

## Research Article

# On Complete Convergence for Weighted Sums of $\varphi$ -Mixing Random Variables

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Some results on complete convergence for weighted sums  $\sum_{i=1}^n a_{ni}X_i$  are presented, where  $\{X_n, n \geq 1\}$  is a sequence of  $\varphi$ -mixing random variables and  $\{a_{ni}, n \geq 1, i \geq 1\}$  is an array of constants. They generalize the corresponding results for *i.i.d* sequence to the case of  $\varphi$ -mixing sequence.

## 1. Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of random variables defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ . Let  $n$  and  $m$  be positive integers. Write  $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ . Given  $\sigma$ -algebras  $\mathcal{B}, \mathcal{R}$  in  $\mathcal{F}$ , let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B | A) - P(B)|. \quad (1.1)$$

Define the  $\varphi$ -mixing coefficients by

$$\varphi(n) = \sup_{k \geq 1} \varphi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty\right), \quad n \geq 0. \quad (1.2)$$

*Definition 1.1.* A random variable sequence  $\{X_n, n \geq 1\}$  is said to be a  $\varphi$ -mixing random variable sequence if  $\varphi(n) \downarrow 0$  as  $n \rightarrow \infty$ .

$\varphi$ -mixing random variables were introduced by Dobrushin [1] and many applications have been found. See, for example, Dobrushin [1], Utev [2], and Chen [3] for central limit

theorem, Herrndorf [4] and Peligrad [5] for weak invariance principle, Sen [6, 7] for weak convergence of empirical processes, Shao [8] for almost sure invariance principles, Hu and Wang [9] for large deviations, and so forth. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired.

Throughout the paper, let  $I(A)$  be the indicator function of the set  $A$ . We assume that  $\phi(x)$  is a positive increasing function on  $(0, \infty)$  satisfying  $\phi(x) \uparrow \infty$  as  $x \rightarrow \infty$  and  $\psi(x)$  is the inverse function of  $\phi(x)$ . Since  $\phi(x) \uparrow \infty$ , it follows that  $\psi(x) \uparrow \infty$ . For easy notation, we let  $\phi(0) = 0$  and  $\psi(0) = 0$ .  $a_n = O(b_n)$  denotes that there exists a positive constant  $C$  such that  $|a_n/b_n| \leq C$ .  $C$  denotes a positive constant which may be different in various places.

Let  $\{X, X_n, n \geq 1\}$  be a sequence of *i.i.d.* random variables and let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be an array of constants. The almost sure limiting behavior of weighted sums  $\sum_{i=1}^n a_{ni}X_i$  was studied by many authors; see, for example, Choi and Sung [10], Cuzick [11], Wu [12], and Sung [13, 14], and so forth.

The main purpose of this paper is to extend the complete convergence for weighted sums  $\sum_{i=1}^n a_{ni}X_i$  of *i.i.d.* random variables to the case of  $\varphi$ -mixing random variables.

**Definition 1.2.** A sequence  $\{X_n, n \geq 1\}$  of random variables is said to be stochastically dominated by a random variable  $X$  if there exists a positive constant  $C$ , such that

$$P(|X_n| > x) \leq CP(|X| > x) \quad (1.3)$$

for all  $x \geq 0$  and  $n \geq 1$ .

**Definition 1.3.** A double array  $\{a_{ni}, n \geq 1, i \geq 1\}$  of real numbers is said to be a Toeplitz array if  $\lim_{n \rightarrow \infty} a_{ni} = 0$  for each  $i \geq 1$  and

$$\sum_{i=1}^{\infty} |a_{ni}| \leq C \quad (1.4)$$

for all  $n \geq 1$ , where  $C$  is a positive constant.

**Lemma 1.4.** Let  $\{X_n, n \geq 1\}$  be a sequence of random variables which is stochastically dominated by a random variable  $X$ . For any  $\alpha > 0$  and  $b > 0$ , the following statement holds:

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C \{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}, \quad (1.5)$$

where  $C$  is a positive constant.

**Lemma 1.5** (cf. [15, Lemma 1.2.8]). Let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables. Let  $X \in L_p(\mathcal{F}_1^k)$ ,  $Y \in L_q(\mathcal{F}_{k+n}^\infty)$ ,  $p \geq 1$ ,  $q \geq 1$ , and  $1/p + 1/q = 1$ . Then

$$|EXY - EXEY| \leq 2(\varphi(n))^{1/p} (E|X|^p)^{1/p} (E|Y|^q)^{1/q}. \quad (1.6)$$

**Lemma 1.6** (cf. [8, Lemma 2.2]). Let  $\{X_n, n \geq 1\}$  be a  $\varphi$ -mixing sequence. Put  $T_a(n) = \sum_{i=a+1}^{a+n} X_i$ . Suppose that there exists an array  $\{C_{a,n}\}$  of positive numbers such that

$$ET_a^2(n) \leq C_{a,n} \quad \text{for every } a \geq 0, n \geq 1. \quad (1.7)$$

Then for every  $q \geq 2$ , there exists a constant  $C$  depending only on  $q$  and  $\varphi(\cdot)$  such that

$$E\left(\max_{1 \leq j \leq n} |T_a(j)|^q\right) \leq C \left[ C_{a,n}^{q/2} + E\left(\max_{a+1 \leq i \leq a+n} |X_i|^q\right) \right] \tag{1.8}$$

for every  $a \geq 0$  and  $n \geq 1$ .

**Lemma 1.7.** Let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ .  $q \geq 2$ . Assume that  $EX_n = 0$  and  $E|X_n|^q < \infty$  for each  $n \geq 1$ . Then there exists a constant  $C$  depending only on  $q$  and  $\varphi(\cdot)$  such that

$$E\left(\max_{1 \leq j \leq n} \left| \sum_{i=a+1}^{a+j} X_i \right|^q\right) \leq C \left[ \sum_{i=a+1}^{a+n} E|X_i|^q + \left( \sum_{i=a+1}^{a+n} EX_i^2 \right)^{q/2} \right] \tag{1.9}$$

for every  $a \geq 0$  and  $n \geq 1$ . In particular, one has

$$E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^q\right) \leq C \left[ \sum_{i=1}^n E|X_i|^q + \left( \sum_{i=1}^n EX_i^2 \right)^{q/2} \right] \tag{1.10}$$

for every  $n \geq 1$ .

*Proof.* By Lemma 1.5, we can see that

$$\begin{aligned} E\left(\sum_{i=a+1}^{a+n} X_i\right)^2 &\leq \sum_{i=a+1}^{a+n} EX_i^2 + 4 \sum_{a+1 \leq i < j \leq a+n} \varphi^{1/2}(j-i) (EX_i^2)^{1/2} (EX_j^2)^{1/2} \\ &\leq \sum_{i=a+1}^{a+n} EX_i^2 + 2 \sum_{k=1}^{n-1} \sum_{i=a+1}^{a+n-k} \varphi^{1/2}(k) (EX_i^2 + EX_{k+i}^2) \\ &\leq \left(1 + 4 \sum_{k=1}^{\infty} \varphi^{1/2}(k)\right) \sum_{i=a+1}^{a+n} EX_i^2 \doteq C_{a,n}, \end{aligned} \tag{1.11}$$

which implies (1.7). By Lemma 1.6, we can get the desired result (1.9) immediately. The proof is complete.  $\square$

**Lemma 1.8.** Assume that the inverse function  $\psi(x)$  of  $\phi(x)$  satisfies

$$\psi(n) \sum_{i=1}^n \frac{1}{\psi(i)} = O(n). \tag{1.12}$$

If  $E[\phi(|X|)] < \infty$ , then

$$\sum_{n=1}^{\infty} \frac{1}{\psi(n)} E|X|I(|X| > \psi(n)) < \infty. \tag{1.13}$$

*Proof.* The proof is similar to that of Lemma 1 by Sung [14]. So we omit it.  $\square$

## 2. Main Results and Their Proofs

**Theorem 2.1.** Let  $\{X, X_n, n \geq 1\}$  be a sequence of identically distributed  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ ,  $EX = 0$ ,  $EX^2 < \infty$ , and  $E[\phi(|X|)] < \infty$ . Assume that the inverse function  $\psi(x)$  of  $\phi(x)$  satisfies (1.12). Let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be an array of constants such that

- (i)  $\max_{1 \leq i \leq n} |a_{ni}| = O(1/\psi(n))$ ;
- (ii)  $\sum_{i=1}^n a_{ni}^2 = O(\log^{-1-\alpha} n)$  for some  $\alpha > 0$ .

Then for any  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon\right) < \infty. \quad (2.1)$$

*Proof.* For each  $n \geq 1$ , denote

$$\begin{aligned} X_j^{(n)} &= X_j I(|X_j| \leq \psi(n)), \quad T_j^{(n)} = \sum_{i=1}^j (a_{ni} X_i^{(n)} - E a_{ni} X_i^{(n)}), \quad 1 \leq j \leq n, \\ A &= \bigcap_{i=1}^n (X_i = X_i^{(n)}) = \bigcap_{i=1}^n (|X_i| \leq \psi(n)), \quad B = \bar{A} = \bigcup_{i=1}^n (X_i \neq X_i^{(n)}) = \bigcup_{i=1}^n (|X_i| > \psi(n)), \\ E_n &= \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon \right). \end{aligned} \quad (2.2)$$

It is easy to check that

$$\begin{aligned} \sum_{i=1}^j a_{ni} X_i &= \sum_{i=1}^j a_{ni} X_i I(|X_i| \leq \psi(n)) + \sum_{i=1}^j a_{ni} X_i I(|X_i| > \psi(n)) \\ &= T_j^{(n)} + \sum_{i=1}^j E a_{ni} X_i^{(n)} + \sum_{i=1}^j a_{ni} X_i I(|X_i| > \psi(n)), \\ E_n &= E_n A + E_n B = \left( \max_{1 \leq j \leq n} \left| T_j^{(n)} + \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| > \varepsilon \right) + E_n B \\ &\subset \left( \max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| \right) + B. \end{aligned} \quad (2.3)$$

Therefore

$$\begin{aligned} P(E_n) &\leq P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right|\right) + P(B) \\ &\leq P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right|\right) + \sum_{i=1}^n P(|X_i| > \psi(n)). \end{aligned} \quad (2.4)$$

Firstly, we will show that

$$\max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.5)$$

It follows from Lemma 1.8 and Kronecker's lemma that

$$\frac{1}{\psi(n)} \sum_{i=1}^n E|X|I(|X| > \psi(i)) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

By  $EX = 0$ , condition (i), (2.6), and  $\psi(n) \uparrow \infty$ , we can see that

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| &= \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i I(|X_i| > \psi(n)) \right| \\ &\leq \sum_{i=1}^n E |a_{ni} X_i| I(|X_i| > \psi(n)) \\ &\leq \sum_{i=1}^n |a_{ni}| E|X| I(|X| > \psi(n)) \\ &\leq \frac{1}{\psi(n)} \sum_{i=1}^n E|X| I(|X| > \psi(i)) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (2.7)$$

which implies (2.5). By (2.4) and (2.5), we can see that, for sufficiently large  $n$ ,

$$P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon\right) \leq P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) + \sum_{i=1}^n P(|X_i| > \psi(n)). \quad (2.8)$$

To prove (2.1), it suffices to show that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) &< \infty, \\ \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X_i| > \psi(n)) &< \infty. \end{aligned} \quad (2.9)$$

By Markov's inequality, Lemma 1.7,  $EX^2 < \infty$ , and condition (ii), we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) &\leq C \sum_{n=1}^{\infty} n^{-1} E\left(\max_{1 \leq j \leq n} |T_j^{(n)}|^2\right) \\
 &\leq C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n E|a_{ni} X_i^{(n)}|^2 \\
 &= C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n a_{ni}^2 EX^2 I(|X| \leq \psi(n)) \quad (2.10) \\
 &\leq C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n a_{ni}^2 \\
 &\leq C \sum_{n=1}^{\infty} n^{-1} \log^{-1-\alpha} n < \infty.
 \end{aligned}$$

It follows from  $E[\phi(|X|)] < \infty$  that

$$\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X_i| > \psi(n)) = \sum_{n=1}^{\infty} P(|X| > \psi(n)) = \sum_{n=1}^{\infty} P(\phi(|X|) > n) \leq CE[\phi(|X|)] < \infty. \quad (2.11)$$

We complete the proof of the theorem.  $\square$

**Theorem 2.2.** Let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$  and let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be an array of real numbers. Let  $\{b_n, n \geq 1\}$  be an increasing sequence of positive integers and let  $\{c_n, n \geq 1\}$  be a sequence of positive real numbers. If for some  $q \geq 2, 0 < t < 2$ , and for any  $\varepsilon > 0$ , the following conditions are satisfied:

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P(|a_{ni} X_i| \geq \varepsilon b_n^{1/t}) < \infty, \quad (2.12)$$

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) < \infty, \quad (2.13)$$

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[ \sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) \right]^{q/2} < \infty, \quad (2.14)$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{ \max_{1 \leq i \leq b_n} \left| \sum_{j=1}^i [a_{nj} X_j - a_{nj} EX_j I(|a_{nj} X_j| < \varepsilon b_n^{1/t})] \right| \geq \varepsilon b_n^{1/t} \right\} < \infty. \quad (2.15)$$

*Proof.* Note that if the series  $\sum_{n=1}^{\infty} c_n$  is convergent, then (2.15) holds. Therefore, we will consider only such sequences  $\{c_n, n \geq 1\}$  for which the series  $\sum_{n=1}^{\infty} c_n$  is divergent.

Let

$$\begin{aligned}
 Y_i^{(n)} &= a_{ni}X_iI(|a_{ni}X_i| < \varepsilon b_n^{1/t}), \quad S'_{ni} = \sum_{j=1}^i Y_j^{(n)}, \quad n \geq 1, \quad i \geq 1, \\
 A &= \bigcap_{i=1}^{b_n} \{Y_i^{(n)} = a_{ni}X_i\}, \quad B = \overline{A} = \bigcup_{i=1}^{b_n} \{Y_i^{(n)} \neq a_{ni}X_i\} = \bigcup_{i=1}^{b_n} (|a_{ni}X_i| \geq \varepsilon b_n^{1/t}), \\
 E_n &= \left\{ \max_{1 \leq i \leq b_n} \left| \sum_{j=1}^i [a_{nj}X_j - a_{nj}EX_jI(|a_{nj}X_j| < \varepsilon b_n^{1/t})] \right| \geq \varepsilon b_n^{1/t} \right\}.
 \end{aligned} \tag{2.16}$$

Therefore

$$\begin{aligned}
 &P \left\{ \max_{1 \leq i \leq b_n} \left| \sum_{j=1}^i [a_{nj}X_{nj} - a_{nj}EX_jI(|a_{nj}X_j| < \varepsilon b_n^{1/t})] \right| \geq \varepsilon b_n^{1/t} \right\} \\
 &= P(E_n) = P(E_nA) + P(E_nB) \leq P(E_nA) + P(B) \\
 &\leq \sum_{i=1}^{b_n} P(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}) + \varepsilon^{-q} b_n^{-q/t} E \left( \max_{1 \leq i \leq b_n} |S'_{ni} - ES'_{ni}| \right)^q.
 \end{aligned} \tag{2.17}$$

Using the  $C_r$  inequality and Jensen's inequality, we can estimate  $E|Y_i^{(n)} - EY_i^{(n)}|^q$  in the following way:

$$E|Y_i^{(n)} - EY_i^{(n)}|^q \leq C|a_{ni}|^q E|X_i|^q I(|a_{ni}X_i| < \varepsilon b_n^{1/t}). \tag{2.18}$$

By (2.17), (2.18), and Lemma 1.7, we can get

$$\begin{aligned}
 &P \left\{ \max_{1 \leq i \leq b_n} \left| \sum_{j=1}^i [a_{nj}X_j - a_{nj}EX_jI(|a_{nj}X_j| < \varepsilon b_n^{1/t})] \right| \geq \varepsilon b_n^{1/t} \right\} \\
 &\leq C \sum_{i=1}^{b_n} P(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}) + Cb_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I(|a_{ni}X_i| < \varepsilon b_n^{1/t}) \\
 &\quad + Cb_n^{-q/t} \left[ \sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I(|a_{ni}X_i| < \varepsilon b_n^{1/t}) \right]^{q/2}.
 \end{aligned} \tag{2.19}$$

Therefore, we can conclude that (2.15) holds by (2.12), (2.13), (2.14), and (2.19). □

**Theorem 2.3.** Let  $1 \leq p \leq 2$  and let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ ,  $EX_n = 0$ , and  $E|X_n|^p < \infty$  for  $n \geq 1$ . Let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be an array of real numbers satisfying the following condition:

$$\sum_{i=1}^n |a_{ni}|^p E|X_i|^p = O(n^\delta) \quad \text{as } n \rightarrow \infty \quad (2.20)$$

for some  $0 < \delta \leq 2/q$  and  $q > 2$ . Then for any  $\varepsilon > 0$  and  $ap \geq 1$ ,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq i \leq n} \left| \sum_{j=1}^i a_{nj} X_j \right| \geq \varepsilon n^\alpha\right) < \infty. \quad (2.21)$$

*Proof.* Take  $c_n = n^{\alpha p - 2}$ ,  $b_n = n$ , and  $1/t = \alpha$  in Theorem 2.2. By (2.20) we have

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P(|a_{ni} X_i| \geq \varepsilon b_n^{1/t}) &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2} \sum_{i=1}^n \frac{|a_{ni}|^p E|X_i|^p}{n^{\alpha p}} \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty, \\ \sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) &\leq \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^n |a_{ni}|^p E|X_i|^p \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty, \\ \sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[ \sum_{i=1}^{b_n} a_{ni}^2 E X_i^2 I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) \right]^{q/2} &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha p q / 2} \left( \sum_{i=1}^n |a_{ni}|^p E|X_i|^p \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha p q / 2 + \delta q / 2} \leq C \sum_{n=1}^{\infty} n^{\alpha p (1 - q/2) - 1} < \infty \end{aligned} \quad (2.22)$$

following from  $\delta q/2 \leq 1$ . By the assumption  $EX_n = 0$  for  $n \geq 1$  and (2.20) we get

$$\begin{aligned} \frac{1}{n^\alpha} \max_{1 \leq i \leq n} \left| \sum_{j=1}^i a_{nj} EX_j I(|a_{nj} X_j| < \varepsilon n^\alpha) \right| &\leq \frac{1}{n^\alpha} \sum_{j=1}^n |a_{nj} EX_j I(|a_{nj} X_j| < \varepsilon n^\alpha)| \\ &= \frac{1}{n^\alpha} \sum_{j=1}^n |a_{nj} EX_j I(|a_{nj} X_j| \geq \varepsilon n^\alpha)| \\ &\leq \frac{1}{n^{\alpha p}} \sum_{j=1}^n |a_{nj}|^p E|X_j|^p \leq C n^{\delta - \alpha p} \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned} \quad (2.23)$$

following from  $\delta < 1$  and  $\alpha p \geq 1$ . We get the desired result by Theorem 2.2 immediately. The proof is completed.  $\square$

**Theorem 2.4.** Let  $1 \leq p \leq 2$  and let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ ,  $EX_n = 0$ , and  $E|X_n|^p < \infty$  for  $n \geq 1$ . Assume that the random variables are stochastically dominated by a random variable  $X$  such that  $E|X|^p < \infty$  and let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be an array of real numbers satisfying the following condition:

$$\sum_{i=1}^n |a_{ni}|^p = O(n^\delta) \quad \text{as } n \rightarrow \infty \tag{2.24}$$

for some  $0 < \delta \leq 2/q$  and  $q > 2$ . Then for any  $\varepsilon > 0$  and  $\alpha p \geq 1$ , (2.21) holds.

*Proof.* The proof is similar to that of Theorem 2.3. We only need to note that

$$\begin{aligned} E|X_n|^p &= \int_0^\infty t^p dP(|X_n| \leq t) \\ &= - \int_0^\infty t^p dP(|X_n| > t) \\ &= - \lim_{t \rightarrow \infty} t^p P(|X_n| > t) + \int_0^\infty P(|X_n| > t) dt^p \\ &= 0 + p \int_0^\infty t^{p-1} P(|X_n| > t) dt \\ &\leq Cp \int_0^\infty t^{p-1} P(|X| > t) dt \\ &= CE|X|^p < \infty \end{aligned} \tag{2.25}$$

for each  $n \geq 1$ . □

**Theorem 2.5.** Let  $\{X_n, n \geq 1\}$  be a sequence of  $\varphi$ -mixing random variables satisfying  $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$  and let  $\{a_{ni}, n \geq 1, i \geq 1\}$  be a Toeplitz array. Assume that the random variables are stochastically dominated by a random variable  $X$ . If for some  $0 < t < 2$  and  $\delta > 1/t$ ,

$$\sup_{i \geq 1} |a_{ni}| = O(n^{1/t-\delta}), \quad E|X|^\beta < \infty, \tag{2.26}$$

where  $\beta = \max(2/\delta, 1 + 1/\delta)$ , then for any  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} P\left(\max_{1 \leq i \leq n} \left| \sum_{j=1}^i a_{nj} X_j \right| \geq \varepsilon n^{1/t}\right) < \infty. \tag{2.27}$$

*Proof.* Take  $c_n = 1$ ,  $b_n = n$  for  $n \geq 1$  and  $q \geq \max(2, 1 + 1/\delta)$  in Theorem 2.2. Then we can see that (2.12) and (2.13) are satisfied. In fact, by (1.4) and (2.26) we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P(|a_{ni} X_i| \geq \varepsilon b_n^{1/t}) &= \sum_{n=1}^{\infty} \sum_{i=1}^n P(|a_{ni} X_i| \geq \varepsilon n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|a_{ni} X| \geq C n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|X| \geq C n^\delta) \\
 &= C \sum_{n=1}^{\infty} n \sum_{k=n}^{\infty} P(C k^\delta \leq |X| < C(k+1)^\delta) \\
 &\leq C \sum_{k=1}^{\infty} k^2 P(C k^\delta \leq |X| < C(k+1)^\delta) \\
 &\leq CE|X|^{2/\delta} < \infty,
 \end{aligned} \tag{2.28}$$

and by Lemma 1.4, (1.5), and (2.26) we have

$$\begin{aligned}
 &\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) \\
 &= \sum_{n=1}^{\infty} n^{-q/t} \sum_{i=1}^n |a_{ni}|^q E|X_i|^q I(|a_{ni} X_i| < \varepsilon n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/t} \sum_{i=1}^n |a_{ni}|^q \left[ E|X|^q I(|a_{ni} X| < \varepsilon n^{1/t}) + \frac{n^{q/t}}{|a_{ni}|^q} P(|a_{ni} X| \geq \varepsilon n^{1/t}) \right] \\
 &\leq C \sum_{n=1}^{\infty} n^{-(1+1/\delta)/t} \sum_{i=1}^n |a_{ni}|^{1+1/\delta} E|X|^{1+1/\delta} + C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|a_{ni} X| \geq \varepsilon n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} n^{-1/t-1} E|X|^{1+1/\delta} \sum_{i=1}^n |a_{ni}| + CE|X|^{2/\delta} \\
 &\leq C \sum_{n=1}^{\infty} n^{-1/t-1} + CE|X|^{2/\delta} < \infty.
 \end{aligned} \tag{2.29}$$

In order to prove that (2.14) holds, we should consider the following two cases.

In the case  $\delta > 1$ , by Lemma 1.4, (1.5), (2.26), and  $C_r$  inequality, we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[ \sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) \right]^{q/2} \\
 &= \sum_{n=1}^{\infty} n^{-q/t} \left[ \sum_{i=1}^n a_{ni}^2 EX_i^2 I(|a_{ni} X_i| < \varepsilon n^{1/t}) \right]^{q/2} \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/2t-q/2\delta t} \left( \sum_{i=1}^n |a_{ni}|^{1+1/\delta} E|X_i|^{1+1/\delta} \right)^{q/2} + C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|a_{ni} X_i| \geq \varepsilon n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/2t-q/2\delta t} n^{(1/\delta)(1/t-\delta)(q/2)} (E|X|^{1+1/\delta})^{q/2} \left( \sum_{i=1}^n |a_{ni}| \right)^{q/2} + CE|X|^{2/\delta} \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/2t-q/2} + CE|X|^{2/\delta} \\
 &= C \sum_{n=1}^{\infty} n^{-(q/2)(1+1/t)} + CE|X|^{2/\delta} < \infty.
 \end{aligned} \tag{2.30}$$

In the case  $0 < \delta \leq 1$ , we can get

$$\begin{aligned}
 & \sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[ \sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I(|a_{ni} X_i| < \varepsilon b_n^{1/t}) \right]^{q/2} \\
 &= \sum_{n=1}^{\infty} n^{-q/t} \left[ \sum_{i=1}^n a_{ni}^2 EX_i^2 I(|a_{ni} X_i| < \varepsilon n^{1/t}) \right]^{q/2} \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/t} n^{(1/t-\delta)(q/2)} \left( \sum_{i=1}^n |a_{ni}| EX_i^2 \right)^{q/2} + C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|a_{ni} X_i| \geq \varepsilon n^{1/t}) \\
 &\leq C \sum_{n=1}^{\infty} n^{-q/2t-q\delta/2} (EX^2)^{q/2} \left( \sum_{i=1}^n |a_{ni}| \right)^{q/2} + CE|X|^{2/\delta} \\
 &\leq C \sum_{n=1}^{\infty} n^{-(q/2)(\delta+1/t)} + CE|X|^{2/\delta} < \infty.
 \end{aligned} \tag{2.31}$$

To complete the proof of the theorem, we only need to prove

$$n^{-1/t} \max_{1 \leq i \leq n} \left| \sum_{j=1}^i a_{nj} EX_j I(|a_{nj} X_j| < \varepsilon n^{1/t}) \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{2.32}$$

Indeed, by Lemma 1.4, it follows that

$$\begin{aligned}
 & n^{-1/t} \max_{1 \leq i \leq n} \left| \sum_{j=1}^i a_{nj} EX_j I(|a_{nj} X_j| < \varepsilon n^{1/t}) \right| \\
 & \leq C n^{-1/t} \sum_{j=1}^n |a_{nj}| E|X| + C \sum_{j=1}^n P(|a_{nj} X| \geq \varepsilon n^{1/t}) \\
 & \leq C n^{-1/t} + C \sum_{j=1}^n P(|a_{nj} X| \geq \varepsilon n^{1/t}) \rightarrow 0, \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{2.33}$$

Thus we get the desired result.  $\square$

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