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Research Article

Weak and Strong Convergence Theorems for Equilibrium Problems and Countable Strict Pseudocontractions Mappings in Hilbert Space

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We introduce two iterative sequence for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a countable family of strict pseudocontractions in Hilbert Space. Then we study the weak and strong convergence of the sequences.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and let T be a self-mapping of C. Then T is said to be a strict pseudocontraction mappings if for all $x, y \in C$, there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

(if (1.1) holds, we also say that T is a κ-strict pseudocontraction). We use $F(T)$ to denote the set of fixed points of T, $\rightharpoonup$ $(\rightarrow)$ to denote weak(strong) convergence, and $W_w(x_n) = \{x : \exists x_{n_k} \rightharpoonup x\}$ to denote the W-limit set of $\{x_n\}$.

Let $f : C \times C \rightarrow R$ be a bifunction where R is the set of real numbers. Then, we consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } f(z, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of such $z \in C$ is denoted by $\mathrm{EP}(f)$. Numerous problems in physics, optimization, and economics can be reduced to find a solution of (1.2). Some methods have been proposed to solve the equilibrium problem (see [1–3]). Recently, S. Takahashi and W. Takahashi [4] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert spaces. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

In this paper, thanks to the condition introduced by Aoyama et al. [5], We introduce two iterative sequence for finding a common element of the set of solutions of an equilibrium problems and the set of fixed points of a countable family of strict pseudocontractions mappings in Hilbert Space. Then we study the weak and strong convergence of the sequences. The additional condition is inspired by Marino and Xu [6] and Kim and Xu [7].

2. Preliminaries

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [3]):

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x,y) + f(y,x) \le 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, that is, for each $x, y, z \in C, \limsup_{t \to 0^+} f(tz + (1-t)x, y) \le f(x,y)$;

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

Let H be a real Hilbert space. Then there hold the following well-known results:

$$\begin{gathered}\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall x, y \in H, \ \forall t \in [0,1]; \\ \|x+y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad \forall x, y \in H.\end{gathered} \tag{2.1}$$

If $\{x_n\}$ is a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - z\|^2 + \|z-y\|^2 \quad \forall y \in H. \tag{2.2}$$

Recall that the nearest point projection P_C from H onto C assigns to each $x \in H$ its nearest point denoted by $P_C x$ in C; that is, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.3}$$

Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the following relation:

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.4}$$

Lemma 2.1 (see [6]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be a κ-strict pseudocontraction such that $F(T) \neq \emptyset$.*

(1) *(Demi-closed principle) T is demi-closed on C, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

(2) *T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leq L\|x - y\| = \frac{1+\kappa}{1-\kappa}\|x - y\| \quad \forall x, y \in C. \tag{2.5}$$

(3) *The fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 (see [5]). *Let C be a nonempty closed convex subset of a Banach space and let $\{T_n\}$ be a sequence of mapping of C into itself. Suppose $\sum_{n=1}^{\infty} \sup_{x \in C}\|T_{n+1}x - T_n x\| < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n \to \infty} T_n y \quad \forall y \in C. \tag{2.6}$$

Then $\lim_{n \to \infty} \sup_{x \in C}\|T_n x - Tx\| = 0$.

Lemma 2.3 (see [8]). *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $W_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \quad \forall n, \tag{2.7}$$

then $x_n \to q$.

Lemma 2.4 (see [9]). *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and(A4). Then, for any $\lambda > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f\langle z, y\rangle + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

Further, if $T_\lambda x = \{z \in C : f\langle z, y\rangle + (1/\lambda)\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following holds:

(1) *$T_\lambda x$ is single-valued;*

(2) *$T_\lambda x$ is firmly nonexpansive, that is,*

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle, \quad \forall x, y \in H; \tag{2.9}$$

(3) $F(T_\lambda) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex.*

3. Weak Convergence Theorems

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a sequence of κ_n-strict pseudocontractions mappings on C into itself with $0 \le \kappa_n < 1$. Assume that $\kappa = \max\{\kappa_n : n \ge 1\}$. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), (A4), and $\mathrm{EP}(f) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequence generated by $x_1 \in C$ and*

$$f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n) T_n z_n, \quad \forall n \ge 1. \tag{3.1}$$

Assume that $\{\alpha_n\} \subset [0,1]$ with $\kappa + \delta < \alpha_n < 1 - \delta$ for all n, where $\delta \in (0,1)$ is a small enough constant, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\sum_{n=1}^{\infty} \sup_{x \in B} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset B of C and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge weakly to an element of $F(T) \cap \mathrm{EP}(f)$.

Proof. Pick $p \in F(T) \cap \mathrm{EP}(f)$. Then from the definition of Tr in Lemma 2.4, we have $z_n = T_{r_n} x_n$, and therefore $\|z_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\|$. It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)(T_n z_n - p) + \alpha_n (z_n - p)\|^2 \\
&= \alpha_n \|z_n - p\|^2 + (1 - \alpha_n)\|T_n z_n - p\|^2 - \alpha_n (1 - \alpha_n)\|z_n - T_n z_n\|^2 \\
&\le \alpha_n \|z_n - p\|^2 + (1 - \alpha_n)\left(\|z_n - p\|^2 + \kappa \|z_n - T_n z_n\|^2\right) \\
&\quad - \alpha_n (1 - \alpha_n)\|z_n - T_n z_n\|^2 \\
&= \|z_n - p\|^2 - (\alpha_n - \kappa)(1 - \alpha_n)\|z_n - T_n z_n\|^2 \\
&\le \|x_n - p\|^2 - (\alpha_n - \kappa)(1 - \alpha_n)\|z_n - T_n z_n\|^2.
\end{aligned} \tag{3.2}$$

Since $\kappa + \delta < \alpha_n < 1 - \delta$ for all n, we get $\|x_{n+1} - p\| \le \|x_n - p\|$; that is, the sequence $\{\|x_n - p\|\}$ is decreasing. Hence $\lim_{n\to\infty} \|x_n - p\|$ exists. In particular, $\{x_n\}$ is bounded. Since T_r is firmly nonexpensive, $\{z_n\}$ is also bounded. Also (3.2) implies that

$$\|z_n - T_n z_n\|^2 \le \frac{1}{\delta^2}\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right). \tag{3.3}$$

Taking the limit as $n \to \infty$ yields that

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0. \tag{3.4}$$

Since $\{z_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty} \sup_{x \in \{z_n\}} \|T_{n+1}x - T_n x\| < \infty. \tag{3.5}$$

We apply Lemma 2.2 to get

$$\begin{aligned} \|z_n - Tz_n\| &\le \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \\ &\le \|z_n - T_n z_n\| + \sup\{\|T_n z - Tz\| : z \in \{z_n\}\} \longrightarrow 0. \end{aligned} \tag{3.6}$$

Next, we claim that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Indeed, let p be an arbitrary element of $F(T)\cap \mathrm{EP}(f)$. Then as above

$$\begin{aligned} \|z_n - p\|^2 &= \|T_r x_n - T_r p\|^2 \\ &\le \langle T_r x_n - T_r p, x_n - p\rangle \\ &= \langle z_n - T_r p, x_n - p\rangle \\ &= \frac{1}{2}\left(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2\right), \end{aligned} \tag{3.7}$$

and hence

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2. \tag{3.8}$$

Therefore, from (3.2), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|z_n - p\|^2 - (\alpha_n - \kappa)(1-\alpha_n)\|z_n - T_n z_n\|^2 \\ &\le \|z_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|x_n - z_n\|^2, \end{aligned} \tag{3.9}$$

and hence

$$\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.10}$$

So, from the existence of $\lim_{n\to\infty}\|x_n - p\|$, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.11}$$

Next, we claim that $W_w(x_n) \subset F(T)\cap \mathrm{EP}(f)$. since $\{x_n\}$ is bounded and H is reflexive, $W_w(x_n)$ is nonempty. Let $w \in W_w(x_n)$ be an arbitrary element. Then a subsequence x_{n_i} of $\{x_n\}$ converges weakly to w. Hence, from (3.11) we know that $z_{n_i} \rightharpoonup w$. As $\|z_n - Tz_n\| \to 0$, we obtain that $Tz_{n_i} \rightharpoonup w$. Let us show $W_w(x_n) \subset \mathrm{EP}(f)$. Since $z_n = T_{r_n}x_n$, we have

$$f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.12}$$

By (A2), we have

$$\frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq f(y, z_n), \tag{3.13}$$

and hence

$$\left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq f(y, z_{n_i}). \tag{3.14}$$

From (A4), we have

$$0 \geq f(y, w) \quad \forall y \in C. \tag{3.15}$$

Then, for $t \in (0,1]$ and $y \in C$, from (A1), and (A4), we also have

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y), \end{aligned} \tag{3.16}$$

Taking $t \to 0^+$ and using (A3), we get

$$f(w, y) \geq 0 \quad \forall y \in C, \tag{3.17}$$

and hence $w \in \mathrm{EP}(f)$. Since T is a strict pseudocontraction mapping, by Lemma 2.1(1) we know that the mapping T is demiclosed at zero. Note that $\|z_n - Tz_n\| \to 0$ and $z_{n_i} \rightharpoonup w$. Thus, $w \in F(T)$. Consequently, we deduce that $w \subset F(T) \cap \mathrm{EP}(f)$. Since w is an arbitrary element, we conclude that $W_w(x_n) \subset F(T) \cap \mathrm{EP}(f)$.

To see that $\{x_n\}$ and $\{z_n\}$ are actually weakly convergent, we take $\overline{x}, \widetilde{x} \in W_w(x_n)$ $(x_{n_i} \rightharpoonup \overline{x}, x_{m_j} \rightharpoonup \widetilde{x})$. Since $\lim_{n\to\infty}\|x_n - p\|$ exist for every $p \in F(T)$, by (2.2), we have

$$\begin{aligned} \lim_{n\to\infty}\|x_n - \widetilde{x}\|^2 &= \lim_{i\to\infty}\|(x_{n_i} - \widetilde{x})\|^2 \\ &= \lim_{i\to\infty}\|x_{n_i} - \overline{x}\|^2 + \|\overline{x} - \widetilde{x}\|^2 \\ &= \lim_{j\to\infty}\left\|x_{m_j} - \overline{x}\right\|^2 + \|\overline{x} - \widetilde{x}\|^2 \\ &= \lim_{j\to\infty}\left\|x_{m_j} - \widetilde{x}\right\|^2 + 2\|\overline{x} - \widetilde{x}\|^2 \\ &= \lim_{n\to\infty}\|x_n - \widetilde{x}\|^2 + 2\|\overline{x} - \widetilde{x}\|^2. \end{aligned} \tag{3.18}$$

Hence $\widetilde{x} = \overline{x}$ and proof is completed. □

4. Strong Convergence Theorems

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of κ_n-strict pseudocontractions mappings on C into itself with $0 \leq \kappa_n < 1$. Assume that $\kappa = \max\{\kappa_n : n \geq 1\}$. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), (A4) and $\mathrm{EP}(f) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, let $\{x_n\}$ and $\{z_n\}$ be sequence generated by $x_0 \in C$ and*

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ z_n \in C \text{ such that } f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C : \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{gathered} \tag{4.1}$$

Assume that $\{\alpha_n\} \subset [0,1]$ with $\kappa + \delta < \alpha_n < 1 - \delta$ for all n, where $\delta \in (0,1)$ is a small enough constant, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\sum_{n=1}^{\infty} \sup_{x\in B} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset B of C and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ Suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}$ converges strongly to $P_{F(T)\cap \mathrm{EP}(f)} x_0$.

Proof. First, we show that C_n is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. For $z \in C_k$, we know that $\|z_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z \rangle \leq 0. \tag{4.2}$$

So C_{k+1} is closed and convex. Then, C_n is closed and convex.

Next, we show by induction that $F(T) \cap \mathrm{EP}(f) \subset C_n$ for all $n \geq 1$. $F(T) \cap \mathrm{EP}(f) \subset C_1$ is obvious. Suppose that $F(T) \cap \mathrm{EP}(f) \subset C_k$ for some $k \geq 1$. Let $p \in F(T) \cap \mathrm{EP}(f) \subset C_k$. Putting $z_n = T_{r_n} y_n$ for all n, we know from (4.1) that

$$\begin{aligned} \|z_n - p\|^2 &= \|T_{r_n} y_n - p\|^2 \\ &\leq \|y_n - p\|^2 \\ &= \|(1-\alpha_n)(T_n x_n - p) + \alpha_n (x_n - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left(\|x_n - p\|^2 + \kappa \|x_n - T_n x_n\|^2\right) \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - (\alpha_n - \kappa)(1-\alpha_n)\|x_n - T_n x_n\|^2 \\
&\le \|x_n - p\|^2 - \delta^2\|x_n - T_n x_n\|^2 \\
&\le \|x_n - p\|^2,
\end{aligned}
\tag{4.3}
$$

and hence $p \in C_{k+1}$. This implies that $F(T) \cap \mathrm{EP}(f) \subset C_n$ for all $n \ge 1$.

This implied that $\{x_n\}$ is well defined.

From $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| \le \|x_0 - y\| \quad \forall y \in C_n. \tag{4.4}$$

Using $F(T) \cap \mathrm{EP}(f) \subset C_n$, we have

$$\|x_0 - x_n\| \le \|x_0 - u\| \quad \forall u \in F(T) \cap \mathrm{EP}(f),\ n \ge 1. \tag{4.5}$$

Then, $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{z_n\}$. In particular,

$$\|x_0 - x_n\| \le \|x_0 - p\| \quad \text{where } p = P_{F(T)\cap \mathrm{EP}(f)} x_0. \tag{4.6}$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|. \tag{4.7}$$

Since $\{\|x_n - x_0\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$. we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0. \tag{4.8}$$

In fact, from (4.8), we have

$$
\begin{aligned}
\|x_n - x_{n+1}\| &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&= \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&\le \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2.
\end{aligned}
\tag{4.9}
$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have that $\|x_n - x_{n+1}\| \to 0$. On the other hand $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\| \longrightarrow 0. \tag{4.10}$$

Further, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.11}$$

From (4.3), we have

$$\|x_n - T_n x_n\|^2 \le \frac{1}{\delta^2}\left(\|x_n - p\|^2 - \|z_n - p\|^2\right). \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}\|x_n - p\|^2 - \|z_n - p\|^2 &= \|x_n\|^2 - \|z_n\|^2 + 2\langle z_n - x_n, p\rangle \\ &\le \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|p\|\|x_n - z_n\|.\end{aligned} \tag{4.13}$$

Then, we have

$$\lim_{n\to\infty}\|x_n - p\|^2 - \|z_n - p\|^2 = 0. \tag{4.14}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{4.15}$$

We apply Lemma 2.2 to get

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - T_n x_n\| + \sup\{\|T_n x - Tx\| : x \in \{x_n\}\} \longrightarrow 0.\end{aligned} \tag{4.16}$$

Lastly, we show that the sequence $\{x_n\}$ converges to $P_{F(T)\cap \mathrm{EP}(f)}x_0$. Since $\{x_n\}$ is bounded and H is reflexive, $W_w(x_n)$ is nonempty. Let $w \in W_w(x_n)$ be an arbitrary element. Then a subsequence x_{n_i} of $\{x_n\}$ converges weakly to w. From Lemma 2.1 and (4.16), we obtain that $\omega_w(x_n) \subset F(T)$. Next, we show $W_w(x_n) \subset \mathrm{EP}(f)$. Let p be an arbitrary element of $F(T)\cap \mathrm{EP}(f)$. From $z_n = T_{r_n}y_n$ and $\|y_n - p\| \le \|x_n - p\|$, we have

$$\begin{aligned}\|z_n - p\|^2 &\le \|T_r y_n - T_r p\|^2 \\ &\le \langle T_r y_n - T_r p, y_n - p\rangle \\ &= \langle z_n - T_r p, y_n - p\rangle \\ &= \frac{1}{2}\left(\|z_n - p\|^2 + \|y_n - p\|^2 - \|y_n - z_n\|^2\right) \\ &= \frac{1}{2}\left(\|z_n - p\|^2 + \|x_n - p\|^2 - \|y_n - z_n\|^2\right),\end{aligned} \tag{4.17}$$

and hence

$$\|y_n - z_n\|^2 \le \|x_n - p\|^2 - \|z_n - p\|^2. \tag{4.18}$$

Therefore, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{4.19}$$

As in the proof of Theorem 3.1, we have

$$f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \ge 0, \quad \forall y \in C. \tag{4.20}$$

By (A2), we have

$$\frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \ge f(y, z_n), \tag{4.21}$$

and hence

$$\langle y - z_{n_i}, \frac{z_{n_i} - y_{n_i}}{r_{n_i}}\rangle \ge f(y, z_{n_i}). \tag{4.22}$$

From (A4), we have

$$0 \ge f(y, w) \quad \forall y \in C. \tag{4.23}$$

Then, for $t \in (0,1]$ and $y \in C$, from (A1) and (A4), we also have

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\le tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\le tf(ty + (1-t)w, y). \end{aligned} \tag{4.24}$$

Taking $t \to 0^+$ and using (A3), we get

$$f(w, y) \ge 0 \quad \forall y \in C, \tag{4.25}$$

and hence $w \in \mathrm{EP}(f)$. Lemma 2.3 and (4.6) ensure the strong convergence of $\{x_n\}$ to $P_{F(T)\cap \mathrm{EP}(f)}x_0$. This completes the proof. □

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