Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 358986, 13 pages
doi:10.1155/2010/358986

Research Article

On Carlitz's Type q-Euler Numbers Associated with the Fermionic P-Adic Integral on $\mathbb{Z}_p$

Min-Soo Kim,[1] Taekyun Kim,[2] and Cheon-Seoung Ryoo[3]

[1] *Department of Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Guseong-dong, Yuseong-Gu, Daejeon 305-701, Republic of Korea*
[2] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea*

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

Received 2 August 2010; Accepted 28 September 2010

Academic Editor: Jewgeni Dshalalow

We consider the following problem in the paper of Kim et al. (2010): "Find Witt's formula for Carlitz's type q-Euler numbers." We give Witt's formula for Carlitz's type q-Euler numbers, which is an answer to the above problem. Moreover, we obtain a new p-adic q-l-function $l_{p,q}(s,\chi)$ for Dirichlet's character χ, with the property that $l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}$ for $n = 0, 1, \ldots$ using the fermionic p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Throughout this paper, let p be an odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the rings of p-adic integers, the field of p-adic numbers, and the field of p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p = p^{-1}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$.

As the definition of q-number, we use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Note that $\lim_{q\to 1}[x]_q = x$ for $x \in \mathbb{Z}_p$, where q tends to 1 in the region $0 < |q-1|_p < 1$.

When one talks of q-analogue, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q = 1 + t \in \mathbb{C}_p$, one normally assumes

$|t|_p < 1$. We will further suppose that $\mathrm{ord}_p(t) > 1/(p-1)$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$.

After Carlitz [1, 2] gave q-extensions of the classical Bernoulli numbers and polynomials, the q-extensions of Bernoulli and Euler numbers and polynomials have been studied by several authors (cf. [1–21]). The Euler numbers and polynomials have been studied by researchers in the field of number theory, mathematical physics, and so on (cf. [1, 2, 9, 11, 13–16, 22, 23]). Recently, various q-extensions of these numbers and polynomials have been studied by many mathematicians (cf. [6–8, 10, 12, 17, 18, 20]). Also, some authors have studied in the several area of q-theory (cf. [3, 4, 16, 19, 24]).

It is known that the generating function of Euler numbers $F(t)$ is given by

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.2}$$

From (1.2), we know the recurrence formula of Euler numbers is given by

$$E_0 = 1, \qquad (E+1)^n + E_n = 0 \quad \text{if } n > 0, \tag{1.3}$$

with the usual convention of replacing E^n by E_n (see [7, 18]).

In [17], the q-extension of Euler numbers $E^*_{n,q}$ are defined as

$$E^*_{0,q} = 1, \qquad (qE^* + 1)^n + E^*_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{1.4}$$

with the usual convention of replacing $(E^*)^n$ by $E^*_{n,q}$.

As the same motivation of the construction in [18], Carlitz's type q-Euler numbers $E_{n,q}$ are defined as

$$E_{0,q} = \frac{2}{[2]_q}, \qquad q(qE+1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{1.5}$$

with the usual convention of replacing E^n by $E_{n,q}$. It was shown that $\lim_{q \to 1} E_{n,q} = E_n$, where E_n is the nth Euler number. In the complex case, the generating function of Carlitz's type q-Euler numbers $F_q(t)$ is given by

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} (-q)^n e^{[n]_q t}, \tag{1.6}$$

where q is a complex number with $|q| < 1$ (see [18]). The remark point is that the series on the right-hand side of (1.6) is uniformly convergent in the wider sense. In p-adic case, Kim et al. [18] could not determine the generating function of Carlitz's type q-Euler numbers and Witt's formula for Carlitz's type q-Euler numbers.

In this paper, we obtain the generating function of Carlitz's type q-Euler numbers in the p-adic case. Also, we give Witt's formula for Carlitz's type q-Euler numbers, which is a partial answer to the problem in [18]. Moreover, we obtain a new p-adic q-l-function $l_{p,q}(s,\chi)$ for Dirichlet's character χ, with the property that

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p,} \tag{1.7}$$

for $n \in \mathbb{Z}^+$ using the fermionic p-adic integral on $\mathbb{Z}_p$.

2. Carlitz's Type q-Euler Numbers in the p-Adic Case

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. Then, the p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(a)d\mu_q(a) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a)q^a, \tag{2.1}$$

(cf. [5–17, 19, 20, 22]). The bosonic p-adic integral on $\mathbb{Z}_p$ is considered as the limit $q \to 1$, that is,

$$I_1(f) = \int_{\mathbb{Z}_p} f(a)d\mu_1(a). \tag{2.2}$$

From (2.1), we have the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \int_{\mathbb{Z}_p} f(a)d\mu_{-1}(a). \tag{2.3}$$

Using (2.3), we can readily derive the classical Euler polynomials, $E_n(x)$, namely

$$2\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{2.4}$$

In particular, when $x = 0$, $E_n(0) = E_n$ is the well-known the Euler numbers (cf. [7, 16, 19]).

By definition of $I_{-1}(f)$, we show that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{2.5}$$

where $f_1(x) = f(x+1)$ (see [7]). By (2.5) and induction, we obtain

$$I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2\sum_{i=0}^{n-1}(-1)^{n-i-1}f(i), \tag{2.6}$$

where $n = 1, 2, \ldots$ and $f_n(x) = f(x+n)$. From (2.6), we note that

$$I_{-1}(f_n) + I_{-1}(f) = 2\sum_{i=0}^{n-1}(-1)^i f(i) \quad \text{if } n \text{ is odd}$$
$$I_{-1}(f_n) - I_{-1}(f) = 2\sum_{i=0}^{n-1}(-1)^{i+1} f(i) \quad \text{if } n \text{ is even}. \tag{2.7}$$

For $x \in \mathbb{Z}_p$ and any integer $i \geq 0$, we define

$$\binom{x}{i} = \begin{cases} \dfrac{x(x-1)\cdots(x-i+1)}{i!} & \text{if } i \geq 1, \\ 1, & \text{if } i = 0. \end{cases} \tag{2.8}$$

It is easy to see that $\binom{x}{i} \in \mathbb{Z}_p$ (see [23, page 172]). We put $x \in \mathbb{C}_p$ with $\mathrm{ord}_p(x) > 1/(p-1)$ and $|1-q|_p < 1$. We define q^x for $x \in \mathbb{Z}_p$ by

$$q^x = \sum_{i=0}^{\infty}\binom{x}{i}(q-1)^i, \qquad [x]_q = \sum_{i=1}^{\infty}\binom{x}{i}(q-1)^{i-1}. \tag{2.9}$$

If we set $f(x) = q^x$ in (2.7), we have

$$I_{-1}(q^x) = \frac{2}{q^n+1}\sum_{i=0}^{n-1}(-1)^i q^i = \frac{2}{q+1} \quad \text{if } n \text{ is odd}$$
$$I_{-1}(q^x) = \frac{2}{q^n-1}\sum_{i=0}^{n-1}(-1)^{i+1} q^i = \frac{2}{q+1} \quad \text{if } n \text{ is even}. \tag{2.10}$$

From (2.10), we note that if $f(x) = q^x$, then $I_{-1}(q^x) = 2/(q+1)$, hence there is no need to consider both (odd and even) cases. Thus, for each $l \in \mathbb{N}$, we obtain $I_{-1}(q^{lx}) = 2/(q^l+1)$. Therefore, we have

$$\begin{aligned} I_{-1}\left(q^x[x]_q^n\right) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l I_{-1}\left(q^{(l+1)x}\right) \\ &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{2}{q^{l+1}+1}. \end{aligned} \tag{2.11}$$

Also, if $f(x) = q^{lx}$ in (2.5), then

$$I_{-1}\left(q^{l(x+1)}\right) + I_{-1}\left(q^{lx}\right) = 2f(0) = 2. \tag{2.12}$$

On the other hand, by (2.12), we obtain that

$$\begin{aligned} I_{-1}\left(q^{x+1}[x+1]_q^n\right)+I_{-1}\left(q^x[x]_q^n\right) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(I_{-1}\left(\left(q^{l+1}\right)^{x+1}\right)+I_{-1}\left(\left(q^{l+1}\right)^{x}\right)\right) \\ &= \frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l=0 \end{aligned} \tag{2.13}$$

is equivalent to

$$\begin{aligned} 0 &= I_{-1}\left(q^{x+1}[x+1]_q^n\right)+I_{-1}\left(q^x[x]_q^n\right) \\ &= qI_{-1}\left(q^x(1+q[x]^n)\right)+I_{-1}\left(q^x[x]_q^n\right) \\ &= qI_{-1}\left(q^x\sum_{l=0}^{n}\binom{n}{l}q^l[x]^l\right)+I_{-1}\left(q^x[x]_q^n\right) \\ &= q\sum_{l=0}^{n}\binom{n}{l}q^lI_{-1}\left(q^x[x]^l\right)+I_{-1}\left(q^x[x]_q^n\right). \end{aligned} \tag{2.14}$$

From the definition of fermionic p-adic integral on $\mathbb{Z}_p$ and (2.11), we can derive

$$\begin{aligned} I_{-1}\left(q^x[x]_q^n\right) &= \int_{\mathbb{Z}_p}[x]_q^nq^x\,d\mu_{-1}(x) \\ &= \lim_{N\to\infty}\sum_{a=0}^{p^N-1}\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^iq^{ia}(-q)^a \\ &= \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\lim_{N\to\infty}\sum_{a=0}^{p^N-1}(-1)^a\left(q^{i+1}\right)^a \\ &= \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{2}{1+q^{i+1}} \end{aligned} \tag{2.15}$$

is equivalent to

$$\begin{aligned} \sum_{n=0}^{\infty}I_{-1}\left(q^x[x]_q^n\right)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{2}{1+q^{i+1}}\frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty}(-q)^ne^{[n]_qt}. \end{aligned} \tag{2.16}$$

From (2.12), (2.13), (2.14), (2.15), and (2.16), it is easy to show that

$$q\sum_{l=0}^{n}\binom{n}{l}q^{l}E_{l,q}+E_{n,q}=\begin{cases}2 & \text{if } n=0,\\ 0 & \text{if } n>0,\end{cases}\tag{2.17}$$

where $E_{n,q}$ are Carlitz's type q-Euler numbers defined by (see [18])

$$F_q(t)=2\sum_{n=0}^{\infty}(-q)^{n}e^{[n]_q t}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}.\tag{2.18}$$

Therefore, we obtain the recurrence formula for the Carlitz's type q-Euler numbers as follows:

$$q(qE+1)^{n}+E_{n,q}=\begin{cases}2 & \text{if } n=0,\\ 0 & \text{if } n>0,\end{cases}\tag{2.19}$$

with the usual convention of replacing E^n by $E_{n,q}$. Therefore, by (2.16), (2.18), and (2.19), we obtain the following theorem, which is a partial answer to the problem in [18].

Theorem 2.1 (Witt's formula for $E_{n,q}$). *For $n \in \mathbb{Z}^{+}$,*

$$E_{n,q}=\frac{1}{(1-q)^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\frac{2}{1+q^{i+1}}=\int_{\mathbb{Z}_p}[x]_q^{n}q^{x}\,d\mu_{-1}(x).\tag{2.20}$$

Carlitz's type q-Euler numbers $E_n = E_{n,q}$ can be determined inductively by

$$q(qE+1)^{n}+E_{n,q}=\begin{cases}2 & \text{if } n=0,\\ 0 & \text{if } n>0,\end{cases}\tag{2.21}$$

with the usual convention of replacing E^n by $E_{n,q}$.

Carlitz type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function $F_q(x,t)$ as follows:

$$F_q(x,t)=2\sum_{k=0}^{\infty}(-1)^{k}q^{k}e^{[k+x]_q t}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.\tag{2.22}$$

In the cases $x=0, E_{n,q}(0)=E_{n,q}$ will be called Carlitz type q-Euler numbers (cf. [8, 19]). One also can see that the generating functions $F_q(x,t)$ are determined as solutions of

$$F_q(x,t)=2e^{[x]_q t}-qe^{t}F_q(x,qt).\tag{2.23}$$

From (2.22), one gets the following.

Lemma 2.2. (1)$F_q(x,t) = 2e^{t/(1-q)}\sum_{j=0}^{\infty}(1/(q-1))^j q^{xj}(1/(1+q^{j+1}))(t^j/j!)$.
(2)$E_{n,q}(x) = 2\sum_{k=0}^{\infty}(-1)^k q^k [k+x]_q^n$.

It is clear from (1) and (2) of Lemma 2.2 that

$$E_{n,q}(x) = \frac{2}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{1+q^{k+1}}q^{xk},$$
$$\begin{aligned}\sum_{k=0}^{m-1}(-1)^k q^k[k+x]_q^n &= \sum_{k=0}^{\infty}(-1)^k q^k[k+x]_q^n - \sum_{k=0}^{\infty}(-1)^{k+m}q^{k+m}[k+m+x]_q^n \\ &= \frac{1}{2}\Big(E_{n,q}(x) + (-1)^{m+1}q^m E_{n,q}(x+m)\Big).\end{aligned} \tag{2.24}$$

From (2.24), we may state the following.

Proposition 2.3. *If $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, then*
(1) $E_{n,q}(x) = (2/(1-q)^n)\sum_{k=0}^{n}\binom{n}{k}((-1)^k/(1+q^{k+1}))q^{xk}$,
(2) $\sum_{k=0}^{m-1}(-1)^k q^k[k+x]_q^n = (1/2)(E_{n,q}(x) + (-1)^{m+1}q^m E_{n,q}(x+m))$.

Proposition 2.4. *For $n \in \mathbb{Z}^+$, the value of $\int_{\mathbb{Z}_p}[x+y]_q^n q^y d\mu_{-1}(y)$ is $n!$ times the coefficient of t^n in the formal expansion of $2\sum_{k=0}^{\infty}(-1)^k q^k e^{[k+x]_q t}$ in powers of t. That is, $E_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n q^y d\mu_{-1}(y)$.*

Proof. From (2.3), we have

$$\int_{\mathbb{Z}_p} q^{k(x+y)}q^y d\mu_{-1}(y) = q^{xk}\lim_{N\to\infty}\sum_{a=0}^{p^N-1}\left(-q^{k+1}\right)^a = \frac{2q^{xk}}{1+q^{k+1}}, \tag{2.25}$$

which leads to

$$\begin{aligned}\int_{\mathbb{Z}_p}[x+y]_q^n q^y d\mu_{-1}(y) &= 2\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(1-q)^n}(-1)^k\int_{\mathbb{Z}_p}q^{k(x+y)}q^y d\mu_{-1}(y) \\ &= \frac{2}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{1+q^{k+1}}q^{xk}.\end{aligned} \tag{2.26}$$

The result now follows by using (1) of Proposition 2.3. □

Corollary 2.5. *If $n \in \mathbb{Z}^+$, then*

$$E_{n,q}(x) = \sum_{k=0}^{n}\binom{n}{k}[x]_q^{n-k}q^{kx}E_{k,q}. \tag{2.27}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and p be a fixed odd prime number. One sets

$$X = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod\, dp^N \right) \right\}, \tag{2.28}$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N$ (cf. [7, 9]). Note that the natural map $\mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}$ induces

$$\pi : X \longrightarrow \mathbb{Z}_p. \tag{2.29}$$

Hereafter, if f is a function on $\mathbb{Z}_p$, one denotes by the same f the function $f \circ \pi$ on X. Namely one considers f as a function on X.

Let χ be the Dirichlet character with an odd conductor $d = d_\chi \in \mathbb{N}$. Then, the generalized Carlitz type q-Euler polynomials attached to χ are defined by

$$E_{n,\chi,q}(x) = \int_X \chi(a) [x+y]_q^n q^y \, d\mu_{-1}(y), \tag{2.30}$$

where $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}_p$. Then, one has the generating function of generalized Carlitz type q-Euler polynomials attached to χ

$$F_{q,\chi}(x,t) = 2\sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!}. \tag{2.31}$$

Now, fixed any $t \in \mathbb{C}_p$ with $\mathrm{ord}_p(t) > 1/(p-1)$ and $|1-q|_p < 1$. From (2.31), one has

$$\begin{aligned} F_{q,\chi}(x,t) &= 2\sum_{m=0}^{\infty} \chi(m)(-q)^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{i(m+x)} \frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{ix} \\ &\quad \times \sum_{j=0}^{d-1} \sum_{l=0}^{\infty} \chi(j+dl)(-q)^{j+dl} q^{i(j+dl)} \frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{q^{i(x+j)}}{1+q^{d(i+1)}} \frac{t^n}{n!}, \end{aligned} \tag{2.32}$$

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. By (2.31) and (2.32), one can derive

$$\begin{aligned}
E_{n,\chi,q}(x) &= \frac{1}{(1-q)^n}\sum_{j=0}^{d-1}\chi(j)(-q)^j\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{i(x+j)}\frac{2}{1+q^{d(i+1)}} \\
&= \frac{1}{(1-q)^n}\sum_{j=0}^{d-1}\chi(j)(-q)^j\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{i(x+j)} \times \lim_{N\to\infty}\sum_{l=0}^{p^N-1}(-1)^l\left(q^{d(i+1)}\right)^l \\
&= \lim_{N\to\infty}\sum_{j=0}^{d-1}\sum_{l=0}^{p^N-1}\chi(j+dl)\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{i(j+dl+x)} \times (-1)^{j+dl}q^{j+dl} \\
&= \lim_{N\to\infty}\sum_{a=0}^{dp^N-1}\chi(a)\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{i(a+x)}(-q)^a \\
&= \int_X \chi(y)[x+y]_q^n q^y d\mu_{-1}(y),
\end{aligned} \tag{2.33}$$

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Therefore, one obtains the following.

Theorem 2.6.

$$E_{n,\chi,q}(x) = \frac{1}{(1-q)^n}\sum_{j=0}^{d-1}\chi(j)(-q)^j\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{i(x+j)}\frac{2}{1+q^{d(i+1)}}, \tag{2.34}$$

where $n \in \mathbb{Z}^+$ *and* $x \in \mathbb{Z}_p$.

Let ω denote the Teichmüller character mod p. For $x \in X^*$, one sets

$$\langle x\rangle = [x]_q\omega^{-1}(x) = \frac{[x]_q}{\omega(x)}. \tag{2.35}$$

Note that since $|\langle x\rangle - 1|_p < p^{-1/(p-1)}$, $\langle x\rangle^s$ is defined by $\exp(s\log_p\langle x\rangle)$ for $|s|_p \le 1$ (cf. [10, 12, 21]). One notes that $\langle x\rangle^s$ is analytic for $s \in \mathbb{Z}_p$.

One defines an interpolation function for Carlitz type q-Euler numbers. For $s \in \mathbb{Z}_p$,

$$l_{p,q}(s,\chi) = \int_{X^*}\langle x\rangle^{-s}\chi(x)q^x d\mu_{-1}(x). \tag{2.36}$$

Then, $l_{p,q}(s,\chi)$ is analytic for $s \in \mathbb{Z}_p$.

The values of this function at nonpositive integers are given by the following.

Theorem 2.7. *For integers $n \geq 0$,*

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)\,[p]_q^n E_{n,\chi_n,q^p}, \tag{2.37}$$

where $\chi_n = \chi\omega^{-n}$. In particular, if $\chi = \omega^n$, then $l_{p,q}(-n,\omega^n) = E_{n,q} - [p]_q^n E_{n,q^p}$.

Proof.

$$\begin{aligned} l_{p,q}(-n,\chi) &= \int_{X^*} \langle x\rangle^n \chi(x) q^x d\mu_{-1}(x) \\ &= \int_X [x]_q^n \chi_n(x) q^x d\mu_{-1}(x) - \int_X [px]_q^n \chi_n(px) q^{px} d\mu_{-1}(px) \\ &= \int_X [x]_q^n \chi_n(x) q^x d\mu_{-1}(x) - [p]_q^n \chi_n(p) \int_X [x]_{q^p}^n \chi_n(x) q^{px} d\mu_{-1}(x). \end{aligned} \tag{2.38}$$

Therefore by (2.30), the theorem is proved. □

Let χ be the Dirichlet character with an odd conductor $d = d_\chi \in \mathbb{N}$. Let F be a positive integer multiple of p and d. Then, by (2.22) and (2.31), we have

$$\begin{aligned} F_{q,\chi}(x,t) &= 2\sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{[m+x]_q t} \\ &= 2\sum_{a=0}^{F-1} \chi(a)(-q)^a \sum_{k=0}^{\infty} (-q)^{Fk} e^{[F]_q [k+((x+a)/F)]_{q^F} t} \\ &= \sum_{n=0}^{\infty} \left([F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q^F}\left(\frac{x+a}{F}\right)\right) \frac{t^n}{n!}. \end{aligned} \tag{2.39}$$

Therefore, we obtain the following

$$E_{n,\chi,q}(x) = [F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q^F}\left(\frac{x+a}{F}\right). \tag{2.40}$$

If $\chi_n(p) \neq 0$, then $(p, d_{\chi_n}) = 1$, so that F/p is a multiple of d_{χ_n}. From (2.40), we derive

$$\begin{aligned} \chi_n(p)\,[p]_q^n E_{n,\chi_n,q^p} &= \chi_n(p)\,[p]_q^n \left[\frac{F}{p}\right]_{q^p}^n \sum_{a=0}^{F/p-1} \chi_n(a)(-q^p)^a E_{n,(q^p)^{F/p}}\left(\frac{a}{F/p}\right) \\ &= [F]_q^n \sum_{\substack{a=0 \\ p\mid a}}^{F} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right). \end{aligned} \tag{2.41}$$

Thus, we have

$$E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p} = [F]_q^n \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right). \tag{2.42}$$

By Corollary 2.5, we easily see that

$$\begin{aligned} E_{n,q^F}\left(\frac{a}{F}\right) &= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a}{F}\right]_{q^F}^{n-k} q^{ka} E_{k,q^F} \\ &= [F]_q^{-n}[a]_q^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}. \end{aligned} \tag{2.43}$$

From (2.42) and (2.43), we have

$$\begin{aligned} E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p} &= [F]_q^n \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right) \\ &= \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^n (-q)^a \sum_{k=0}^{\infty} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}, \end{aligned} \tag{2.44}$$

since $\chi_n(a) = \chi(a)\omega^{-n}(a)$. From Theorem 2.7 and (2.44), we have

$$l_{p,q}(-n,\chi) = \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^n (-q)^a \sum_{k=0}^{\infty} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}, \tag{2.45}$$

for $n \in \mathbb{Z}^+$. Therefore, we have the following theorem.

Theorem 2.8. *Let F be a positive integer multiple of p and $d = d_\chi$, and let*

$$l_{p,q}(s,\chi) = \int_{X^*} \langle x\rangle^{-s} \chi(x) q^x \, d\mu_{-1}(x), \quad s \in \mathbb{Z}_p. \tag{2.46}$$

Then, $l_{p,q}(s,\chi)$ is analytic for $s \in \mathbb{Z}_p$ and

$$l_{p,q}(s,\chi) = \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^{-s} (-q)^a \sum_{k=0}^{\infty} \binom{-s}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}. \tag{2.47}$$

Furthermore, for $n \in \mathbb{Z}^+$

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}. \tag{2.48}$$

Acknowledgments

The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (2010-0001654). The second author was supported by the research grant of Kwangwoon University in 2010.

References

[1] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.
[2] L. Carlitz, "q-Bernoulli and Eulerian numbers," *Transactions of the American Mathematical Society*, vol. 76, pp. 332–350, 1954.
[3] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, "Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 135–160, 2009.
[4] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, "On the higher-order w-q-Genocchi numbers," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 39–57, 2009.
[5] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.
[6] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[7] T. Kim, "On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 779–792, 2007.
[8] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.
[9] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.
[10] T. Kim, "On p-adic q-l-functions and sums of powers," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 1472–1481, 2007.
[11] T. Kim, "On the multiple q-Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.
[12] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.
[13] T. Kim, "On a p-adic interpolation function for the q-extension of the generalized Bernoulli polynomials and its derivative," *Discrete Mathematics*, vol. 309, no. 6, pp. 1593–1602, 2009.
[14] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.
[15] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.
[16] T. Kim, "Barnes-type multiple q-zeta functions and q-Euler polynomials," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 25, Article ID 255201, 11 pages, 2010.
[17] T. Kim, L.-C. Jang, Y.-H. Kim, and S.-H. Rim, "New approach to q-Euler numbers and polynomials," *Advances in Difference Equations*, vol. 2010, Article ID 431436, 9 pages, 2010.
[18] T. Kim, Y.-H. Kim, and B. Lee, "Note on Carlitz's type q-Euler numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 13, no. 2, pp. 149–155, 2010.
[19] H. Ozden, I. N. Cangul, and Y. Simsek, "Multivariate interpolation functions of higher-order q-Euler numbers and their applications," *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.
[20] H. Ozden, Y. Simsek, S.-H. Rim, and I. N. Cangul, "A note on p-adic q-Euler measure," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 233–239, 2007.
[21] J. Satoh, "q-analogue of Riemann's ζ-function and q-Euler numbers," *Journal of Number Theory*, vol. 31, no. 3, pp. 346–362, 1989.
[22] M. Cenkci, Y. Simsek, and V. Kurt, "Multiple two-variable p-adic q-L-function and its behavior at $s = 0$," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 447–459, 2008.

[23] A. M. Robert, *A Course in* p-*Adic Analysis*, vol. 198 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.

[24] N. K. Govil and V. Gupta, “Convergence of q-Meyer-König-Zeller-Durrmeyer operators,” *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 97–108, 2009.