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Research Article

Asymptotics for the Moment Convergence of U-Statistics in LIL

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Let U_n be a U-statistic based on a symmetric kernel $h(x, y)$ and i.i.d. samples $\{X, X_n; n \geq 1\}$. In this paper, the exact moment convergence rates in the law of the iterated logarithm and the law of the logarithm of U_n are obtained, which extend previous results concerning partial sums.

1. Introduction and Main Result

Let $h(x, y)$ be a real-valued Borel measurable function, symmetric in its arguments. Define a U-statistic based on an independent and identically distributed (i.i.d.) sequence $\{X, X_n; n \geq 1\}$ and kernel function h as follows:

$$U_n := \frac{\sum_{1 \leq i < j \leq n} h(X_i, X_j)}{\binom{n}{2}}, \quad n \geq 2. \tag{1.1}$$

This class of U-statistics was introduced by Hoeffding [1] and Halmos [2] in the 1940s, and we have witnessed a rapid development in asymptotic theory of U-statistics since then (see Koroljuk and Borovskich [3] and Serfling [4] for more details).

It is well known that, initiating from the work of Gut and Spătaru [5], many authors devoted themselves to the research of precise asymptotics. Recently, Zhou et al. [6] studied the precise asymptotics of a special kind of statistics, which includes the U-statistics, Von-Mises statistics, linear processes, moving average processes, error variance estimates in linear models and power sums. One of their main results is as follows, which reflects the exact probability convergence rate in the law of the iterated logarithm.

Theorem A. *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with mean zero and variance one. Let $T_n = T_n(X_1, \ldots, X_n)$ be a random function or statistic satisfying $T_n = S_n + R_n$, where $S_n = \sum_{i=1}^{n} X_i$. If $\mathsf{E}|R_n|^2 < \infty$, then for any $b > -1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\left(|T_n| \geq \varepsilon\sqrt{2n\log\log n}\right) = \frac{1}{(b+1)\sqrt{\pi}} \Gamma\left(b + \frac{3}{2}\right), \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function and $\log n = \log(n \vee e), n \geq 0$.

Since Theorem A requires a strong condition, that is, $\mathsf{E}|R_n|^2 < \infty$, Yan and Su [7] investigated the precise asymptotics of U-statistics under minimal conditions and got the following result.

Theorem B. *Let U_n be a U-statistic given by (1.1). Suppose that for some $0 < \delta \leq 1$, $\zeta_1 = \mathsf{E}(\widetilde{h}_1(X_1))^2 > 0$, $\mathsf{E}(\widetilde{h}_1(X_1))^{2+\delta} < \infty$ and $\mathsf{E}|h(X_1, X_2)|^{(4+\delta)/3} < \infty$, where $\widetilde{h}_1(x) = \mathsf{E}(h(X_1, X_2) \mid X_1 = x) - \mu$ and $\mu = \mathsf{E}h(X_1, X_2)$. Then for any $b > -1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\left(|U_n| \geq \varepsilon\sqrt{2n^{-1}\log\log n}\right) = \frac{1}{(b+1)\sqrt{\pi}} \Gamma\left(b + \frac{3}{2}\right). \tag{1.3}$$

On the other hand, for the i.i.d. sequence $\{X, X_n; n \geq 1\}$, it is noted that Chow [8] first introduced the well-known complete moment convergence and gave the result as follows.

Theorem C. *Suppose that $\mathsf{E}X = 0$. For $0 < p < 2$, $r > 1$ and $rp \geq 1$, if $\mathsf{E}(|X|^{rp} + |X|\log(1 + |X|)) < \infty$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2-1/p} \mathsf{E}\left\{\max_{1 \leq k \leq n} |S_k| - \varepsilon n^{1/p}\right\}_+ < \infty, \tag{1.4}$$

where $\{x\}_+ = x \vee 0$.

Inspired by them, in this paper, we aim to establish a moment version of Theorem B for U-statistics. Our main result reads as follows.

Theorem 1.1. *Let U_n be a U-statistic given by (1.1). Suppose that $\mathsf{E}h(X_1, X_2) = 0$ and $\mathsf{E}|h(X_1, X_2)|^2 < \infty$. Then for any $b > -1/2$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=2}^{\infty} \frac{(\log\log n)^{b-1/2}}{n \log n} \mathsf{E}\left\{\sqrt{n}|U_n| - \varepsilon\sqrt{2\log\log n}\right\}_+ = \frac{2^{-1/2-b}\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}, \tag{1.5}$$

where N is a normal random variable with mean zero and variance $4\zeta_1$.

Remark 1.2. Here we consider the moment convergence rates of U-statistic in the law of the iterated logarithm, extending the results of Zhou et al. [6] and Yan and Su [7] for exact probability convergence rates and reflecting the convergence rates of the law of the iterated logarithm more directly.

By some modifications, we can get the following result easily.

Theorem 1.3. *Under the assumptions of Theorem 1.1, One has that for $d > 0$ and $1/2 < b+1/d < 1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n=2}^{\infty} \frac{(\log n)^{bd-d/2}}{n} \mathsf{E}\left\{ n^{1/2}|U_n| - \varepsilon (\log n)^{d/2} \right\}_+ = \frac{d\mathsf{E}|N|^{2b+2/d}}{(bd+1)(2bd+2-d)}. \tag{1.6}$$

Remark 1.4. Note that in our theorem, we assume $E|h(X_1, X_2)|^2 < \infty$, which is stronger than the condition imposed by Yan and Su [7], and required only to use a moment bound of Chen [9] given in Lemma 2.1. However, the assumption $\mathsf{E}(\widetilde{h}_1(X_1))^{2+\delta} < \infty$ in Yan and Su [7] is weakened.

2. Proof of Theorem 1.1

Note that $\mathsf{E}|h(X_1, X_2)|^2 < \infty$ readily implies $\zeta_1 = \mathsf{E}(\widetilde{h}_1(X_1))^2 < \infty$. Thus without loss of generality, assume $\mathsf{E}(\widetilde{h}_1(X_1))^2 = 1/4$. In the sequel, let C denote a positive constant whose value possibly varies from place to place and the notation of $[x]$ means the integer part of x.

We first introduce some useful lemmas, which are known as the moment inequality of U-statistics and the Toeplitz lemma, respectively.

Lemma 2.1 (Chen [9]). *Let U_n be given by (1.1). Suppose that $Eh(X_1, X_2) = 0$ and $E|h(X_1, X_2)|^q < \infty$ for $q \geq 2$. Then there exists a constant D_q depending only on q such that*

$$\mathsf{E}|U_n|^q \leq D_q n^{-q} \mathsf{E}|h(X_1, X_2)|^q. \tag{2.1}$$

Lemma 2.2 (Stout [10]). *Let $\{a_{ni}\}$ be a matrix of real numbers and $\{x_i\}$ a sequence of real numbers. Let $x_i \to x$ as $i \to \infty$. Then*

$$\begin{gathered} \sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty \quad \forall n \geq 1, \\ \sum_{i=1}^{\infty} a_{ni} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \\ a_{ni} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \text{for each } i \geq 1 \end{gathered} \tag{2.2}$$

imply that

$$\sum_{i=1}^{\infty} a_{ni} x_i \longrightarrow x \quad \text{as } n \longrightarrow \infty. \tag{2.3}$$

In what follows, for $M > 4$ and $0 < \varepsilon < 1/4$, we set $a(\varepsilon) = [\exp(\exp(M/\varepsilon^2))]$. The proof is very much modeled for proving results in the area of precise asymptotics, and hence Theorem 1.1 follows immediately by applying the following propositions.

Proposition 2.3. *For any $b > -1/2$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n \log n} \mathsf{E}\left\{ |N| - \varepsilon\sqrt{2\log\log n} \right\}_+ = \frac{2^{-1/2-b}\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}, \tag{2.4}$$

where N is defined as above.

Proof. Notice that

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n \log n} \mathsf{E}\left\{ |N| - \varepsilon\sqrt{2\log\log n} \right\}_+ \\
&\quad = \lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \int_{e^e}^{\infty} \frac{(\log\log y)^{b-1/2}}{y \log y} \int_{\varepsilon\sqrt{2\log\log y}}^{\infty} \mathsf{P}\{|N| \ge x\}\, dx\, dy \\
&\quad = 2^{1/2-b} \lim_{\varepsilon \searrow 0} \int_{\sqrt{2}\varepsilon}^{\infty} z^{2b} \int_{z}^{\infty} \mathsf{P}\{|N| \ge x\}\, dx\, dz \\
&\quad = 2^{1/2-b} \lim_{\varepsilon \searrow 0} \int_{\sqrt{2}\varepsilon}^{\infty} \mathsf{P}\{|N| \ge x\} \int_{\sqrt{2}\varepsilon}^{x} z^{2b}\, dz\, dx \\
&\quad = \frac{2^{1/2-b}}{2b+1} \lim_{\varepsilon \searrow 0} \int_{\sqrt{2}\varepsilon}^{\infty} \mathsf{P}\{|N| \ge x\} x^{2b+1}\, dx \\
&\quad = \frac{2^{-1/2-b}\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}.
\end{aligned} \tag{2.5}$$

□

Proposition 2.4. *For $b > -1/2$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n \le a(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n \log n} \left| \mathsf{E}\left\{ |N| - \varepsilon\sqrt{2\log\log n} \right\}_+ - \mathsf{E}\left\{ n^{1/2}|U_n| - \varepsilon\sqrt{2\log\log n} \right\}_+ \right| = 0. \tag{2.6}$$

Proof. Set $\Delta_n = \sup_{x\in R}|\mathsf{P}(|N| \geq x) - \mathsf{P}(n^{1/2}|U_n| \geq x)|$. Then, from the central limit theorem for U-statistics (cf. Koroljuk and Borovskich [3]), it follows that $\Delta_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\left|\mathsf{E}\left\{|N|-\varepsilon\sqrt{2\log\log n}\right\}_+ - \mathsf{E}\left\{n^{1/2}|U_n|-\varepsilon\sqrt{2\log\log n}\right\}_+\right| \\
&\quad= \varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\left|\int_0^\infty \mathsf{P}\left(|N| \geq x+\varepsilon\sqrt{2\log\log n}\right)dx\right. \\
&\qquad\qquad\left. -\int_0^\infty \mathsf{P}\left(n^{1/2}|U_n| \geq x+\varepsilon\sqrt{2\log\log n}\right)dx\right| \\
&\quad\leq \sqrt{2}\varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\int_0^\infty\left|\mathsf{P}\left(|N| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)\right. \\
&\qquad\qquad\left. -\mathsf{P}\left(n^{1/2}|U_n| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)\right|dx \\
&\quad\leq \sqrt{2}\varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}(P_{n1}+P_{n2}),
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
P_{n1} &:= \int_0^{(\log\log n)^{-1/2}\Delta_n^{-1/2}}\left|\mathsf{P}\left(|N| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)-\mathsf{P}\left(n^{1/2}|U_n| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)\right|dx, \\
P_{n2} &:= \int_{(\log\log n)^{-1/2}\Delta_n^{-1/2}}^{\infty}\left|\mathsf{P}\left(|N| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)-\mathsf{P}\left(n^{1/2}|U_n| \geq (x+\varepsilon)\sqrt{2\log\log n}\right)\right|dx.
\end{aligned}\tag{2.8}$$

Thus, for P_{n1}, by applying Lemma 2.2, we have

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}P_{n1} \leq \varepsilon^{2b+1}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\Delta_n^{1/2} \\
&\quad\leq M^{b+1/2}\frac{1}{(\log\log a(\varepsilon))^{b+1/2}}\sum_{n\leq a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\Delta_n^{1/2} \longrightarrow 0, \quad \text{as } \varepsilon \searrow 0.
\end{aligned}\tag{2.9}$$

As for P_{n2}, coupled with Markov's inequality and Lemma 2.1 with $q=2$, then an application of Lemma 2.2 provides

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n\le a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}P_{n2}\\
&\le C\varepsilon^{2b+1}\sum_{n\le a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\int_{(\log\log n)^{-1/2}\Delta_n^{-1/2}}^{\infty}\left(\frac{1}{(x+\varepsilon)^2\log\log n}+\frac{n^{-1}}{(x+\varepsilon)^2\log\log n}\right)dx\\
&\le C\varepsilon^{2b+1}\sum_{n\le a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\int_{(\log\log n)^{-1/2}\Delta_n^{-1/2}}^{\infty}\frac{1}{(x+\varepsilon)^2\log\log n}dx\\
&\le C\varepsilon^{2b+1}\sum_{n\le a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\Delta_n^{1/2}\longrightarrow 0,\quad \text{as } \varepsilon\searrow 0.
\end{aligned}\tag{2.10}$$

Hence (2.6) holds true. □

Proposition 2.5. *For $0<\varepsilon<1/4$ and $b>-1/2$, one has uniformly*

$$\lim_{M\to\infty}\varepsilon^{2b+1}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\mathsf{E}\left\{|N|-\varepsilon\sqrt{2\log\log n}\right\}_{+}=0.\tag{2.11}$$

Proof. Note that for k large enough,

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\int_0^{\infty}\mathsf{P}\left\{|N|\ge\varepsilon\sqrt{2\log\log n}+x\right\}dx\\
&\le C\varepsilon^{2b+1}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\int_0^{\infty}\mathsf{P}\left\{|N|\ge(x+\varepsilon)\sqrt{2\log\log n}\right\}dx\\
&\le C\varepsilon^{2b+1}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\int_0^{\infty}\frac{\mathsf{E}|N|^k}{(x+\varepsilon)^k(\log\log n)^{k/2}}dx\\
&\le C\varepsilon^{2b-k+2}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b-k/2}}{n\log n}=CM^{b-(k-2)/2}\longrightarrow 0,
\end{aligned}\tag{2.12}$$

when $M\to\infty$, uniformly for $0<\varepsilon<1/4$. □

Proposition 2.6. *Under the assumptions of Theorem 1.1, one has*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n>a(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\mathsf{E}\left\{n^{1/2}|U_n|-\varepsilon\sqrt{2\log\log n}\right\}_{+}=0.\tag{2.13}$$

Proof. Notice that by virtue of Lemma 2.1 with $q = 2$, it follows that

$$\begin{aligned}
&\varepsilon^{2b+1} \sum_{n>a(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \mathsf{E}\left\{ n^{1/2}|U_n| - \varepsilon\sqrt{2\log\log n} \right\}_+ \\
&\quad = \varepsilon^{2b+1} \sum_{n>a(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \int_{\varepsilon\sqrt{2\log\log n}}^{\infty} \mathsf{P}\left(n^{1/2}|U_n| \ge x \right) dx \\
&\quad \le C\varepsilon^{2b+1} \sum_{n>a(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n^2\log n} \int_{\varepsilon\sqrt{2\log\log n}}^{\infty} \frac{1}{x^2} dx \\
&\quad \le C\varepsilon^{2b} \sum_{n>a(\varepsilon)} \frac{(\log\log n)^{b-1}}{n^2\log n} \le C\varepsilon^{2b} \sum_{n>a(\varepsilon)} \frac{1}{n^{3/2}} \\
&\quad \le C\varepsilon^{2b} a(\varepsilon)^{-1/2} \longrightarrow 0, \quad \text{as } \varepsilon \searrow 0.
\end{aligned} \tag{2.14}$$

□

Proof of Theorem 1.1. Theorem 1.1 follows from Propositions 2.3–2.6 by using the triangle inequality immediately. □

3. Proof of Theorem 1.3

By some simple modifications, Theorem 1.3 can be got similarly. For completeness, we state the similar Propositions 3.1–3.4 in the following without details.

Proposition 3.1. *For $d > 0$ and $b + 1/d > 1/2$, one has*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n=1}^{\infty} \frac{(\log n)^{bd-d/2}}{n} \mathsf{E}\left\{ |N| - \varepsilon(\log n)^{d/2} \right\}_+ = \frac{d\mathsf{E}|N|^{2b+2/d}}{(bd+1)(2bd+2-d)}. \tag{3.1}$$

Proposition 3.2. *For $d > 0$ and $b + 1/d > 1/2$, one has*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n\le c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} \left| \mathsf{E}\left\{ |N| - \varepsilon(\log n)^{d/2} \right\}_+ - \mathsf{E}\left\{ n^{1/2}|U_n| - \varepsilon(\log n)^{d/2} \right\}_+ \right| = 0, \tag{3.2}$$

where $c(\varepsilon) = [\exp(M/\varepsilon^2)]$.

Proposition 3.3. *For $d > 0$ and $b + 1/d > 1/2$, one has*

$$\lim_{M\to\infty} \lim_{\varepsilon\searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n>c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} \mathsf{E}\left\{ |N| - \varepsilon(\log n)^{d/2} \right\}_+ = 0. \tag{3.3}$$

Proposition 3.4. *For $d > 0$ and $1/2 < b + 1/d < 1$, one has*

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{2b+(2/d)-1}\sum_{n>c(\varepsilon)}\frac{(\log n)^{bd-d/2}}{n}\mathsf{E}\left\{n^{1/2}|U_n|-\varepsilon(\log n)^{d/2}\right\}_+=0. \tag{3.4}$$

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