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Research Article

A General Iterative Method of Fixed Points for Mixed Equilibrium Problems and Variational Inclusion Problems

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The purpose of this paper is to investigate the problem of finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusions with set-valued maximal monotone mappings and inverse-strongly monotone mappings, and the set of fixed points of a family of finitely nonexpansive mappings in the setting of Hilbert spaces. We propose a new iterative scheme for finding the common element of the above three sets. Our results improve and extend the corresponding results of the works by Zhang et al. (2008), Peng et al. (2008), Peng and Yao (2009), as well as Plubtieng and Sriprad (2009) and some well-known results in the literature.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm being denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, 2^H denoting the family of all subsets of H and leting C be a closed convex subset of H. A mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$, for all $x, y \in C$. We use $F(S)$ to denote the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. It is assumed throughout the paper that S is a nonexpansive mapping such that $F(S) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is *contraction* on C if there exists a constant $\alpha \in [0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \le \alpha \|x - y\|$. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \ge \overline{\gamma} \|x\|^2 \quad \forall x \in H. \tag{1.1}$$

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. We consider the following *variational inclusion problem*, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.2}$$

where θ is the zero vector in H. The set of solutions of problem (1.2) is denoted by $I(A, M)$.

If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the subdifferential of ϕ, then the variational inclusion problem (1.2) is equivalent to find $u \in H$ such that

$$\langle Au, v - u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall v, y \in H, \tag{1.3}$$

which is called the *mixed quasivariational inequality* (see [1]).

If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.4}$$

then the variational inclusion problem (1.2) is equivalent to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in H. \tag{1.5}$$

This problem is called *Hartman-Stampacchia variational problem* (see [2–4]).

A set-valued mapping $M : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply that $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph of $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies that $f \in Mx$.

Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the *resolvent operator* $J_{M,\lambda}$ that is associate with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \tag{1.6}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive, and 1-inverse-strongly monotone (see [5, 6]).

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [7] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x) \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by $\mathrm{MEP}(F,\varphi)$. We see that x is a solution of problem (1.7) implying that $x \in \operatorname{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi = 0$, then the mixed equilibrium problem (1.7) becomes the following *equilibrium problem* that is to find $x \in C$ such that

$$F(x,y) \geq 0 \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by EP(*F*). Given a mapping $T : C \to H$, let $F(x,y) = \langle Tx, y-x\rangle$ for all $x,y \in C$. Then $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y-z\rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.8). Some methods have been proposed to solve the equilibrium problem (see [8–21]).

In 2008, Zhang et al. [6] introduced an iterative scheme for finding a common element of the set of solutions to the variational inclusion problem with a multivalued maximal monotone mapping and an inverse-strongly monotone mapping and the set of fixed points of nonexpansive mapping in Hilbert spaces. The iterative scheme is $x_0 = x \in H$, and:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n x + (1-\alpha_n) S y_n \end{aligned} \tag{1.9}$$

for all $n \geq 0$. They proved the strong convergence theorem under some mind conditions. In the same year, Peng et al. [22] introduced an iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse-strongly monotone mapping, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping in Hilbert spaces. The sequence $\{x_n\}$ is generated as follows:

$$\begin{gathered} x_1 \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.10}$$

They proved that if $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions, then $\{x_n\}$ converges strongly to $z \in F(S) \cap I(A,M)$, where $z = P_{F(S)\cap I(A,M)} f(z)$.

In 2009, Plubtieng and Sriprad [23] introduced an iterative method for finding a common element of the set of common fixed points of a countable family of nonexpansive mapping, the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse-strongly monotone mappings, and the set of solutions of an equilibrium problem in Hilbert spaces. Starting with an arbitrary $x_1 \in H$, define the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{gathered}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B) S y_n, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{1.11}$$

where A is an inverse-strongly monotone mapping and B is a bounded linear operator on H. They proved that if the sequences $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then $\{x_n\}$ is generated by (1.11) converging strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad \forall x \in F(S) \cap \mathrm{EP}(F) \cap I(A, M), \tag{1.12}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x)\rangle, \tag{1.13}$$

where h is a potential function for γf, that is, $h^1(x) = \gamma f(x)$, for all $x \in H$.

Let $T_i : C \to C$, where $i = 1, 2, \ldots, N$, be a family of finitely nonexpansive mappings. Let the mapping $W_n : C \to C$ be defined by

$$\begin{gathered}
U_{n,0} = I, \\
U_{n,1} = \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1}) I, \\
U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\
\vdots \\
U_{n,N-1} = \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\
W_n = U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I,
\end{gathered} \tag{1.14}$$

where $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\} \in [0, 1]$. Such a mapping W_n is called the *W-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in [24, Lemma 3.1], it is shown that $F(W_n) = \bigcap_{i=1}^{N} F(T_i)$.

The concept of W-mappings was introduced in [25, 26]. It is now one of the main tools in studying convergence of iterative methods for approaching common fixed points of nonlinear mappings; more recent progresses can be found in [24, 27, 28] and the references cited therein.

Following from W-mappings, Peng and Yao [29] introduced iterative schemes based on the extragradient method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of nonexpansive mappings, and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. The sequence $\{x_n\}$ is generated by

$$
\begin{gathered}
x_1 = x \in C, \\
F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda A u_n), \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n P_C(u_n - \lambda A y_n), \quad \forall n \in \mathbb{N},
\end{gathered}
\tag{1.15}
$$

where A is monotone and Lipschitz continuous mapping and B is an inverse-strongly monotone mapping. They proved the weak convergence theorem if the sequences $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions.

In this paper, motivated by the above results and the iterative schemes considered by Zhang et al. in [6], Peng et al. in [22], Peng and Yao in [29], and Plubtieng and Sriprad in [23], we present a new general iterative scheme for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusions with set-valued maximal monotone mapping and inverse-strongly monotone mapping, and the set of fixed points of a family of finitely nonexpansive mappings in the setting of Hilbert spaces. Then, we prove strong convergence theorem under some mind conditions. Furthermore, by using above result, an iterative algorithm for solution of an optimization problem was obtained. The results presented in this paper extend and improve the results of Zhang et al. [6], Peng et al. [22], Peng and Yao [29], Plubtieng and Sriprad [23], and some authors.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence. In a real Hilbert space H, we have

$$
\begin{gathered}
\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \\
\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\end{gathered}
\tag{2.1}
$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$
\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \tag{2.2}
$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$
\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}
$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties:

$$P_C x \in C,$$
$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$
$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H, y \in C$.

Recall that a mapping A of H into itself is called *β-inverse-strongly monotone* if there exists a positive real number β such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{2.5}$$

for all $u, v \in H$. It is obvious that any β-inverse-strongly monotone mapping A is $(1/\beta)$-Lipschitz monotone and continuous mapping. We also have that, for all $u, v \in H$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.\end{aligned} \tag{2.6}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from H into itself.

It is also known that H satisfies the Opial condition [30], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty}\|x_n - x\| < \liminf_{n \to \infty}\|x_n - y\| \tag{2.7}$$

holds for every $y \in H$ with $x \neq y$.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ, and the set C.

(A1) $F(x, x) = 0$ for all $x \in C$.

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$.

(A3) For each $x, y, z \in C, \lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$.

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

(A5) For each $y \in C, x \mapsto F(x, y)$ is weakly upper semicontinuous.

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$, such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z) \tag{2.8}$$

(B2) C is a bounded set.

We need the following lemmas for proving our main result.

Lemma 2.1 (Peng and Yao [31]). *Let C be a nonempty closed convex subset of H. Let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C \right\} \tag{2.9}$$

for all $z \in H$. Then, the following hold.

(1) *For each $x \in H$, $T_r(x) \neq \emptyset$.*

(2) *T_r is single valued.*

(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$.*

(4) *$F(T_r) = MEP(F,\varphi)$.*

(5) $\mathrm{MEP}(F,\varphi)$ *is closed and convex.*

Lemma 2.2 (Xu [32]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.10}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 (Osilike and Igbokwe [33]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.11}$$

Lemma 2.4 (Colao et al. [28]). *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^{N}$ be a family of finitely nonexpansive mappings of C into itself and let $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$ $(i = 1, \ldots, N)$. Moreover for every integer $n \geq 1$, let W and W_n be the W-mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$, respectively. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty} \|W_n x - W x\| = 0. \tag{2.12}$$

Lemma 2.5 (Suzuki [34]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.6 (Marino and Xu [35]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.7 (Brézis [5]). *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

Remark 2.8. Lemma 2.7 implies that $I(A, M)$ is closed and convex if $M : H \to 2^H$ is a maximal monotone mapping and $A : H \to H$ is a Lipschitz continuous mapping.

Lemma 2.9 (Zhang et al. [6]). *$u \in H$ is a solution of variational inclusion* (1.2) *if and only if $u = J_{M,\lambda}(u - \lambda Au)$, for all $\lambda > 0$, that is,*

$$I(A, M) = F(J_{M,\lambda}(I - \lambda A)), \quad \forall\lambda > 0. \tag{2.13}$$

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of a mixed equilibrium problem, and the set of solutions of a variational inclusion problem for an inverse-strongly monotone mapping in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let f be a contraction of H into itself with coefficient $\alpha \in (0, 1)$. Let A be an β-inverse-strongly monotone mapping of H into itself, $M : H \to 2^H$ be a maximal monotone mapping, and let B be a strongly bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into H such that $\Omega := \cap_{n=1}^{N} F(T_i) \cap I(A, M) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$ and let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{gathered}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = J_{M,\lambda}(u_n - \lambda Au_n), \\
v_n = J_{M,\lambda}(y_n - \lambda Ay_n), \\
x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)W_n v_n
\end{gathered} \tag{3.1}$$

for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1), \{r_n\} \subset (0, \infty)$, and $\lambda \in (0, 2\beta)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iv) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega(I - B + \gamma f)(z)$, which is the unique solution of the variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad x \in \Omega. \tag{3.2}$$

Proof. Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < (1-\beta_n)\|B\|^{-1}$ for all n. We assume that $\|I - B\| \leq 1 - \overline{\gamma}$. Since B is linear bounded self-adjoint operator on H, we have

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.3}$$

Observe that

$$\langle ((1-\beta_n)I - \alpha_n B)x, x\rangle = 1 - \beta_n - \alpha_n\langle Bx, x\rangle \geq 1 - \beta_n - \alpha_n\|B\| \geq 0, \tag{3.4}$$

this shows that $(1-\beta_n)I - \alpha_n B$ is positive. It follows that

$$\begin{aligned} \left\|(1-\beta_n)I - \alpha_n B\right\| &= \sup\left\{\left|\langle ((1-\beta_n)I - \alpha_n B)x, x\rangle\right| : x \in H, \|x\| = 1\right\} \\ &= \sup\left\{1 - \beta_n - \alpha_n\langle Bx, x\rangle : x \in H, \|x\| = 1\right\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}. \end{aligned} \tag{3.5}$$

Let $p \in \Omega$, let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1, and let $u_n = T_{r_n}x_n$. For any $n \in \mathbb{N}$, we have

$$\left\|u_n - p\right\| = \left\|T_{r_n}x_n - T_{r_n}p\right\| \leq \left\|x_n - p\right\|. \tag{3.6}$$

Since $p \in \Omega$, we have $p = J_{M,\lambda}(p - \lambda Ap)$. From $J_{M,\lambda}$ and $I - \lambda A$ being nonexpansive, then we have

$$\begin{aligned} \left\|v_n - p\right\| &= \left\|J_{M,\lambda}(y_n - \lambda Ay_n) - J_{M,\lambda}(p - \lambda Ap)\right\| \\ &\leq \left\|(y_n - \lambda Ay_n) - (p - \lambda Ap)\right\| \\ &\leq \left\|y_n - p\right\| \\ &= \left\|J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(p - \lambda Ap)\right\| \\ &\leq \left\|(u_n - \lambda Au_n) - (p - \lambda Ap)\right\| \\ &\leq \left\|u_n - p\right\| \\ &\leq \left\|x_n - p\right\| \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)W_n v_n-p\| \\
&= \|\alpha_n(\gamma f(x_n)-Bp)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n B)(W_n v_n-p)\| \\
&\le \alpha_n\|\gamma f(x_n)-Bp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|v_n-p\| \\
&\le \alpha_n\|\gamma f(x_n)-Bp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\| \\
&\le \alpha_n\|\gamma f(x_n)-\gamma f(p)\|+\alpha_n\|\gamma f(p)-Bp\|+(1-\alpha_n\overline{\gamma})\|x_n-p\| \\
&\le \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|+(1-\alpha_n\overline{\gamma})\|x_n-p\| \\
&= (1-(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|+(\overline{\gamma}-\gamma\alpha)\alpha_n\frac{\|\gamma f(p)-Bp\|}{\overline{\gamma}-\gamma\alpha}
\end{aligned} \tag{3.8}$$

for every $n \in \mathbb{N}$. It follows by mathematical induction that

$$\|x_{n+1}-p\| \le \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-Bp\|}{\overline{\gamma}-\gamma\alpha}\right\}, \quad n \ge 1. \tag{3.9}$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{W_n v_n\}$, $\{f(x_n)\}$, $\{y_n\}$, and $\{v_n\}$, are all bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Observing that $u_n = T_{r_n}x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \operatorname{dom}\varphi$, we get

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0 \quad \forall y \in C, \tag{3.10}$$

$$F(u_{n+1},y)+\varphi(y)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \ge 0 \quad \forall y \in C. \tag{3.11}$$

Take $y = u_{n+1}$ in (3.10) and $y = u_n$ in (3.11); by using condition (A2), we obtain

$$\left\langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \ge 0. \tag{3.12}$$

Thus $\langle u_{n+1}-u_n, u_n-u_{n+1}+x_{n+1}-x_n+(1-r_n/r_{n+1})(u_{n+1}-x_{n+1})\rangle \ge 0$. Without loss of generality, let us assume that there exists a real number c such that $r_n > c$, for all $n \ge 1$. Then, we have

$$\|u_{n+1}-u_n\|^2 \le \|u_{n+1}-u_n\|\left\{\|x_{n+1}-x_n\|+\left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\right\}, \tag{3.13}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\le \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_1, \end{aligned} \tag{3.14}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

On the other hand, again since $J_{M,\lambda}$ and $I - \lambda A$ are nonexpansive, we obtain

$$\begin{aligned} \|v_{n+1} - v_n\| &= \left\| J_{M,\lambda}(y_{n+1} - \lambda A y_{n+1}) - J_{M,\lambda}(y_n - \lambda A y_n) \right\| \\ &\le \left\| (y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n) \right\| \\ &\le \left\| y_{n+1} - y_n \right\| \\ &= \| J_{M,\lambda}(u_{n+1} - \lambda A u_{n+1}) - J_{M,\lambda}(u_n - \lambda A u_n) \| \\ &\le \left\| (u_{n+1} - \lambda A u_{n+1}) - (u_n - \lambda A y_n) \right\| \\ &\le \|u_{n+1} - u_n\|. \end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\|v_{n+1} - v_n\| \le \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_1. \tag{3.16}$$

Define the sequence $\{z_n\}$ by $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$, for all $n \ge 1$. Then, observe that

$$\begin{aligned} z_n &= \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) W_n v_n - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n B) W_n v_n}{1 - \beta_n}. \end{aligned} \tag{3.17}$$

It follows that

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + \left((1-\beta_{n+1})I - \alpha_{n+1}B\right)W_{n+1}v_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n) + \left((1-\beta_n)I - \alpha_n B\right)W_n v_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})W_{n+1}v_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}BW_{n+1}v_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)W_n v_n}{1-\beta_n} + \frac{\alpha_n BW_n v_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - BW_{n+1}v_{n+1}\right) \\
&\quad + \frac{\alpha_n}{1-\beta_n}\left(BW_n v_n - \gamma f(x_n)\right) + W_{n+1}v_{n+1} - W_n v_n.
\end{aligned}
\tag{3.18}
$$

From the definition of W_n, since T_i and $U_{n,i}$, $i = 1,2,\ldots,N$, are nonexpansive, we have

$$
\begin{aligned}
\|W_{n+1}v_n - W_n v_n\| &= \|\lambda_{n+1,N}T_N U_{n+1,N-1}v_n + (1-\lambda_{n+1,N})v_n - \lambda_{n,N}T_N U_{n,N-1}v_n - (1-\lambda_{n,N})v_n\| \\
&\leq |\lambda_{n+1,N} - \lambda_{n,N}|\|v_n\| + \|\lambda_{n+1,N}T_N U_{n+1,N-1}v_n - \lambda_{n,N}T_N U_{n,N-1}v_n\| \\
&\leq |\lambda_{n+1,N} - \lambda_{n,N}|\|v_n\| + \|\lambda_{n+1,N}(T_N U_{n+1,N-1}v_n - T_N U_{n,N-1}v_n)\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}v_n\| \\
&\leq 2M_2|\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N}\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\|,
\end{aligned}
\tag{3.19}
$$

where M_2 is an approximate constant such that $M_2 \geq \max\{\sup_{n\geq 1}\{\|v_n\|\}, \{\sup_{n\geq 1}\{\|T_m U_{n,m-1}v_n\|\} \mid m = 1,2,\ldots,N\}\}$.

Since $0 < \lambda_{n_i} \leq 1$ for all $n \geq 1$ and $i = 1,2,\ldots,N$, we compute

$$
\begin{aligned}
&\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| \\
&\quad = \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n + (1-\lambda_{n+1,N-1})v_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n - (1-\lambda_{n,N-1})v_n\| \\
&\quad \leq |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|v_n\| + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n\| \\
&\quad \leq |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|v_n\| + \|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}v_n - T_{N-1}U_{n,N-2}v_n)\| \\
&\qquad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}v_n\| \\
&\quad \leq 2M_2|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + \|U_{n+1,N-2}v_n - U_{n,N-2}v_n\|.
\end{aligned}
\tag{3.20}
$$

It follows that

$$
\begin{aligned}
\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| &\le 2M_2|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + 2M_2|\lambda_{n+1,N-2} - \lambda_{n,N-2}| \\
&\quad + \|U_{n+1,N-3}v_n - U_{n,N-3}v_n\| \\
&\le 2M_2\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| + \|U_{n+1,1}v_n - U_{n,1}v_n\| \\
&= 2M_2\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\quad + \|\lambda_{n+1,1}T_1v_n + (1-\lambda_{n+1,1})v_n - \lambda_{n,1}T_1v_n - (1-\lambda_{n,1})v_n\| \\
&\le 2M_2\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned}
\tag{3.21}
$$

Substituting (3.21) into (3.19) yields that

$$
\begin{aligned}
\|W_{n+1}v_n - W_nv_n\| &\le 2M_2|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M_2\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\le 2M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|,
\end{aligned}
\tag{3.22}
$$

and hence

$$
\begin{aligned}
\|W_{n+1}v_{n+1} - W_nv_n\| &\le \|W_{n+1}v_{n+1} - W_{n+1}v_n\| + \|W_{n+1}v_n - W_nv_n\| \\
&\le \|v_{n+1} - v_n\| + 2M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned}
\tag{3.23}
$$

Combining (3.16) and (3.23), we obtain

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|BW_nv_n\| + \|\gamma f(x_n)\|\right) \\
&\quad + \|W_{n+1}v_{n+1} - W_nv_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|BW_nv_n\| + \|\gamma f(x_n)\|\right) \\
&\quad + \|v_{n+1} - v_n\| + 2M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|BW_nv_n\| + \|\gamma f(x_n)\|\right) \\
&\quad + \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1 + 2M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned}
\tag{3.24}
$$

So

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|BW_{n+1}v_{n+1}\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|BW_n v_n\| + \left\|\gamma f(x_n)\right\|\right) + \frac{1}{c}|r_{n+1} - r_n|M_1 + 2M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \tag{3.25}$$

Conditions (i)–(iv) imply that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.26}$$

Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.27}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.28}$$

From (ii), (3.14), (3.16), and (3.28), we also have $\|u_{n+1} - u_n\| \to 0$ and $\|v_{n+1} - v_n\| \to 0$ as $n \to \infty$. We note that

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n B\right)W_n v_n - x_n \\
&= \alpha_n\gamma f(x_n) - \alpha_n Bx_n + \alpha_n Bx_n + \beta_n x_n + \left((1-\beta_n)I - \alpha_n B\right)W_n v_n \\
&\quad - \left((1-\beta_n)I - \alpha_n B\right)x_n + \left((1-\beta_n)I - \alpha_n B\right)x_n - x_n \\
&= \alpha_n\left(\gamma f(x_n) - Bx_n\right) + \left((1-\beta_n)I - \alpha_n B\right)(W_n v_n - x_n) \\
&\leq \alpha_n\left(\gamma f(x_n) - Bx_n\right) + \left(1-\beta_n - \alpha_n\overline{\gamma}\right)(W_n v_n - x_n)
\end{aligned} \tag{3.29}$$

It follows that

$$\left(1-\beta_n - \alpha_n\overline{\gamma}\right)\|x_n - W_n v_n\| \leq \alpha_n\left\|\gamma f(x_n) - Bx_n\right\| + \|x_n - x_{n+1}\|, \tag{3.30}$$

From (i), (iii), and (3.28), we obtain

$$\lim_{n\to\infty}\|x_n - W_n v_n\| = 0. \tag{3.31}$$

Next, we shall show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. For any $p \in \Omega$, since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - p\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 &\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}\Big(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2\Big).
\end{aligned} \tag{3.32}$$

It follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.33}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n B)(W_n v_n - p)\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\|v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\|u_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\Big(\|x_n - p\|^2 - \|x_n - u_n\|^2\Big) \\
&= \alpha_n\|\gamma f(x_n) - Bp\|^2 + (1-\alpha_n\overline{\gamma})\|x_n - p\|^2 - (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_n\|^2.
\end{aligned} \tag{3.34}$$

Then, we obtain

$$\begin{aligned}
(1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_n\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + (1-\alpha_n\overline{\gamma})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.35}$$

By (i), (iii), and (3.28) imply that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.36}$$

Since $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.37}$$

We note that, by (3.34), nonexpansiveness of $J_{M,\lambda}$ and the inverse-strong monotonicity of A imply that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\|v_n - p\|^2 \\
&= \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|J_{M,\lambda}(y_n - \lambda Ay_n) - J_{M,\lambda}(p - \lambda Ap)\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|(y_n - \lambda Ay_n) - (p - \lambda Ap)\| \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\overline{\gamma})\left\{\|y_n - p\|^2 + \lambda(\lambda-2\beta)\|Ay_n - Ap\|^2\right\} \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\lambda(\lambda-2\beta)\|Ay_n - Ap\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\lambda(\lambda-2\beta)\|Ay_n - Ap\|^2.
\end{aligned} \tag{3.38}$$

It follows from (i), (iii), and (3.28) that

$$\begin{aligned}
0 &\le (1-\beta_n-\alpha_n\overline{\gamma})\lambda(2\beta-\lambda)\|Ay_n - Ap\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \longrightarrow 0,
\end{aligned} \tag{3.39}$$

which implies that

$$\|Ay_n - Ap\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.40}$$

On the other hand, since $J_{M,\lambda}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|v_n - p\|^2 &= \|J_{M,\lambda}(y_n - \lambda A y_n) - J_{M,\lambda}(p - \lambda A p)\|^2 \\
&\le \langle (y_n - \lambda A y_n) - (p - \lambda A p), v_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|(y_n - \lambda A y_n) - (p - \lambda A p)\|^2 + \|v_n - p\|^2 \\
&\qquad - \|(y_n - \lambda A y_n) - (p - \lambda A p) - (v_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|y_n - p\|^2 + \|v_n - p\|^2 - \|(y_n - v_n) - \lambda(A y_n - A p)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|y_n - p\|^2 + \|v_n - p\|^2 - \|y_n - v_n\|^2 \\
&\qquad + 2\lambda\langle y_n - v_n, A y_n - A p\rangle - \lambda^2\|A y_n - A p\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|y_n - p\|^2 + \|v_n - p\|^2 - \|y_n - v_n\|^2 \\
&\qquad + 2\lambda\|y_n - v_n\|\|A y_n - A p\| - \lambda^2\|A y_n - A p\|^2 \Big\},
\end{aligned} \tag{3.41}$$

which yields that

$$\|v_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda\|y_n - v_n\|\|A y_n - A p\|. \tag{3.42}$$

From (3.34) and (3.42), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})\|v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\Big\{ \|y_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda\|y_n - v_n\|\|A y_n - A p\| \Big\} \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\|^2 \\
&\quad - (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - v_n\|^2 \\
&\quad + 2\lambda(1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - v_n\|\|A y_n - A p\| \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - v_n\|^2 \\
&\quad + 2\lambda(1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - v_n\|\|A y_n - A p\|.
\end{aligned} \tag{3.43}$$

Hence, we get

$$\begin{aligned}(1-\beta_n-\alpha_n\overline{\gamma})\left\|y_n-v_n\right\|^2 &\leq \alpha_n\left\|\gamma f(x_n)-Bp\right\|^2+\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\\ &\quad+2\lambda(1-\beta_n-\alpha_n\overline{\gamma})\left\|y_n-v_n\right\|\left\|Ay_n-Ap\right\|\\ &\leq \alpha_n\left\|\gamma f(x_n)-Bp\right\|^2+\|x_n-x_{n+1}\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)\\ &\quad+2\lambda(1-\beta_n-\alpha_n\overline{\gamma})\left\|y_n-v_n\right\|\left\|Ay_n-Ap\right\|.\end{aligned} \tag{3.44}$$

By (i), (iii), (3.28), and (3.40), we have

$$\lim_{n\to\infty}\left\|y_n-v_n\right\|=0. \tag{3.45}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\left\|Au_n-Ap\right\|=0. \tag{3.46}$$

By the same idea in (3.42), (3.45) and using (3.46), then we obtain that

$$\lim_{n\to\infty}\left\|u_n-y_n\right\|=0. \tag{3.47}$$

From

$$\|W_nv_n-v_n\|\leq\|W_nv_n-x_n\|+\|x_n-u_n\|+\left\|u_n-y_n\right\|+\left\|y_n-v_n\right\|, \tag{3.48}$$

hence

$$\lim_{n\to\infty}\|W_nv_n-v_n\|=0, \tag{3.49}$$

and also

$$\|v_n-x_n\|\leq\left\|v_n-y_n\right\|+\left\|y_n-u_n\right\|+\|u_n-x_n\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.50}$$

Observe that $P_\Omega(I-B+\gamma f)$ is a contraction of H into itself. Indeed, for all $x,y\in H$, we have

$$\begin{aligned}\left\|P_\Omega(I-B+\gamma f)(x)-P_\Omega(I-B+\gamma f)(y)\right\| &\leq\left\|(I-B+\gamma f)(x)-(I-B+\gamma f)(y)\right\|\\ &\leq\|I-B\|\left\|x-y\right\|+\gamma\left\|f(x)-f(y)\right\|\\ &\leq(1-\overline{\gamma})\left\|x-y\right\|+\gamma\alpha\left\|x-y\right\|\\ &=(1-(\overline{\gamma}-\gamma\alpha))\left\|x-y\right\|.\end{aligned} \tag{3.51}$$

Since H is complete, there exists a unique fixed point $z \in H$ such that $z = P_{\Omega}(I - B + \gamma f)z$. Next, we show that

$$\limsup_{n\to\infty} \langle (B - \gamma f)z, z - x_n \rangle \leq 0. \tag{3.52}$$

Indeed, we can choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i\to\infty} \left\langle (B - \gamma f)z, z - v_{n_i} \right\rangle = \limsup_{n\to\infty} \langle (B - \gamma f)z, z - v_n \rangle. \tag{3.53}$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $v \in C$. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup v$. From $\|W_n v_n - v_n\| \to 0$, we obtain $W_n v_{n_i} \rightharpoonup v$. Let us show that $v \in \mathrm{MEP}(F, \varphi)$. Since $u_n = T_{r_n} x_n \in \operatorname{dom} \varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.54}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n), \quad \forall y \in C, \tag{3.55}$$

and hence

$$\varphi(y) - \varphi(u_n) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.56}$$

From $\|x_n - u_n\| \to 0$, $\|x_n - W_n v_n\| \to 0$, and $\|W_n v_n - v_n\| \to 0$, we get $u_{n_i} \rightharpoonup v$. Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$, it follows by (A4) and the weakly lower semicontinuity of φ that

$$F(y, v) + \varphi(v) - \varphi(y) \leq 0, \quad \forall y \in C. \tag{3.57}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)v$. Since $y \in C$ and $v \in C$, we have $y_t \in C$, and hence $F(y_t, v) + \varphi(v) - \varphi(y_t) \leq 0$. So, from (A1), (A4), and the convexity of φ, we have

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1 - t)F(y_t, v) + t\varphi(y) + (1 - t)\varphi(v) - \varphi(y_t) \\ &\leq t\big(F(y_t, y) + \varphi(y) - \varphi(y_t)\big). \end{aligned} \tag{3.58}$$

Dividing by t, we get $F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semicontinuity of φ, we have $F(v, y) + \varphi(y) - \varphi(v) \geq 0$ for all $y \in C$, and hence $v \in \mathrm{MEP}(F, \varphi)$.

Next, we show that $v \in F(W_n) = \bigcap_{n=1}^{N} F(T_i)$. Assume that $v \notin \bigcap_{n=1}^{N} F(T_i)$. Since $v_{n_i} \rightharpoonup v$ and $W_n v \neq v$, from Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|v_{n_i} - v\| &< \liminf_{i\to\infty}\|v_{n_i} - W_n v\| \\ &\leq \liminf_{i\to\infty}(\|v_{n_i} - W_n v_{n_i}\| + \|W_n v_{n_i} - W_n v\|) \\ &\leq \liminf_{i\to\infty}\|v_{n_i} - v\|,\end{aligned} \tag{3.59}$$

which is a contradiction. Thus, we obtain $v \in F(W_n) = \bigcap_{n=1}^{N} F(T_i)$.

Next, we show that $v \in I(A, M)$. In fact, having A as β-inverse-strongly monotone, implies that A is $(1/\beta)$-Lipschitz continuous monotone mapping and that domain of A is equal to H. It follows from Lemma 2.7 that $M + A$ is a maximal monotone. Let $(y, g) \in G(M + A)$, that is, $g - Ay \in M(y)$. Since $v_{n_i} = J_{M,\lambda}(y_{n_i} - \lambda A y_{n_i})$, we have $y_{n_i} - \lambda A y_{n_i} \in (I + \lambda M)(v_{n_i})$, that is,

$$\frac{1}{\lambda}(y_{n_i} - v_{n_i} - \lambda A y_{n_i}) \in M(v_{n_i}). \tag{3.60}$$

With $M + A$ being a maximal monotone, we have

$$\left\langle y - v_{n_i}, g - Ay - \frac{1}{\lambda}(y_{n_i} - v_{n_i} - \lambda A y_{n_i})\right\rangle \geq 0, \tag{3.61}$$

and so

$$\begin{aligned}\langle y - v_{n_i}, g\rangle &\geq \left\langle y - v_{n_i}, Ay + \frac{1}{\lambda}(y_{n_i} - v_{n_i} - \lambda A y_{n_i})\right\rangle \\ &= \left\langle y - v_{n_i}, Ay - Av_{n_i} + Av_{n_i} - Ay_{n_i} + \frac{1}{\lambda}(y_{n_i} - v_{n_i})\right\rangle \\ &\geq 0 + \langle y - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle + \left\langle y - v_{n_i}, \frac{1}{\lambda}(y_{n_i} - v_{n_i})\right\rangle.\end{aligned} \tag{3.62}$$

It follows from $\|y_n - v_n\| \to 0$, $\|Ay_n - Av_n\| \to 0$, and $v_{n_i} \rightharpoonup v$ that

$$\lim_{i\to\infty}\langle y - v_{n_i}, g\rangle = \langle y - v, g\rangle \geq 0. \tag{3.63}$$

It follows from the maximal monotonicity of $M + A$ that $0 \in (M + A)(v)$, that is, $v \in I(A, M)$. This implies that $v \in \Omega$.

Since $z = P_\Omega(I - B + \gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle (B - \gamma f)z, z - x_n\rangle &= \limsup_{n\to\infty}\langle (B - \gamma f)z, z - v_n\rangle \\ &= \lim_{i\to\infty}\langle (B - \gamma f)z, z - v_{n_i}\rangle = \langle (B - \gamma f)z, z - v\rangle \leq 0.\end{aligned} \tag{3.64}$$

By (3.49), (3.50), and the last inequality, we have

$$\limsup_{n\to\infty}\langle \gamma f z - Bz, W_n v_n - z\rangle \leq 0. \tag{3.65}$$

Finally, we show that $\{x_n\}$ converges strongly to z. Indeed, from (3.1), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\alpha_n\gamma f(x_n)+\beta_n x_n+\big((1-\beta_n)I-\alpha_n B\big)W_n v_n - z\right\|^2\\
&= \left\|\alpha_n(\gamma f(x_n)-Bz)+\beta_n(x_n-z)+\big((1-\beta_n)I-\alpha_n B\big)(W_n v_n - z)\right\|^2\\
&= \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\left\|\beta_n(x_n-z)+\big((1-\beta_n)I-\alpha_n B\big)(W_n v_n - z)\right\|^2\\
&\quad + 2\langle \beta_n(x_n-z)+\big((1-\beta_n)I-\alpha_n B\big)(W_n v_n-z), \alpha_n\big(\gamma f(x_n)-Bz\big)\rangle\\
&\leq \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\big(\beta_n\|x_n-z\|+\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\|v_n-z\|\big)^2\\
&\quad + 2\alpha_n\beta_n\langle x_n-z,\gamma f(x_n)-Bz\rangle+2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(x_n)-Bz\rangle\\
&\leq \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\big(\beta_n\|x_n-z\|+\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\|x_n-z\|\big)^2\\
&\quad + 2\alpha_n\beta_n\langle x_n-z,\gamma f(x_n)-\gamma f(z)\rangle+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad + 2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(x_n)-\gamma f(z)\rangle\\
&\quad + 2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(z)-Bz\rangle\\
&\leq \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\big(1-\alpha_n\overline{\gamma}\big)^2\|x_n-z\|^2\\
&\quad + 2\alpha_n\beta_n\gamma\|x_n-z\|\left\|f(x_n)-f(z)\right\|+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad + 2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\gamma\|W_n v_n-z\|\left\|f(x_n)-f(z)\right\|\\
&\quad + 2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(z)-Bz\rangle\\
&\leq \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\big(1-\alpha_n\overline{\gamma}\big)^2\|x_n-z\|^2\\
&\quad + 2\alpha_n\beta_n\gamma\alpha\|x_n-z\|^2+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad + 2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\gamma\alpha\|x_n-z\|^2+2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(z)-Bz\rangle\\
&= \alpha_n^2\left\|\gamma f(x_n)-Bz\right\|^2+\Big(1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2+2\alpha_n\gamma\alpha-2\alpha_n^2\overline{\gamma}\gamma\alpha\Big)\|x_n-z\|^2\\
&\quad + 2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle+2\alpha_n\big(1-\beta_n-\alpha_n\overline{\gamma}\big)\langle W_n v_n-z,\gamma f(z)-Bz\rangle\\
&= \Big(1-\alpha_n\Big(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\Big)\Big)\|x_n-z\|^2+\alpha_n\sigma_n,
\end{aligned} \tag{3.66}$$

where $\sigma_n = \alpha_n\|\gamma f(x_n)-Bz\|^2+2\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle+2(1-\beta_n-\alpha_n\overline{\gamma})\langle W_n v_n-z,\gamma f(z)-Bz\rangle$. By (3.65), we get $\limsup_{n\to\infty}\sigma_n \leq 0$. Hence by Lemma 2.2 to (3.66), we conclude that $x_n \to z$. This completes the proof. □

Using Theorem 3.1, we obtain the following corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$. Let A be an β-inverse-strongly monotone mapping of H into itself and let $M : H \to 2^H$ be a maximal monotone mapping such that $\Theta := F(T) \cap I(A,M) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) T J_{M,\lambda}(y_n - \lambda A y_n) \end{gathered} \tag{3.67}$$

for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{r_n\} \subset (0,\infty)$, and $\lambda \in (0,2\beta)$ satisfy the conditions (i)–(iii) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $z = P_\Theta \gamma f(z)$.

Proof. Taking $T_i = T$ for $i = 1,2,\ldots,N$, $B = I$, and $\varphi = 0$, in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$. Let A be an β-inverse-strongly monotone mapping of C into H and let B be a strongly bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into H such that $\Upsilon := \bigcap_{n=1}^{N} F(T_i) \cap \mathrm{VI}(C,A) \cap \mathrm{MEP}(F,\varphi) \neq \emptyset$ and let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda A u_n), \\ v_n = P_C(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) W_n v_n \end{gathered} \tag{3.68}$$

For every $n \geq 1$. where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{r_n\} \subset (0,\infty)$, and $\lambda \in (0,2\beta)$ satisfy the condition (i)–(iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $z \in \Upsilon$ which is the unique solution of the variational inequality

$$\langle (B - \gamma f) z, z - x\rangle \leq 0, \quad x \in \Upsilon. \tag{3.69}$$

Equivalently, one has $z = P_\Upsilon(I - B + \gamma f)(z)$.

Proof. From Theorem 3.1 put $M = \partial\delta_C$; then $J_{M,\lambda} = P_C$. So we have $y_n = P_C(u_n - \lambda A u_n)$ and $v_n = P_C(y_n - \lambda A y_n)$. The conclusion of Corollary 3.3 can be obtained from Theorem 3.1 immediately. □

4. Application

In this section, we study a kind of optimization problem by using the result of this paper. We will give an iterative algorithm of solution for the following optimization problem with nonempty set of solutions:

$$\min h(x), \quad x \in C, \tag{4.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined convex subset C of a Hilbert space H. We denote by $M(h)$ the set of solutions of (4.1). Let $F : C \times C \to \mathbb{R}$ be a bifunction defined by $F(x, y) = h(y) - h(x)$. We consider the equilibrium problem (1.8); it is obvious that $\mathrm{EP}(F) = \mathrm{Min}(h)$. Therefore, from Theorem 3.1, we give the following corollary.

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $h : C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$. Let A be an β-inverse-strongly monotone mapping of H into itself, let $M : H \to 2^H$ be a maximal monotone mapping, and let B be a strongly bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into H such that $\Phi := \bigcap_{n=1}^{N} F(T_i) \cap I(A, M) \cap \mathrm{Min}(h) \neq \emptyset$ and let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{aligned} h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ v_n = J_{M,\lambda}(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B) W_n v_n \end{aligned} \tag{4.2}$$

for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{r_n\} \subset (0, \infty)$, and $\lambda \in (0, 2\beta)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(iv) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then $\{x_n\}$ converges strongly to $z \in \Phi$, where $z = P_\Phi(I - B + \gamma f)(z)$, which is the unique solution of the variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad x \in \Phi. \tag{4.3}$$

Proof. From Theorem 3.1 put $F(u_n, y) = h(y) - h(u_n)$ and $\varphi \equiv 0$. The conclusion of Corollary 4.1 can be obtained from Theorem 3.1 immediately. □

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