Research Article

Characterization of P -Core and Absolute Equivalence of Double Sequences

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The P -core of a double sequence has been defined and it is studied by many authors. In this paper, we have determined two permutations π_1 and π_2 on the set of natural numbers for which $P\text{-core}A_{\pi_1}(x) = P\text{-core}A_{\pi_2}(x)$ for all $x \in \ell_\infty^2$.

1. Introduction and Preliminaries

A double sequence $x = [x_{jk}]_{j,k=0}^\infty$ is said to be convergent in the Pringsheim sense or P -convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > N$ [1]. In this case, we write $P\text{-}\lim x = l$. By c_2 , we mean the space of all P -convergent sequences.

A double sequence x is said to be bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j, k , that is, if

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty. \quad (1.1)$$

By ℓ_∞^2 we will denote the set of all bounded double sequences. We note that in contrast to the case for single sequences, a convergent double sequence needs not to be bounded. So, by c_2^∞ we will denote the space of all real bounded and convergent double sequences.

Let $A = [a_{jk}^{mn}]_{j,k=0}^\infty$ be a four-dimensional infinite matrix of real numbers for all $m, n = 0, 1, \dots$. The sums

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk} \quad (1.2)$$

are called the A -transforms of the double sequence x and we will denote it by $[Ax]$. We say that a sequence x is A -summable to the limit l if the A -transform of x exists for all $m, n = 0, 1, \dots$ and convergent to l in the Pringsheim sense, that is,

$$\lim_{p,q \rightarrow \infty} \sum_{j=0}^p \sum_{k=0}^q a_{jk}^{mn} x_{jk} = y_{mn}, \quad (1.3)$$

$$\lim_{m,n \rightarrow \infty} y_{mn} = l.$$

Móricz and Rhoades [2] have defined almost convergence of a double sequence as follows.

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ of real numbers is said to be almost convergent to a limit l if

$$\lim_{p,q \rightarrow \infty} \left| \frac{1}{pq} \sum_{j=0}^p \sum_{k=0}^q x_{j+s,k+t} - l \right| = 0 \quad (1.4)$$

uniformly in s, t . By f_2 we denote the set of all almost convergent double sequences.

Recall that Knopp's Core of a single bounded sequence x is the closed interval $[\liminf x, \limsup x]$ in [3, page 138]. In the sense of Knopp's Core, P -core of a double sequence was introduced by Patterson as the closed interval $[P\text{-}\liminf x, P\text{-}\limsup x]$ in [4], where the definitions of $P\text{-}\liminf x$ (Pringsheim limit inferior) and $P\text{-}\limsup x$ (Pringsheim limit superior) are given as follows. Let $\alpha_n = \sup_n \{x_{jk} : j, k \geq n\}$ and $\beta_n = \inf_n \{x_{jk} : j, k \geq n\}$. Then

$$P\text{-}\limsup x = \begin{cases} +\infty, & \alpha_n = +\infty \text{ for each } n, \\ \inf_n \alpha_n, & \alpha_n < \infty \text{ for some } n, \end{cases} \quad (1.5)$$

$$P\text{-}\liminf x = \begin{cases} -\infty, & \beta_n = -\infty \text{ for each } n, \\ \sup_n \beta_n, & \beta_n > -\infty \text{ for some } n. \end{cases}$$

After this definition, this concept has been studied by many authors. For example see in [5–11] and the others.

Let $\mathbb{N}$ denote the set of all natural numbers. A bijective function $\pi : \mathbb{N} \rightarrow \mathbb{N}$ is said to be a permutation. In this paper we have determined two permutations π_1 and π_2 for which $P\text{-core}(A\pi_1(x)) = P\text{-core}(A\pi_2(x))$ for all $x \in \ell_{\infty}^2$.

A two-dimensional matrix transformation is said to be regular (see [3, page 64]) if it maps every convergent sequence into a convergent sequence with the same limit. In 1926, Robison presented a four-dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness. A four-dimensional matrix $A = [a_{jk}^{mn}]$ is said to be RH -regular if it maps every bounded P -convergent sequence into a P -convergent sequence with the same P -limit.

Lemma 1.1 (see [12, 13]). *The four-dimensional matrix A is bounded-regular or RH-regular if and only if*

$$\begin{aligned} P - \lim_{m,n} a_{jk}^{mn} &= 0 \quad (j, k = 0, 1, \dots), \\ P - \lim_{m,n} \sum_{j,k} a_{jk}^{mn} &= 1, \\ P - \lim_{m,n} \sum_j |a_{jk}^{mn}| &= 0 \quad (k = 0, 1, \dots), \\ P - \lim_{m,n} \sum_k |a_{jk}^{mn}| &= 0 \quad (j = 0, 1, \dots), \\ \sum_{j,k} |a_{jk}^{mn}| &\leq C < \infty \quad (m, n = 0, 1, \dots). \end{aligned} \tag{1.6}$$

Lemma 1.2 (see [4]). *If A is a four-dimensional matrix, then for all real-valued double sequences x ,*

$$P - \limsup(Ax) \leq P - \limsup(x), \tag{1.7}$$

if and only if A is RH-regular and

$$P - \lim_{m,n} \sum_{j,k} |a_{jk}^{mn}| = 1. \tag{1.8}$$

Now let us state the definition given in [14] for absolutely equivalent RH-regular matrices.

Definition 1.3. Two RH-regular matrices A and B are said to be absolutely equivalent for a given class of sequences $x = [x_{jk}]$ whenever

$$P - \lim(Ax - Bx) = 0. \tag{1.9}$$

This means that Ax and Bx have the same limit or neither Ax nor Bx have a limit but their difference goes to zero.

The following proposition, lemma, and theorem characterizing the relationship between absolutely equivalent matrices and the P -core are given in [14].

Proposition 1.4. *A necessary and sufficient condition for the RH-regular matrices $A = [a_{jk}^{mn}]$ and $B = [b_{jk}^{mn}]$ to be absolutely equivalent for all bounded sequences is that*

$$P - \lim_{m,n} \sum_{j,k} |a_{jk}^{mn} - b_{jk}^{mn}| = 0. \tag{1.10}$$

Lemma 1.5. *If two double sequences $x = [x_{jk}]$ and $y = [y_{jk}]$ are such that $P - \lim_{j,k} |x_{jk} - y_{jk}| = 0$, then $P\text{-core}(x) = P\text{-core}(y)$.*

Theorem 1.6. $P\text{-core}(Ax) \subseteq P\text{-core}(x)$ for all bounded sequences $x = [x_{jk}]$ if and only if A is RH -regular and is absolutely equivalent to a nonnegative matrix $B = [b_{jk}^{mn}]$ for all bounded sequences.

If $A \in (f_2, c_2^\infty)$, then the four-dimensional matrix $A = [a_{jk}^{mn}]$ is said to be strongly RH -regular. In [2] the characterization of strongly RH -regular has been given as follows.

Theorem 1.7. A matrix $A = [a_{jk}^{mn}]$ is strongly RH -regular if and only if A is RH -regular and

$$\begin{aligned} P\text{-}\lim_{m,n} \sum_{j,k} \left| a_{jk}^{mn} - a_{j+1,k}^{mn} \right| &= 0, \\ P\text{-}\lim_{m,n} \sum_{j,k} \left| a_{jk}^{mn} - a_{j,k+1}^{mn} \right| &= 0. \end{aligned} \quad (1.11)$$

2. The Main Results

Theorem 2.1. If $A = [a_{jk}^{mn}]$ is a strongly Rx -regular matrix and π_1, π_2 are two permutations such that

$$1 \leq |\pi_1(k) - \pi_2(k)| \leq M \quad \forall k \in \mathbb{N}, \quad (2.1)$$

then $P\text{-core}(A\pi_1(x)) = P\text{-core}(A\pi_2(x))$ for all $x \in \ell_\infty^2$, where M is a positive integer and

$$A\pi_1(x) = \sum_{j,k} a_{\pi_1(j),\pi_1(k)}^{mn} x_{jk}, \quad A\pi_2(x) = \sum_{j,k} a_{\pi_2(j),\pi_2(k)}^{mn} x_{jk}. \quad (2.2)$$

Proof. In the light of Lemma 1.5, it is enough to show that $P\text{-}\lim |A\pi_1(x) - A\pi_2(x)| = 0$.

Let $m(k) = \min\{\pi_1(k), \pi_2(k)\}$ and $M(k) = \max\{\pi_1(k), \pi_2(k)\}$. Then, it is clear that $|\pi_1(k) - \pi_2(k)| = M(k) - m(k)$ for each $k \in \mathbb{N}$. Now, for $j, k, m, n \in \mathbb{N}$, we can write

$$\begin{aligned} \left| a_{\pi_1(j),\pi_1(k)}^{mn} - a_{\pi_2(j),\pi_2(k)}^{mn} \right| &= \left| a_{M(j),M(k)}^{mn} - a_{m(j),m(k)}^{mn} \right| \\ &\leq \left| a_{M(j),M(k)}^{mn} - a_{M(j)-1,M(k)-1}^{mn} \right| \\ &\quad + \left| a_{M(j)-1,M(k)-1}^{mn} - a_{M(j)-2,M(k)-2}^{mn} \right| \\ &\quad + \cdots + \left| a_{m(j)+1,m(k)+1}^{mn} - a_{m(j),m(k)}^{mn} \right| \\ &= \sum_{r=1}^{M(j)-m(j)} \sum_{s=1}^{M(k)-m(k)} \left| a_{m(j)+r,m(k)+s}^{mn} - a_{m(j)+r-1,m(k)+s-1}^{mn} \right|. \end{aligned} \quad (2.3)$$

On the other hand, since A is strongly RH -regular, by an easy calculation it can be seen that

$$\begin{aligned}
 |A\pi_1(x) - A\pi_2(x)| &\leq \|x\| \sum_{j,k} \left| a_{\pi_1(j),\pi_1(k)}^{mn} - a_{\pi_2(j),\pi_2(k)}^{mn} \right| \\
 &\leq \|x\| \sum_{j,k} \sum_{r=1}^{M(j)-m(j)} \sum_{s=1}^{M(k)-m(k)} \left| a_{m(j)+r,m(k)+s}^{mn} - a_{m(j)+r-1,m(k)+s-1}^{mn} \right| \\
 &\leq \|x\| \sum_{j,k} \sum_{r=1}^M \sum_{s=1}^M \left| a_{m(j)+r,m(k)+s}^{mn} - a_{m(j)+r-1,m(k)+s-1}^{mn} \right| \quad (2.4) \\
 &= \|x\| \sum_{r=1}^{M(j)-m(j)} \sum_{s=1}^{M(k)-m(k)} \sum_{j,k} \left| a_{m(j)+r,m(k)+s}^{mn} - a_{m(j)+r-1,m(k)+s-1}^{mn} \right| \\
 &\leq 4\|x\| \sum_{r=1}^M \sum_{s=1}^M \sum_{j,k} \left| a_{j,k}^{mn} - a_{j+1,k}^{mn} \right|.
 \end{aligned}$$

By the same way, one can also see that

$$|A\pi_1(x) - A\pi_2(x)| \leq 4\|x\| \sum_{r=1}^M \sum_{s=1}^M \sum_{j,k} \left| a_{j,k}^{mn} - a_{j,k+1}^{mn} \right|. \quad (2.5)$$

Now, conditions (1.11) imply that $P - \lim |A\pi_1(x) - A\pi_2(x)| = 0$. This completes the proof. $\square$

Here, let us specialize the permutations π_1 and π_2 . Let $I_r = [2^{r-1}, 2^r - 1] = \{k \in \mathbb{N} : 2^{r-1} \leq k \leq 2^r - 1\}$, $r \in \mathbb{N}$, and π_1 be a permutation on I_1 such that $\pi_1(1) = 1$ and

$$\pi_1(2^{r-1} + z) = \begin{cases} 2^{r-1} + z + 1 & \text{if } z \text{ is even,} \\ 2^{r-1} + z - 1 & \text{if } z \text{ is odd,} \end{cases} \quad (2.6)$$

for $z = 0, 1, 2, \dots, 2^{r-1} - 1$ (see in [15, 16]). Also, let choose the permutation π_2 such that $\pi_2(k) = k$ for all $k \in \mathbb{N}$. Then, we have the following theorem.

Theorem 2.2. *If $A = [a_{j,k}^{mn}]$ is a strongly RH -regular nonnegative matrix and π_1, π_2 are two permutations as above, then $P\text{-core}(A\pi_1(x)) = P\text{-core}(x)$ for all $x \in \ell_\infty^2$.*

Proof. By the same technique used in Theorem 2.1, one can see that $P - \lim |A\pi_1(x) - A\pi_2(x)| = 0$. So, Lemma 1.5 implies that $P\text{-core}(A\pi_1(x)) = P\text{-core}(A\pi_2(x))$. But, since A is an RH -regular nonnegative matrix, $P\text{-core}(A(x)) = P\text{-core}(x)$ for all $x \in \ell_\infty^2$. This step completes the proof. $\square$

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