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Research Article

A Converse of Minkowski's Type Inequalities

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We formulate and prove a converse for a generalization of the classical Minkowski's inequality. The case when $0 < p < 1$ is also considered. Applying the same technique, we obtain an analog converse theorem for integral Minkowski's type inequality.

1. Introduction

If $p > 1$, $a_i \geq 0$, and $b_i \geq 0$ $(i = 1, \ldots, n)$ are real numbers, then by the classical Minkowski's inequality

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}. \tag{1.1}$$

This inequality was published by Minkowski [1, pages 115–117] hundred years ago in his famous book "Geometrie der Zahlen."

It is also known (see [2]) that for $0 < p < 1$ the above inequality is satisfied with "$\geq$" instead of "$\leq$".

Many extensions and generalizations of Minkowski's inequality can be found in [2, 3]. We want to point out the following inequality:

$$\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)^p\right)^{1/p} \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^p\right)^{1/p}, \tag{1.2}$$

where $p > 1$ and $a_{ij} \geq 0$ ($i = 1, \dots, m; j = 1, \dots, n$) are real numbers. Furthermore, if $0 < p < 1$, then the inequality (1.2) is satisfied with "$\geq$" instead of "$\leq$" [2, Theorem 24, page 30]. In both cases, equality holds if and only if all columns $(a_{1j}, a_{2j}, \dots, a_{mj})$, $j = 1, 2, \dots, n$, are proportional.

An extension of inequality (1.2) was formulated by Ingham and Jessen (see [2, pages 31-32]). In 1948, Tôyama [4] published a converse of the inequality of Ingham and Jessen (see also a recent paper [5] for a weighted version of Tôyama's inequality). Namely, Tôyama showed that if $0 < q < p$ and $a_{ij} \geq 0$ ($i = 1, \dots, m; j = 1, \dots, n$) are real numbers, then

$$\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{q/p}\right)^{1/q} \leq (\min(m,n))^{1/q-1/p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{1/p}. \tag{1.3}$$

The main result of this paper gives a converse of inequality (1.2). On the other hand, our result may be regarded as a nonsymmetric analogue of the above inequality, and it is given as follows.

Theorem 1.1. *Let $p > 0$, $q > 0$, and $a_{ij} \geq 0$ ($i = 1, \dots, m; j = 1, \dots, n$) be real numbers. Then for $p \geq 1$ we have*

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p} \leq C\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{1/p}, \tag{1.4}$$

where C is a positive constant given by

$$C = \begin{cases} m^{1-1/q} & \text{if } 1 \leq p \leq q, \\ (\min(m,n))^{1/q-1/p}m^{1-1/q} & \text{if } 1 \leq q < p, \\ m^{1-1/p} & \text{if } 0 < q \leq 1 \leq p. \end{cases} \tag{1.5}$$

If $0 < p < 1$, then

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p} \geq K\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{1/p}, \tag{1.6}$$

where K is a positive constant given by

$$K = \begin{cases} m^{1-1/q} & \text{if } 0 < q \leq p < 1, \\ (\min(m,n))^{1/q-1/p}m^{1-1/q} & \text{if } 0 < p < q < 1, \\ m^{1-1/p} & \text{if } 0 < p < 1 \leq q. \end{cases} \tag{1.7}$$

Inequality (1.4) *with* $1 \le p \le q$ *and inequality* (1.6) *with* $0 < q \le p < 1$ *are sharp for all* m *and* n, *and they are attained for* $a_{ij} = a$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. *If* $m \le n$, *then inequality* (1.4) *is sharp in the cases when* $1 \le q < p$ *and* $0 < q \le 1 \le p$. *In both cases the equalities are attained for*

$$a_{ij} = \begin{cases} a, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.8}$$

When $m \le n$, *the equalities in* (1.6) *concerned with* $0 < p < q < 1$ *and* $0 < p < 1 \le q$ *are also attained for previously defined values* a_{ij}.

Remark 1.2. Note that, proceeding as in the proof of Theorem 1.1, we can prove similar inequalities to (1.4) and (1.6) with $\sum_{j=1}^{n}(\sum_{i=1}^{m})$ instead of $\sum_{i=1}^{m}(\sum_{j=1}^{n})$ on the left-hand side of these inequalities. For example, such an inequality concerning the case when $1 \le q < p$ (i.e., (1.4)) is

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{p}\right)^{1/p} \le n^{1-1/p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{q}\right)^{p/q}\right)^{1/p}. \tag{1.9}$$

The above inequality is sharp if $n \le m$, but it is not in spirit of a converse of Minkowski's type inequality.

The following consequence of Theorem 1.1 for $m = 2$ and $q = 2$ can be viewed as a converse of Minkowski's inequality (1.1).

Corollary 1.3. *Let* $n \ge 1$, $p > 0$, *and let* $a_j \ge 0$, $b_j \ge 0$ $(j = 1, \ldots, n)$ *be real numbers. Then for* $p \ge 1$

$$\left(\sum_{j=1}^{n} a_j^{p}\right)^{1/p} + \left(\sum_{j=1}^{n} b_j^{p}\right)^{1/p} \le 2^{1-\min\{1/2,1/p\}}\left(\sum_{j=1}^{n}\left(a_j^2 + b_j^2\right)^{p/2}\right)^{1/p}. \tag{1.10}$$

If $0 < p < 1$, *then*

$$\left(\sum_{j=1}^{n} a_j^{p}\right)^{1/p} + \left(\sum_{j=1}^{n} v_j^{p}\right)^{1/p} \ge 2^{1-1/p}\left(\sum_{j=1}^{n}\left(a_j^2 + b_j^2\right)^{p/2}\right)^{1/p}. \tag{1.11}$$

Remark 1.4. It is well known that Minkowski's inequality is also true for complex sequences as well. More precisely, if $p \ge 1$ and u_i, v_i $(i = 1, \ldots, n)$ are arbitrary complex numbers, then

$$\left(\sum_{j=1}^{n}\left|u_j + v_j\right|^{p}\right)^{1/p} \le \left(\sum_{j=1}^{n}\left|u_j\right|^{p}\right)^{1/p} + \left(\sum_{j=1}^{n}\left|v_j\right|^{p}\right)^{1/p}. \tag{1.12}$$

Note that the above inequality with $u_j = a_j \in \mathbb{R}$ and $v_j = ib_j$, $b_j \in \mathbb{R}$, for each $j = 1, 2, \ldots, n$, becomes

$$\left(\sum_{j=1}^{n}\left(a_j^2 + b_j^2\right)^{p/2}\right)^{1/p} \le \left(\sum_{j=1}^{n} a_j^p\right)^{1/p} + \left(\sum_{j=1}^{n} v_j^p\right)^{1/p}. \tag{1.13}$$

We see that the first inequality of Corollary 1.3 may be actually regarded as a converse of the previous inequality.

2. Proof of Theorem 1.1

Lemma 2.1 (see [2, page 26]). *If $u_1, u_2, \ldots u_k, s, r$ are nonnegative real numbers and $0 < s < r$, then*

$$\left(u_1^s + u_2^s + \cdots + u_k^s\right)^{1/s} \ge \left(u_1^r + u_2^r + \cdots + u_k^r\right)^{1/r}. \tag{2.1}$$

Proof of Theorem 1.1. In our proof we often use the well-known fact that the scale of power means is nondecreasing (see [2]). More precisely, if $a_1, a_2, \ldots, a_k$ are nonnegative integers and $0 < \alpha \le \beta < +\infty$, then

$$\left(\frac{\sum_{i=1}^{k} a_i^{\alpha}}{k}\right)^{1/\alpha} \le \left(\frac{\sum_{i=1}^{k} a_i^{\beta}}{k}\right)^{1/\beta}. \tag{2.2}$$

In all the cases, for each $i = 1, 2, \ldots, m$, we denote that

$$a_i := \left(\sum_{j=1}^{n} a_{ij}^p\right)^{1/p}. \tag{2.3}$$

We will consider all the six cases related to the inequalities (1.4) and (1.6).

Case 1 ($1 \le p \le q$). The inequality between power means of orders $q/p \ge 1$ and 1 for m positive numbers b_i, $i = 1, 2, \ldots, m$, states that

$$\left(\frac{\sum_{i=1}^{m} b_i^{q/p}}{m}\right)^{p/q} \ge \frac{\sum_{i=1}^{m} b_i}{m}, \tag{2.4}$$

whence for any fixed $j = 1, 2, \ldots n$, after substitution of $b_i = a_{ij}^p$, $i = 1, 2, \ldots m$, we obtain

$$\left(a_{1j}^q + a_{2j}^q + \cdots + a_{mj}^q\right)^{p/q} \ge m^{(p/q)-1}\left(a_{1j}^p + a_{2j}^p + \cdots + a_{mj}^p\right), \tag{2.5}$$

whence after summation over j we find that

$$\begin{aligned}\sum_{j=1}^{n}\left(a_{1j}^{q}+a_{2j}^{q}+\cdots+a_{mj}^{q}\right)^{p/q} &\geq m^{(p/q)-1}\sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}^{p}\\ &= m^{(p/q)-1}\sum_{i=1}^{m}a_{i}^{p}.\end{aligned}\tag{2.6}$$

Because $p \geq 1$, the inequality between power means of orders p and 1 implies that

$$\sum_{i=1}^{m}a_{i}^{p} \geq m^{1-p}\left(\sum_{i=1}^{m}a_{i}\right)^{p}.\tag{2.7}$$

The above inequality and (2.6) immediately yield

$$m^{1-1/q}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{1/p} \geq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p}.\tag{2.8}$$

Case 2 ($1 \leq q < p$). If $m \leq n$, then $C = m^{1-1/p}$ in (1.4), and a related proof is the same as that for the following case when $0 < q \leq 1 \leq p$.

Now suppose that $m > n$. By the inequality for power means of orders $p/q \geq 1$ and 1, we obtain

$$\begin{aligned}&\left(\frac{\sum_{j=1}^{n}\left(a_{1j}^{q}+a_{2j}^{q}+\cdots+a_{mj}^{q}\right)^{p/q}}{n}\right)^{q/p}\\ &\quad\geq \frac{\sum_{j=1}^{n}\left(a_{1j}^{q}+a_{2j}^{q}+\cdots+a_{mj}^{q}\right)}{n} = \frac{m}{n}\cdot\frac{\sum_{i=1}^{m}\left(a_{i1}^{q}+a_{i2}^{q}+\cdots+a_{in}^{q}\right)}{m}.\end{aligned}\tag{2.9}$$

Next, by the inequality for power means (of orders $q \geq 1$ and 1), we obtain

$$\frac{\sum_{i=1}^{m}\left(a_{i1}^{q}+a_{i2}^{q}+\cdots+a_{in}^{q}\right)}{m} \geq \left(\frac{\sum_{i=1}^{m}\left(a_{i1}^{q}+a_{i2}^{q}+\cdots+a_{in}^{q}\right)^{1/q}}{m}\right)^{q}.\tag{2.10}$$

For any fixed $i \in \{1,2,\ldots,m\}$ the inequality (2.1) of Lemma 2.1 with $s = p > q = r$ implies that

$$\left(a_{i1}^{q}+a_{i2}^{q}+\cdots+a_{in}^{q}\right)^{1/q} \geq \left(a_{i1}^{p}+a_{i2}^{p}+\cdots+a_{in}^{p}\right)^{1/p}.\tag{2.11}$$

Obviously, inequalities (2.9), (2.10), and (2.11) immediately yield

$$n^{1-q/p}\cdot m^{q-1}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{q/p}\geq\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p}\right)^{q}, \tag{2.12}$$

which is actually inequality (1.4) with the constant $C=n^{1/q-1/p}\cdot m^{1-1/q}$.

Case 3 ($0<q\leq 1\leq p$). By inequality (2.1) with $r=q$ and $s=p$, for each $j=1,2,\ldots,n$, we obtain

$$\left(a_{1j}^{q}+a_{2j}^{q}+\cdots+a_{mj}^{q}\right)^{p/q}\geq a_{1j}^{p}+a_{2j}^{p}+\cdots+a_{mj}^{p}, \tag{2.13}$$

whence after summation over j, we have

$$\begin{aligned}&\sum_{j=1}^{n}\left(a_{1j}^{q}+a_{2j}^{q}+\cdots+a_{mj}^{q}\right)^{p/q}\\&\quad\geq\sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}^{p}=\sum_{i=1}^{m}\left(a_{i1}^{p}+a_{i2}^{p}+\cdots+a_{in}^{p}\right)=\sum_{i=1}^{m}a_{i}^{p}.\end{aligned} \tag{2.14}$$

By the inequality for power means (of orders $p\geq 1$ and 1), we get

$$\left(\frac{\sum_{i=1}^{m}a_{i}^{p}}{m}\right)^{1/p}\geq\frac{\sum_{i=1}^{m}a_{i}}{m} \tag{2.15}$$

or equivalently

$$\left(\sum_{i=1}^{m}a_{i}^{p}\right)^{1/p}\geq m^{(1/p)-1}\sum_{i=1}^{m}a_{i}=m^{(1/p)-1}\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p}. \tag{2.16}$$

The above inequality and (2.14) immediately yield

$$m^{1-1/p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{q}\right)^{p/q}\right)^{1/p}\geq\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p}, \tag{2.17}$$

as desired.

Case 4 ($0<q\leq p<1$). The proof can be obtained from those of Case 1, by replacing "≥" with "≤" in each related inequality.

Case 5 ($0 < p < q < 1$). If $m \le n$, then the proof is the same as that for Case 6. If $m > n$, then the proof can be obtained from those of Case 2, by replacing "$\ge$" with "$\le$" in each related inequality.

Case 6 ($0 < p < 1 \le q$). For any fixed $j = 1, 2, \ldots, n$, inequality (2.1) of Lemma 2.1 with $r = q$ and $s = p$ gives

$$\left(a_{1j}^q + a_{2j}^q + \cdots + a_{mj}^q\right)^{p/q} \le a_{1j}^p + a_{2j}^p + \cdots + a_{mj}^p, \tag{2.18}$$

whence after summation over j, we get

$$\sum_{j=1}^{n}\left(a_{1j}^q + a_{2j}^q + \cdots + a_{mj}^q\right)^{p/q} \le \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^p = \sum_{i=1}^{m} a_i^p. \tag{2.19}$$

As $1/p > 1$, for positive integers $b_1, b_2, \ldots, b_m$, there holds

$$\frac{\sum_{i=1}^m b_i}{m} \le \left(\frac{\sum_{i=1}^m b_i^{1/p}}{m}\right)^p, \tag{2.20}$$

whence for any fixed $j = 1, 2, \ldots n$, after substitution of $b_i = a_i^p$, $i = 1, 2, \ldots m$, we obtain

$$\left(\sum_{i=1}^{m} a_i^p\right)^{1/p} \le m^{(1/p)-1}\sum_{i=1}^{m} a_i = m^{(1/p)-1}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^p\right)^{1/p}. \tag{2.21}$$

The above inequality and (2.19) immediately yield

$$m^{1-1/p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^q\right)^{p/q}\right)^{1/p} \le \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^p\right)^{1/p}, \tag{2.22}$$

and the proof is completed. □

3. The Integral Analogue of Theorem 1.1

Let (X, Σ, μ) be a measure space with a positive Borel measure μ. For any $0 < p < +\infty$ let $L^p = L^p(\mu)$ denote the usual Lebesgue space consisting of all μ-measurable complex-valued functions $f : X \to \mathbb{C}$ such that

$$\int_X |f|^p d\mu < +\infty. \tag{3.1}$$

Recall that the usual norm $\|\cdot\|_p$ of $f \in L^p$ is defined as $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$ if $p \geq 1$; $\|f\|_p = \int_X |f|^p d\mu$ if $0 < p < 1$.

The following result is the integral analogue of Theorem 1.1.

Theorem 3.1. *For given* $0 < p < \infty$ *let* $u_1, u_2, \ldots, u_m$ *be arbitrary functions in* L^p*. Then, if* $1 \leq p < +\infty$*, we have*

$$\|u_1\|_p + \cdots + \|u_m\|_p \leq m^{1-\min\{1/2,1/p\}} \left\| \sqrt{|u_1|^2 + \cdots + |u_m|^2} \right\|_p. \tag{3.2}$$

If $0 < p < 1$*, then*

$$\|u_1\|_p + \cdots + \|u_m\|_p \geq m^{1-1/p} \left\| \sqrt{|u_1|^2 + \cdots + |u_m|^2} \right\|_p. \tag{3.3}$$

Both inequalities are sharp

For $1 < p \leq 2$ *the equality in* (3.2) and (3.3) is attained if $u_1 = u_2 = \cdots = u_m$ a.e. on X. If $p > 2$ or $0 < p < 1$, then the equality is attained for $u_i = \chi_{E_i}$, where E_i are μ-measurable sets with $i = 1, 2, \ldots, m$, such that $\mu(E_1) = \mu(E_2) = \cdots = \mu(E_n)$ and $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

Proof. The proof of each inequality is completely similar to the corresponding one given in Theorem 1.1 with a fixed $q = 2$. For clarity, we give here only a proof related to the case when $1 \leq p \leq 2$. Applying the inequality between power means of orders $2/p \geq 1$ and 1 to the functions $|u_i|^p$ $(i = 1, \ldots, m)$, we have

$$\left(\sum_{i=1}^{m} |u_i|^2 \right)^{p/2} \geq m^{(p/2)-1} \left(\sum_{i=1}^{m} |u_i|^p \right). \tag{3.4}$$

Integrating the above relation, we obtain

$$\int_X \left(\sum_{i=1}^{m} |u_i|^2 \right)^{p/2} d\mu \geq m^{(p/2)-1} \left(\sum_{i=1}^{m} \int_X |u_i|^p d\mu \right), \tag{3.5}$$

which can be written in the form

$$\begin{aligned} \left\| \sqrt{|u_1|^2 + \cdots + |u_m|^2} \right\|_p &\geq m^{1/2-1/p} \left(\sum_{i=1}^{m} \int_X |u_i|^p d\mu \right)^{1/p} \\ &= \sqrt{m} \left(\frac{\sum_{i=1}^{m} \|u_i\|_p^p}{m} \right)^{1/p} \\ &\geq \sqrt{m} \cdot \frac{\sum_{i=1}^{m} \|u_i\|_p}{m}. \end{aligned} \tag{3.6}$$

Obviously, the above inequality yields (3.2) for $1 < p \leq 2$. □

Corollary 3.2. *Let $p \geq 1$, and let $w = u + iv$ be a complex function in L^p. Then there holds the sharp inequality*

$$\|u\|_p + \|v\|_p \leq 2^{1-\min(1/2,1/p)} \|u + iv\|_p. \tag{3.7}$$

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