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Research Article

Approximate Behavior of Bi-Quadratic Mappings in Quasinormed Spaces

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We obtain the generalized Hyers-Ulam stability of the bi-quadratic functional equation $f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)=4[f(x,z)+f(x,w)+f(y,z)+f(y,w)]$ in quasinormed spaces.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms as follows

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

Hyers [2] proved the stability problem of additive mappings in Banach spaces. Rassias [3] provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded: let $f : E \to E$ be a mapping from a normed vector space E into a Banach space E subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|x\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. The above inequality provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. Găvruţa [4] provided a further generalization of Hyers-Ulam theorem. A square norm on an inner product space satisfies the important parallelogram equality:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.2}$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is called the quadratic functional equation whose solution is said to be a quadratic mapping. A generalized stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [7] proved the generalized stability of the quadratic functional equation, and Park [8] proved the generalized stability of the quadratic functional equation in Banach modules over a C^*-algebra.

Throughout this paper, let X and Y be vector spaces.

Definition 1.1. A mapping $f : X \times X \to Y$ is called *bi-quadratic* if f satisfies the system of the following equations:

$$\begin{aligned} f(x+y,z) + f(x-y,z) &= 2f(x,z) + 2f(y,z), \\ f(x,y+z) + f(x,y-z) &= 2f(x,y) + 2f(x,z). \end{aligned} \tag{1.4}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) := ax^2y^2$ is a solution of (1.4).

For a mapping $f : X \times X \to Y$, consider the functional equation:

$$\begin{aligned} &f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) \\ &\quad = 4[f(x,z) + f(x,w) + f(y,z) + f(y,w)]. \end{aligned} \tag{1.5}$$

Definition 1.2 (see [9, 10]). Let X be a real linear space. A *quasinorm* is real-valued function on X satisfying the following

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from the condition (iii) that

$$\left\| \sum_{i=1}^{2m} x_i \right\| \leq K^m \sum_{i=1}^{2m} \|x_i\|, \qquad \left\| \sum_{i=1}^{2m+1} x_i \right\| \leq K^{m+1} \sum_{i=1}^{2m+1} \|x_i\| \tag{1.6}$$

for all $m \geq 1$ and all $x_1, x_2, \ldots, x_{2m+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasinormed space* if $\|\cdot\|$ is a quasinorm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a *p-norm* $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p \tag{1.7}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [10] (see also [9]), each quasinorm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. In [11], Tabor has investigated a version of Hyers-Rassias-Gajda theorem (see also [3, 12]) in quasi-Banach spaces. Since then, the stability problems have been investigated by many authors (see [13–18]).

The authors [19] solved the solutions of (1.4) and (1.5) as follows.

Theorem A. *A mapping $f : X \times X \to Y$ satisfies* (1.4) *if and only if there exist a multi-additive mapping $M : X \times X \times X \times X \to Y$ such that $f(x, y) = M(x, x, y, y)$ and $M(x, y, z, w) = M(y, x, z, w) = M(x, y, w, z)$ for all $x, y, z, w \in X$.*

Theorem B. *A mapping $f : X \times X \to Y$ satisfies* (1.4) *if and only if it satisfies* (1.5).

In this paper, we investigate the generalized Hyers-Ulam stability of (1.4) and (1.5) in quasi-Banach spaces.

2. Stability of (1.4) and (1.5) in Quasi-normed Spaces

Throughout this section, assume that X is a quasinormed space with quasinorm $\|\cdot\|_X$ and that Y is a p-Banach space with p-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$.

Let $\varphi : X \times X \times X \to [0, \infty)$ and $\psi : X \times X \times X \to [0, \infty)$ be two functions such that

$$\lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, z) = 0, \qquad \lim_{n\to\infty} \frac{1}{4^n}\psi(2^n x, y, z) = 0, \qquad \lim_{n\to\infty} \frac{1}{4^n}\varphi(x, y, 2^n z) = 0,$$
$$\lim_{n\to\infty} \frac{1}{4^n}\psi(x, 2^n y, 2^n z) = 0 \tag{2.1}$$

for all $x, y, z \in X$.

Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying

$$M(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^{pj}} \varphi\left(2^j x, 2^j y, z\right)^p < \infty, \tag{2.2}$$

$$N(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^{pj}} \psi\left(x, 2^j y, 2^j z\right)^p < \infty \tag{2.3}$$

for all $x, y, z \in X$.

Theorem 2.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x+y,z)+f(x-y,z)-2f(x,z)-2f(y,z)\|_Y \le \varphi(x,y,z), \tag{2.4}$$

$$\|f(x,y+z)+f(x,y-z)-2f(x,y)-2f(x,z)\|_Y \le \psi(x,y,z), \tag{2.5}$$

and let $f(x,0)=0$ and $f(0,y)=0$ for all $x,y,z \in X$. Then there exist two bi-quadratic mappings $F_1, F_2 : X \times X \to Y$ such that

$$\|f(x,y)-F_1(x,y)\|_Y \le \frac{1}{4}M(x,x,y)^{1/p}, \tag{2.6}$$

$$\|f(x,y)-F_2(x,y)\|_Y \le \frac{1}{4}N(x,y,y)^{1/p} \tag{2.7}$$

for all $x,y \in X$.

Proof. Letting $y=x$ in (2.4), we get

$$\left\|f(x,z)-\frac{1}{4}f(2x,z)\right\|_Y \le \frac{1}{4}\varphi(x,x,z) \tag{2.8}$$

for all $x,z \in X$. Thus we have

$$\left\|\frac{1}{4^j}f(2^jx,z)-\frac{1}{4^{j+1}}f(2^{j+1}x,z)\right\|_Y \le \frac{1}{4^{j+1}}\varphi\left(2^jx,2^jx,z\right) \tag{2.9}$$

for all $x,z \in X$. Replacing z by y in the above inequality, we obtain

$$\left\|\frac{1}{4^j}f(2^jx,y)-\frac{1}{4^{j+1}}f\left(2^{j+1}x,y\right)\right\|_Y \le \frac{1}{4^{j+1}}\varphi\left(2^jx,2^jx,y\right) \tag{2.10}$$

for all $x,y \in X$. Since Y is a p-Banach space, for given integers l,m $(0 \le l < m)$, we see that

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^lx,y)-\frac{1}{4^m}f(2^mx,y)\right\|_Y^p &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^jx,y\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x,y\right)\right\|_Y^p \\ &\le \frac{1}{4^p}\sum_{j=l}^{m-1}\frac{1}{4^{pj}}\varphi\left(2^jx,2^jx,y\right)^p\end{aligned} \tag{2.11}$$

for all $x,y \in X$. By (2.2) and (2.11), the sequence $\{(1/4^j)f(2^jx,y)\}$ is a Cauchy sequence for all $x,y \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^jx,y)\}$ converges for all $x,y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty}\frac{1}{4^j}f\left(2^jx,y\right) \tag{2.12}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.11), one can obtain the inequality (2.6). By (2.4) and (2.5), we get

$$\begin{aligned}\left\| \frac{1}{4^j} f\left(x+y, 2^j z\right) + \frac{1}{4^j} f\left(x-y, 2^j z\right) - 2\frac{1}{4^j} f\left(x, 2^j z\right) - 2\frac{1}{4^j} f\left(y, 2^j z\right) \right\|_Y &\le \frac{1}{4^j} \varphi\left(x, y, 2^j z\right), \\ \left\| \frac{1}{4^j} f(2^j x, y+z) + \frac{1}{4^j} f(2^j x, y-z) - 2\frac{1}{4^j} f\left(2^j x, y\right) - 2\frac{1}{4^j} f\left(2^j x, z\right) \right\|_Y &\le \frac{1}{4^j} \psi\left(2^j x, y, z\right)\end{aligned} \tag{2.13}$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (2.1), F_1 is bi-quadratic.

Next, setting $z = y$ in (2.5),

$$\left\| f(x, y) - \frac{1}{4} f(x, 2y) \right\|_Y \le \frac{1}{4} \psi(x, y, y) \tag{2.14}$$

for all $x, y \in X$. By the same method as above, define $F_2 : X \times X \to Y$ by $F_2(x, y) := \lim_{j \to \infty}(1/4^j) f(x, 2^j y)$ for all $x, y \in X$. By the same argument as above, F_2 is a bi-quadratic mapping satisfying (2.7). □

Corollary 2.2. *Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned}\| f(x+y, z) + f(x-y, z) - 2f(x, z) - 2f(y, z) \|_Y &\le \delta, \\ \| f(x, y+z) + f(x, y-z) - 2f(x, y) - 2f(x, z) \|_Y &\le \varepsilon,\end{aligned} \tag{2.15}$$

and let $f(x, 0) = 0$ and $f(0, y) = 0$ for all $x, y, z \in X$. Then there exist two bi-quadratic mappings $F_1, F_2 : X \times X \to Y$ such that

$$\begin{aligned}\| f(x, y) - F_1(x, y) \|_Y &\le \frac{\delta}{\sqrt[p]{4^p - 1}}, \\ \| f(x, y) - F_2(x, y) \|_Y &\le \frac{\varepsilon}{\sqrt[p]{4^p - 1}}\end{aligned} \tag{2.16}$$

for all $x, y \in X$.

Proof. In Theorem 2.1, putting $\varphi(x, y, z) := \delta$ and $\psi(x, y, z) := \varepsilon$ for all $x, y, z \in X$, we get the desired result. □

From now on, let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function such that

$$\lim_{n \to \infty} \frac{1}{16^n} \varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0, \tag{2.17}$$

$$L(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{16^{pj}} \varphi\left(2^j x, 2^j y, 2^j z, 2^j w\right)^p < \infty \tag{2.18}$$

for all $x, y, z, w \in X$.

We will use the following lemma in order to prove Theorem 2.4.

Lemma 2.3 (see [20]). *Let* $0 < p \leq 1$ *and let* $x_1, x_2, \ldots, x_n$ *be nonnegative real numbers. Then*

$$\left(\sum_{j=1}^{n} x_j\right)^p \leq \sum_{j=1}^{n} {x_j}^p. \tag{2.19}$$

Theorem 2.4. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\begin{aligned} &\| f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\ &\quad -4[f(x,z) - f(x,w) - f(y,z) - f(y,w)] \|_Y \leq \varphi(x,y,z,w), \end{aligned} \tag{2.20}$$

and let $f(x,0) = 0$ *and* $f(0,y) = 0$ *for all* $x, y, z, w \in X$*. Then there exists a unique bi-quadratic mapping* $F : X \times X \to Y$ *such that*

$$\| f(x,y) - F(x,y) \|_Y \leq \frac{1}{16} L(x,x,y,y)^{1/p} \tag{2.21}$$

for all $x, y \in X$*.*

Proof. Letting $y = x$ and $w = z$ in (2.20), we have

$$\left\| f(x,z) - \frac{1}{16} f(2x, 2z) \right\|_Y \leq \frac{1}{16} \varphi(x,x,z,z) \tag{2.22}$$

for all $x, z \in X$. Thus we obtain

$$\left\| \frac{1}{16^j} f(2^j x, 2^j z) - \frac{1}{16^{j+1}} f(2^{j+1} x, 2^{j+1} z) \right\|_Y \leq \frac{1}{16^{j+1}} \varphi\left(2^j x, 2^j x, 2^j z, 2^j z\right) \tag{2.23}$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{16^j} f(2^j x, 2^j y) - \frac{1}{16^{j+1}} f(2^{j+1} x, 2^{j+1} y) \right\|_Y \leq \frac{1}{16^{j+1}} \varphi\left(2^j x, 2^j x, 2^j y, 2^j y\right) \tag{2.24}$$

for all $x, y \in X$ and all j. By Lemma 2.3, for given integers l, m $(0 \leq l < m)$, we get

$$\begin{aligned} \left\| \frac{1}{16^l} f(2^l x, 2^l y) - \frac{1}{16^m} f(2^m x, 2^m y) \right\|_Y^p &\leq \sum_{j=l}^{\mathrm{m}-1} \left\| \frac{1}{16^j} f\left(2^j x, 2^j y\right) - \frac{1}{16^{j+1}} f\left(2^{j+1} x, 2^{j+1} y\right) \right\|_Y^p \\ &\leq \frac{1}{16} \sum_{j=l}^{m-1} \frac{1}{16^{pj}} \varphi\left(2^j x, 2^j x, 2^j y, 2^j y\right)^p \end{aligned} \tag{2.25}$$

for all $x, y \in X$. By (2.18) and (2.25), the sequence $\{(1/16^j) f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{(1/16^j) f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{16^j} f\left(2^j x, 2^j y\right) \tag{2.26}$$

for all $x, y \in X$.

By (2.20), we have

$$\begin{aligned}
&\left\| \frac{1}{16^j} f\left(2^j(x+y), 2^j(z+w)\right) + \frac{1}{16^j} f\left(2^j(x+y), 2^j(z-w)\right) \right. \\
&\quad + \frac{1}{16^j} f\left(2^j(x-y), 2^j(z+w)\right) + \frac{1}{16^j} f\left(2^j(x-y), 2^j(z-w)\right) \\
&\quad \left. -\frac{4}{16^j} f\left(2^j x, 2^j z\right) - \frac{4}{16^j} f\left(2^j x, 2^j w\right) - \frac{4}{16^j} f\left(2^j y, 2^j z\right) - \frac{4}{16^j} f\left(2^j y, 2^j w\right) \right\|_Y \\
&\le \frac{1}{16^j} \varphi\left(2^j x, 2^j y, 2^j z, 2^j w\right)
\end{aligned} \tag{2.27}$$

for all $x, y, z, w \in X$ and all j. Letting $j \to \infty$ and using (2.17), we see that F satisfies (1.5). By Theorem B, we obtain that F is bi-quadratic. Setting $l = 0$ and taking $m \to \infty$ in (2.25), one can obtain the inequality (2.21). If $G : X \times X \to Y$ is another bi-quadratic mapping satisfying (2.21), we obtain

$$\begin{aligned}
&\|F(x,y) - G(x,y)\|_Y^p \\
&\quad = \frac{1}{16^{pn}} \|F(2^n x, 2^n y) - G(2^n x, 2^n y)\|_Y^p \\
&\quad \le \frac{1}{16^{pn}} \|F(2^n x, 2^n y) - f(2^n x, 2^n y)\|_Y^p + \frac{1}{16^{pn}} \|f(2^n x, 2^n y) - G(2^n x, 2^n y)\|_Y^p \\
&\quad \le \frac{1}{8} \frac{1}{16^{pn}} L(2^n x, 2^n x, 2^n y, 2^n y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty
\end{aligned} \tag{2.28}$$

for all $x, y \in X$. Hence the mapping F is the unique bi-quadratic mapping, as desired. □

Corollary 2.5. *Let ε be a nonnegative real number. Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\
&\quad -4[f(x,z) - f(x,w) - f(y,z) - f(y,w)]\|_Y \le \varepsilon,
\end{aligned} \tag{2.29}$$

and let $f(x,0) = 0$ *and* $f(0,y) = 0$ *for all* $x, y, z, w \in X$. *Then there exists a unique bi-quadratic mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\|_Y \le \frac{\varepsilon}{\sqrt[p]{16^p - 1}} \tag{2.30}$$

for all $x, y \in X$.

Proof. In Theorem 2.4, putting $\varphi(x,y,z,w) := \varepsilon$ for all $x, y, z, w \in X$, we get the desired result. □

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