condition: for every $x \neq 0$, there exists a positive integer $n_x$ such that $kx \notin A$ for all integer $k$ with $|k| \geq n_x$, and such that $kx \in A$ for all integer $k$ with $|k| < n_x$. Let $f : X \times X \to Y$ be a mapping such that

$$J_1 f(x, y, z) = 0, \quad J_2 f(x, y, z) = 0$$

for all $x, y, z \in X \setminus A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$F(x, y) = f(x, y)$$

for all $x, y \in X \setminus A$. Moreover, the equality

$$F(x, y) = f(x, y)$$

holds for all $(x, y) \in (X \times X) \setminus (A \times A)$.

**Proof.** Note that $J_1 f(x, y, z) = 0, J_2 f(x, y, z) = 0, A_1(x, y) = 0$, and $A_2(x, y) = 0$ for all $x, y \in X \setminus A$. Let $((f(0, y) + f(0, -y))/2) = c \in Y$ for any $y \in X \setminus A$. From (3.2), we get the equality

$$f(0, y) + f(0, -y) = f(x, 0) + f(-x, 0) + \frac{1}{2} (J_1 f(x, -x, y) + J_1 f(x, -x, -y) - J_2 f(x, y, -y))$$

for all $x, y \in X \setminus A$, and we know that the equality

$$\frac{f(0, y) + f(0, -y)}{2} = \frac{f(x, 0) + f(-x, 0)}{2} = c$$

(3.6)
holds for all \( x, y \in X \setminus A \). From (3.2), we have

\[
f_1(x, y) = \frac{f_1(2x, y) + A_1(x, y)}{2} - \frac{A_1(x, -y)}{16},
\]

\[
f_1(2x, y) = \frac{f_1(2x, 2y) + A_2(2x, y)}{4} - \frac{A_2(-2x, y)}{32},
\]

\[
f(x, y) = \frac{f(0, y) - f(0, -y)}{2} + \frac{f(0, y) + f(0, -y)}{2} + \frac{f(x, 0) - f(-x, 0)}{2}
\]

\[+ f_1(x, y) - \frac{1}{4}(2J_1 f(x, -x, y) + J_2 f(x, y, -y) - J_2 f(-x, y, -y)),
\]

\[
f(x, 0) - f(-x, 0) = \frac{f(2x, 0) - f(-2x, 0)}{2}
\]

\[+ \frac{1}{8}(4J_2 f(x, y, -y) - 4J_1 f(-x, y, -y) - 2J_1 f(2x, y, -y)
\]

\[+ 2J_1 f(-2x, y, -y) + A_1(x, y) + A_1(x, -y)),
\]

\[
f(0, y) - f(0, -y) = \frac{f(0, 2y) - f(0, -2y)}{2}
\]

\[+ \frac{1}{8}(4J_1 f(x, -x, y) - 4J_1 f(x, -x, -y) - 2J_1 f(x, -x, 2y)
\]

\[+ 2J_1 f(-x, -x, -2y) + A_2(x, y) + A_2(-x, y))
\]

for all \( x, y \in X \setminus A \). From the above equalities, we obtain the equalities

\[
f_1(x, y) = \frac{f_1(2x, y)}{2},
\]

\[
f(x, 0) - f(-x, 0) = \frac{f(2x, 0) - f(-2x, 0)}{2},
\]

\[
f(0, y) - f(0, -y) = \frac{f(0, 2y) - f(0, -2y)}{2},
\]

\[
f_1(x, y) = \frac{f_1(2^n x, 2^n y)}{4^n},
\]

\[
f(x, 0) = \frac{f(x, 0) - f(-x, 0)}{2} + \frac{f(x, 0) + f(-x, 0)}{2} = \frac{f(2^n x, 0) - f(-2^n x, 0)}{2^{n+1}} + c,
\]

\[
f(0, y) = \frac{f(0, y) - f(0, -y)}{2} + \frac{f(0, y) + f(0, -y)}{2} = \frac{f(0, 2^n y) - f(0, -2^n y)}{2^{n+1}} + c,
\]

\[
f(x, y) = \frac{f_1(2^n x, 2^n y)}{4^n} + \frac{f(0, 2^n y) - f(0, -2^n y) + f(2^n x, 0) - f(-2^n x, 0)}{2^{n+1}} + c
\]

for all \( x, y \in X \setminus A \) and \( n \in \mathbb{N} \).
Let $A_x$ be the set defined by $A_x = \{ n \in \mathbb{N} \mid nx \notin A \}$ for each $x \neq 0$. From the above equalities, we can define $F : X \times X \to Y$ by

$$F(x, y) := \begin{cases} \frac{f_1(2^k x, 2^k y)}{4^k} + \frac{f(0, 2^k y) - f(0, -2^k y) + f(2^k x, 0) - f(-2^k x, 0)}{2^{k+1}} + c, & \text{for some } 2^k \in A_x \cap A_y \text{ if } x, y \neq 0, \\ \frac{f(2^k x, 0) - f(-2^k x, 0)}{2^{k+1}} + c, & \text{for some } 2^k \in A_x \text{ if } x \neq 0, y = 0, \\ \frac{f(0, 2^k y) - f(0, -2^k y)}{2^{k+1}} + c & \text{for some } 2^k \in A_y \text{ if } x = 0, y \neq 0, \\ c & \text{if } x, y = 0. \end{cases}$$ (3.12)

From the definition of $F$, we get the equalities

$$F(x, y) = f(x, y), \quad F(0, y) = f(0, y), \quad F(x, 0) = f(x, 0)$$ (3.13)

for all $x, y \in X \setminus A$. By (3.10), we get the equality

$$f(x, y) - F(x, y) = \frac{1}{2} \left[ J_2 f \left( x, (2^k + 2) y, -2^k y \right) - J_2 F \left( x, (2^k + 2) y, -2^k y \right) \right] = 0$$ (3.14)

for all $x \in X \setminus A$ and $y \neq 0$, where $2^k \in A_y$. And also we get the equality

$$f(x, y) - F(x, y) = \frac{1}{2} \left[ J_1 f \left( (2^k + 2) x, -2^k x, y \right) - J_1 F \left( (2^k + 2) x, -2^k x, y \right) \right] = 0$$ (3.15)

for all $x \neq 0$ and $y \in X \setminus A$, where $2^k \in A_x$. Hence the equality

$$f(x, y) = F(x, y)$$ (3.16)

holds for all $(x, y) \in (X \times X) \setminus (A \times A)$. From (3.8), (3.9), (3.10), and the definition of $F$, we easily get

$$J_1 F(x, -x, y) = 0, \quad J_1 F(x, -x, 0) = 0, \quad J_1 F(0, 0, y) = 0,$$

$$J_1 F(x, 0, y) = 0, \quad J_1 F(x, 0, 0) = 0, \quad J_1 F(0, 0, 0) = 0$$ (3.17)
for all \( x, y \neq 0 \). And we obtain
\[
J_1F(x, y, 0) = \frac{J_2f(2^{k-1}(x + y), z, -z) - J_2f(-2^{k-1}(x + y), z, -z)}{2^{k+1}} \\
+ \frac{-J_2f(2^k x, z, -z) + J_2f(-2^k x, z, -z)}{2^{k+2}} \\
+ \frac{J_1f(-2^k y, z, -z) + J_1f(2^k x, 2^k y, z) - J_1f(-2^k x, -2^k y, z)}{2^{k+2}} \\
+ \frac{J_1f(2^k x, 2^k y, -z) - J_1f(-2^k x, -2^k y, -z)}{2^{k+2}} = 0
\] (3.18)

for all \( x, y \neq 0 \) with \( x + y \neq 0 \), where \( 2^k \in A_x \cap A_y \cap A_{x+y} \) and \( z \not\in A \). From this, we have
\[
J_1F(x, y, z) = \frac{J_1f(2^k x, 2^k y, 2^k z) - J_1f(-2^k x, -2^k y, 2^k z)}{4^{k+1}} \\
+ \frac{J_1f(-2^k x, -2^k y, -2^k z) - J_1f(2^k x, 2^k y, -2^k z)}{4^{k+1}} + J_1F(x, y, 0) = 0
\] (3.19)

for all \( x, y, z \neq 0 \) with \( x + y \neq 0 \), where \( 2^k \in A_x \cap A_y \cap A_z \). From the above equalities, we get
\[
J_1F(x, y, z) = 0
\] (3.20)

for all \( x, y, z \in X \). By the similar method, we have
\[
J_2F(x, y, z) = 0
\] (3.21)

for all \( x, y, z \in X \). Hence \( F \) is a bi-Jensen mapping. Let \( F' \) be another bi-Jensen mapping satisfying
\[
F'(x, y) = f(x, y) = F(x, y)
\] (3.22)

for all \((x, y) \in (X \times X) \setminus (A \times A)\). Using the above equality, we show that the equalities
\[
F'(x, y) - F(x, y) = \frac{1}{2} (J_1F((k + 2)x, -kx, y) - J_1F((k + 2)x, -kx, y)) = 0,
\] (3.23)

\[
F'(0, y) - F(0, y) = \frac{1}{2} (J_1F(kx, -kx, y) - J_1F(kx, -kx, y)) = 0
\]

hold for all \( x \neq 0 \) and \( y \in X \) as we desired, where \( k \in A_x \). \( \square \)
Corollary 3.2. Let \( f : X \times X \rightarrow Y \) be a mapping such that

\[
J_1 f(x, y, z) = 0, \quad J_2 f(x, y, z) = 0
\]

for all \( x, y, z \in X \setminus \{0\} \). Then there exists a unique bi-Jensen mapping \( F : X \times X \rightarrow Y \) such that

\[
F(x, y) = f(x, y)
\]

for all \( (x, y) \neq (0, 0) \).

Example 3.3. Let \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be the mapping defined by

\[
f(x, y) := \begin{cases} 
(x + 3)(y + 4) & \text{for } (x, y) \neq (0, 0), \\
1 & \text{for } (x, y) = (0, 0),
\end{cases}
\]

and let \( F \) be the mapping defined by \( F(x, y) := (x + 3)(y + 4) \) for all \( x, y \in \mathbb{R} \). Then the mappings \( f, F \) satisfy the conditions of Corollary 3.2 with \( f(0, 0) \neq F(0, 0) \).

Now, we prove the Hyers-Ulam stability of a bi-Jensen functional equation on the punctured domain \( X \setminus A \).

Theorem 3.4. Let \( \varepsilon > 0 \) and \( x_0 \in X \setminus A \). Let \( f : X \times X \rightarrow Y \) be a mapping such that

\[
\| J_1 f(x, y, z) \| \leq \varepsilon, \quad \| J_2 f(x, y, z) \| \leq \varepsilon
\]

for all \( x, y, z \in X \setminus A \). Then there exists a unique bi-Jensen mapping \( F : X \times X \rightarrow Y \) such that

\[
\| f(x, y) - F(x, y) \| \leq \frac{17}{2} \varepsilon
\]

holds for all \( (x, y) \in (X \times X) \setminus (A \times A) \) with \( F(0, 0) = (f(x_0, 0) + f(-x_0, 0))/2 \). The mapping \( F : X \times X \rightarrow Y \) is given by

\[
F(x, y) := \lim_{j \to \infty} \left( f(x, y) + \frac{f(0, y) + f(0, 0)}{2^{j+1}} \right) + \frac{f(x_0, 0) + f(-x_0, 0)}{2}
\]

for all \( x, y \in X \).
Proof. By (3.27), we get

\[
\left\|\frac{f_1(2^i x, 2^i y)}{4^i} - \frac{f_1(2^{i+1} x, 2^{i+1} y)}{4^{i+1}}\right\| = \frac{1}{4^{i+2}} \left\| A_1(2^i x, 2^i y) - A_1(2^i x, -2^i y) + \frac{1}{2} A_2(2^{i+1} x, 2^i y) - \frac{1}{2} A_2(-2^{i+1} x, 2^i y)\right\| \leq \frac{3\varepsilon}{4^{i+1}},
\]

\[
\left\|\frac{f(0, 2^i y) - f(0, -2^i y)}{2^{i+1}} - \frac{f(0, 2^{i+1} y) - f(0, -2^{i+1} y)}{2^{i+2}}\right\| = \frac{1}{2^{i+4}} \left\| 4 f_1(0, -2^i y) - 4 f_1(0, -2^{i+1} y) - 2 f_1(x, -x, 2^{i+1} y) + 2 f_1(x, -x, 2^{i+1} y) + A_2(x, 2^i y) + A_2(-x, 2^i y)\right\| \leq \frac{5\varepsilon}{2^{i+2}},
\]

\[
\left\|\frac{f(0, y) + f(0, -y)}{2} - \frac{f(x, 0) + f(-x, 0)}{2}\right\| = \frac{1}{4} \left\| J_1 f(x, -x, y) + J_1 f(x, -x, -y) - J_2 f(x, y, -y) - J_2 f(-x, y, -y)\right\| \leq \varepsilon
\]

(3.30)

for all \(x, y \in X \setminus A\) and \(j \in \mathbb{N}\). For given integers \(l, m (0 \leq l < m)\), we have

\[
\left\|\frac{f_1(2^l x, 2^l y)}{4^l} - \frac{f_1(2^m x, 2^m y)}{4^m}\right\| \leq \sum_{j=l}^{m-1} \frac{3\varepsilon}{4^{j+1}},
\]

(3.31)

\[
\left\|\frac{f(0, 2^l y) - f(0, -2^l y)}{2^{l+1}} - \frac{f(0, 2^m y) - f(0, -2^m y)}{2^{m+1}}\right\| \leq \sum_{j=l}^{m-1} \frac{5\varepsilon}{2^{j+2}},
\]

(3.32)

\[
\left\|\frac{f(2^l x, 0) - f(-2^l x, 0)}{2^{l+1}} - \frac{f(2^m x, 0) - f(-2^m x, 0)}{2^{m+1}}\right\| \leq \sum_{j=l}^{m-1} \frac{5\varepsilon}{2^{j+2}},
\]

(3.33)

\[
\left\|\frac{f(x, 0) + f(-x, 0)}{2} - \frac{f(0, 2^m y) + f(0, -2^m y)}{2}\right\| \leq \varepsilon,
\]

(3.34)

\[
\left\|\frac{f(0, y) + f(0, -y)}{2} - \frac{f(2^m x, 0) + f(-2^m x, 0)}{2}\right\| \leq \varepsilon
\]

(3.35)

for all \(x, y \in X \setminus A\). The sequences \(\{(f_1(2^l x, 2^l y))/4^l\}, \{(f(0, 2^l y) - f(0, -2^l y))/2^{l+1}\}, \) and \(\{(f(2^l x, 0) - f(-2^l x, 0))/2^{l+1}\}\) are Cauchy sequences for all \(x, y \in X \setminus A\). Since \(Y\) is complete, the above sequences converge for all \(x, y \in X \setminus A\). From (3.34) and (3.35), we have

\[
\lim_{j \to \infty} \frac{f(0, 2^j y) + f(0, -2^j y)}{2^{j+1}} = \lim_{j \to \infty} \frac{f(2^j x, 0) + f(-2^j x, 0)}{2^{j+1}} = 0
\]

(3.36)
for all \( x, y \in X \). Using the inequalities (3.31)–(3.35) and the above equality, we can define the mappings \( F_1, F_2, F_3 : X \times X \to Y \) by

\[
F_1(x, y) := \lim_{j \to \infty} \frac{f(2^j x, 2^j y)}{2^j},
\]

\[
F_2(x, y) := \lim_{j \to \infty} \frac{f(0, 2^j y)}{2^j} = \lim_{j \to \infty} \frac{f(0, 2^j y) - f(0, -2^j y)}{2^{j+1}},
\]

\[
F_3(x, y) := \lim_{j \to \infty} \frac{f(2^j x, 0)}{2^j} = \lim_{j \to \infty} \frac{f(2^j x, 0) - f(-2^j x, 0)}{2^{j+1}}.
\]

for all \( x, y \in X \). By (3.27) and the definition of \( F_1 \), we obtain

\[
J_1 F_1(x, y, z) = \lim_{j \to \infty} \left[ \frac{J_1 f(2^j x, 2^j y, 2^j z) - J_1 f(-2^j x, 2^j y, 2^j z)}{4^{j+1}} - \frac{J_1 f(2^j x, 2^j y, -2^j z) - J_1 f(-2^j x, 2^j y, -2^j z)}{4^{j+1}} \right] = 0,
\]

\[
J_2 F_1(x, y, z) = \lim_{j \to \infty} \left[ \frac{J_2 f(2^j x, 2^j y, 2^j z) - J_2 f(-2^j x, 2^j y, 2^j z)}{4^{j+1}} - \frac{J_2 f(2^j x, 2^j y, -2^j z) - J_2 f(-2^j x, 2^j y, -2^j z)}{4^{j+1}} \right] = 0
\]

for all \( x, y, z \neq 0 \). Since \( J_2 F_2(x, y, -y) = 0 \) and

\[
J_2 F_2(x, y, z) = \lim_{j \to \infty} \left( \frac{J_1 f(w, -w, 2^{j-1}(y + z))}{2^j} - \frac{J_1 f(w, -w, 2^j y)}{2^{j+1}} - \frac{J_1 f(w, -w, 2^j z)}{2^{j+1}} + \frac{J_2 f(w, 2^j y, 2^j z)}{2^{j+1}} + \frac{J_2 f(-w, 2^j y, 2^j z)}{2^{j+1}} \right) = 0
\]

for all \( x, y, z \neq 0 \) with \( y + z \neq 0 \), where \( w \notin A \), we have

\[
J_1 F_2(x, y, z) = 0, \quad J_2 F_2(x, y, z) = 0
\]

(3.40)

for all \( x, y, z \neq 0 \). Similarly, the equalities

\[
J_1 F_3(x, y, z) = 0, \quad J_2 F_3(x, y, z) = 0
\]

(3.41)
hold for all $x, y, z \neq 0$. By Lemma 3.1, there exist bi-Jensen mappings $F_1', F_2', F_3' : X \times X \to Y$ such that

$$f_1'(x, y) = F_1(x, y), \quad f_2'(x, y) = F_2(x, y), \quad f_3'(x, y) = F_3(x, y)$$

(3.42)

for all $(x, y) \neq (0, 0)$. Since the equalities

$$F'_1(0, 0) = \frac{F'_1(x, 0) + F'_1(-x, 0)}{2} = \frac{F_1(x, 0) + F_1(-x, 0)}{2} = F_1(0, 0),$$

$$F'_2(0, 0) = \frac{F'_2(x, 0) + F'_2(-x, 0)}{2} = \frac{F_2(x, 0) + F_2(-x, 0)}{2} = F_2(0, 0),$$

$$F'_3(0, 0) = \frac{F'_3(x, 0) + F'_3(-x, 0)}{2} = \frac{F_3(x, 0) + F_3(-x, 0)}{2} = F_3(0, 0)$$

(3.43)

hold, $F_1, F_2, F_3$ are bi-Jensen mappings. Putting $l = 0$ and taking $m \to \infty$ in (3.31), (3.32), and (3.33), one can obtain the inequalities

$$\left\| f_1(x, y) - F_1(x, y) \right\| \leq \varepsilon, \quad \left\| \frac{1}{2} (f(0, y) - f(0, -y)) - F_2(x, y) \right\| \leq \frac{5\varepsilon}{2},$$

$$\left\| \frac{1}{2} (f(x, 0) - f(-x, 0)) - F_3(x, y) \right\| \leq \frac{5\varepsilon}{2}$$

(3.44)

for all $x, y \in X \setminus A$. By (3.30) and the above equalities, we get

$$\left\| f(x, y) - F(x, y) \right\| \leq \left\| f(x, y) - f_1(x, y) - f(0, y) - \frac{f(x, 0) - f(-x, 0)}{2} \right\|$$

$$+ \left\| \frac{f(0, y) + f(0, -y)}{2} - \frac{f(x_0, 0) + f(-x_0, 0)}{2} \right\| + \left\| f_1(x, y) - F_1(x, y) \right\|$$

$$+ \left\| \frac{f(0, y) - f(-x, 0)}{2} - F_2(x, y) \right\| + \left\| \frac{f(x, 0) - f(-x, 0)}{2} - F_3(x, y) \right\|$$

$$\leq \left\| -\frac{1}{2} I_1 f(x, -x, y) - \frac{1}{4} I_2 f(x, y, -y) + \frac{1}{4} I_2 f(-x, y, -y) \right\| + 7\varepsilon$$

$$\leq 8\varepsilon$$

(3.45)

for all $x, y \in X \setminus A$, where $F$ is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + \frac{f(x_0, 0) + f(-x_0, 0)}{2}$$

(3.46)
and $z \notin A$. By (3.45), we get the inequalities

$$
\|f(x, y) - F(x, y)\| = \frac{1}{2} \|J_1 f((k+2)x, -kx, y) + f((k+2)x, y) - F((k+2)x, y) + f(-kx, y) - F(-kx, y)\| \leq \frac{17}{2} \varepsilon,
$$

$$
\|f(0, y) - F(0, y)\| = \frac{1}{2} \|J_1 f(kx, -kx, y) + f(kx, y) - F(kx, y) + f(-kx, y) - F(-kx, y)\| \leq \frac{17}{2} \varepsilon,
$$

(3.47)

for all $x \neq 0$ and $y \notin A$, where $k \in A_x$, and the inequalities

$$
\|f(x, y) - F(x, y)\| \leq \frac{17}{2} \varepsilon,
$$

$$
\|f(x, 0) - F(x, 0)\| \leq \frac{17}{2} \varepsilon
$$

(3.48)

for all $y \neq 0$ and $x \notin A$. Hence $F$ is a bi-Jensen mapping satisfying (3.28).

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (3.28) with $F'(0, 0) = F(0, 0)$. By Lemma 2.1, we have

$$
\|F(x, y) - F'(x, y)\|
\leq \left\| \frac{1}{4^n} (f - F)(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right) ((F - f)(2^n x, 0) + (F - f)(0, 2^n y)) \right\|
\leq \frac{17 \varepsilon}{2^{n-1}}
$$

(3.49)

for all $x, y \in X \setminus A$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X \setminus A$. By Lemma 3.1, $F = F'$ as we desired. 

Example 3.5. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$
f(x, y) := \begin{cases} 
\frac{\varepsilon}{2} & \text{if } (x, y) = (0, 0), \\
0 & \text{if } (x, y) \neq (0, 0).
\end{cases}
$$

(3.50)

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by $F(x, y) = 1$ for all $x, y \in X$. Then $f$ satisfies the conditions in Theorem 3.4, and $F$ is a bi-Jensen mapping satisfying (3.28) but $F(0, 0) \neq f(0, 0)$.
Corollary 3.6. Let \( f : X \times X \to Y \) be a mapping satisfying (3.13) and (3.27) for all \( x, y, z \in X \setminus \{0\} \). Then there exists a bi-Jensen mapping \( F : X \times X \to Y \) such that

\[
\| f(x, y) - F(x, y) \| \leq 8\varepsilon
\]

for all \( (x, y) \neq (0, 0) \).

Proof. Let \( F_2, F_3 \) be as in the proof of Theorem 3.4. By (3.30), we obtain

\[
\| f(0, y) - F(0, y) \| \leq \left\| \frac{f(0, y) + f(0, -y) - f(x_0, 0) - f(-x_0, 0)}{2} \right\| + \left\| \frac{f(0, y) - f(0, -y) - F_2(x, y)}{2} \right\| \leq \frac{7\varepsilon}{2},
\]

\[
\| f(x, 0) - F(x, 0) \| \leq \left\| \frac{f(x, 0) + f(-x, 0) - f(0, y) + f(0, -y)}{2} \right\| + \left\| \frac{f(x, 0) + f(0, -y) - f(x_0, 0) - f(-x_0, 0)}{2} \right\| + \left\| \frac{f(x, 0) - f(-x, 0) - F_3(x, y)}{2} \right\| \leq \frac{9\varepsilon}{2}
\]

for \( x, y \neq 0 \). From the above inequalities and (3.45), we get the inequality

\[
\| f(x, y) - F(x, y) \| \leq 8\varepsilon
\]

for all \( (x, y) \neq (0, 0) \). \( \square \)

References


