Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 346308, 4 pages
doi:10.1155/2010/346308

Research Article

On Nonhomogeneous A-Harmonic Equations and 1-Harmonic Equations

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Received 31 December 2009; Accepted 7 February 2010

Academic Editor: Shusen Ding

We prove a characterization of a nonhomogeneous A-harmonic equation and describe its generalization. We also point out its connection with 1-Harmonic equation.

1. Introduction

Both A-harmonic equations and p-harmonic geometry are rich subjects [1–5]. Many results on both topics have been derived, respectively, but there are very few papers relating both subjects. In this paper, we will connect these two subjects by extending several results from 1-Harmonic functions to A-Harmonic functions.

We consider the following setting: a C^1 function $f : R^n \to R$ is said to be A-harmonic if it is a weak solution of A-harmonic equation

$$\operatorname{div} A\left(x, \frac{\nabla f}{|\nabla f|}\right) = 0, \tag{1.1}$$

where $|\nabla f|$ is the length of the gradient ∇f of f, and for a C^2 function f without a critical point, $\operatorname{div}(\nabla f/|\nabla f|)$ is said to be the 1-tension field of f.

Let Ω be an open subset of R^n, $n \geq 2$. Consider the following second-order divergence-type elliptic equation:

$$\operatorname{div} A(x, \nabla f(x)) = 0, \tag{1.2}$$

where $A : \Omega \times R^n \rightarrow R^n$ satisfies

(i) $|A(x,u)| \leq \lambda w(x)|u|^{p-1}$,

(ii) $\langle A(x,u), u\rangle \geq \lambda^{-1} w(x)|u|^p$,

where $1 < p < \infty, \lambda \geq 1$ are two fixed constants, and $w(x)$ is called a weight if $w \in L^1_{\text{loc}}(R^n)$ and $w > 0$ a.e. Also, in general $d\mu = w dx$ where w is a weight.

In this paper, we characterize subsolutions of (1.1) and indicate its generalization to (1.2). We first recall some results in homogeneous A-harmonic equations in the following section, followed by the main results in Section 3. Some open problems are discussed in the last section.

2. Homogeneous A-Harmonic Equations

Definition 2.1. A function $f \in W^{1,r}_{\text{loc}}(\Omega, w)$, $\max\{1, p-1\} \leq r < p$ is called a very weak solution of (1.2) if

$$\int_\Omega < A(x, \nabla f), \nabla\phi > dx = 0 \tag{2.1}$$

for all $\phi \in W^{1,(r/(r-p+1))}(\Omega, w)$ with compact support.

It was shown in [4] that very weak solutions of (1.2) in fact weak solutions of (1.2) in the usual sense.

Definition 2.2. u an A-harmonic tensor in Ω of u satisfies the A-harmonic equation (2.1) in Ω.

Consider the space of differential l-forms $D'(\Omega, \Lambda^l)$, $u \in D'(\Omega, \Lambda^l)$ being an A-harmonic tensor in a domain $\Omega \subset R^n$, and $du \in L^s_{\text{loc}}(\Omega, \Lambda^{l+1})$, $l = 0, 1, \ldots, n$. Assume that $\sigma > 1$, $1 < s < \infty$, and $w \in A_r$ for some $r > 1$. Then the following local weighted Poincare inequality for A-harmonic tensors was proved in [3]. There exists a constant C independent of u such that

$$\left(\frac{1}{\mu(B)}\int_B |u - u_B|^s d\mu\right)^{1/s} \leq C|B|^{1/n}\left(\frac{1}{|B|}\int_{\sigma B} |du|^s dx\right)^{1/s} \tag{2.2}$$

for all balls B with $\sigma B \subset \Omega$, where the measure μ is defined by $d\mu = w(x)dx$, and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$.

3. Characterizations of Nonhomogeneous A-Harmonic Equations

In what follows, we prove an A-harmonic analog of 1-harmonic equations.

Lemma 3.1. *Let M be a complete noncompact Riemannian. For any $x_0 \in M$ and any pair of positive numbers s, t with $s < t$, there exist a rotational symmetric Lipschitz continuous function $\psi(x) = \psi(x, s, t)$ and a constant $c_1 > 0$ (independent of x_0, s, t) with the following properties:*

(i) $\psi \equiv 1$ *on $B(x_0; s)$ and $\psi \equiv 0$ off $B(x_0; s)$,*

(ii) $|\nabla\psi| \leq c_1/(t-s)$, *a.e. on M.*

Proof. See Andreotti and Vesentini [6], Yau [7], and Karp [8]. □

Theorem 3.2. *Let Ω be a domain in $\mathfrak{R}^n$ containing a ball $B(x_0, r)$ of radius r, centered at x_0, and let $g : \Omega \to \mathfrak{R}$ be a continuous function with $g \geq 0$, and*

$$c = \inf_{x \in B(x_0, r/2)} g(x). \tag{3.1}$$

Let $f : \Omega \to \mathfrak{R}$ be a C^1 weak solution of

$$\operatorname{div} A\left(x, \frac{\nabla f}{|\nabla f|}\right) = g(x) \quad on \quad \Omega, \tag{3.2}$$

then the infimum c satisfies $0 \leq c \leq \lambda k 2^n / r$, where k only depends on w, ψ, and c_1.

Proof. Let $\psi \geq 0$ be as in Lemma 3.1, and $M = \mathfrak{R}^n$, $t = r$, $s = r/2$. Choose ψ to be a test function. Then by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{B(x_0, r/2\ )} C\psi(x)dx \leq \int_{B(x_0, r/2\ )} g(x)\psi(x)dx \leq \int_{B(x_0, r\ )} g(x)\psi(x)dx \\ - \int_{B(x_0, r)} A\left(x \frac{\nabla f}{|\nabla f|}\right)\psi(x)dx \leq \int_{B(x_0, r)} \lambda w(x)|\nabla \psi|dx \end{aligned} \tag{3.3}$$

Hence, $cVol(B(x_0, r/2)) \leq (\lambda c_1 c_2)/rVol(B(x_0, r))$.

Where c_2 only depends on w and ψ. □

Corollary 3.3. *Let $f : \mathfrak{R}^n \to \mathfrak{R}$ be a C^1 weak subsolution of A-harmonic equation* (1.1) *with constant 1-tension field c, that is, $0 \leq \operatorname{div} A(x, \nabla f / |\nabla f|) = c$ in the distribution sense. Then f is an A-harmonic function.*

Remark 3.4. In a similar fashion, the above results can be extended to the nonhomogeneous equation $\operatorname{div} A(x, \nabla f) = g(x)$ by using Sobolev Imbedding Theorem.

4. Further Discussions

It would be interesting to find similar results of Section 2 for nonhomogeneous A-harmonic equations. It would also be interesting to seek analogs of 1-harmonic applications in calibration geometry. The extension of 1-harmonic functions to A-harmonic functions on hyperbolic spaces and their associated spaces could be explored.

To conclude this paper, we state another A-harmonic extension of 1-harmonic result [5], that is an immediate consequence of Corollary 3.3.

Theorem 4.1. *Let $f \in H^{1,1}_{\text{loc}}(R^n)$, and $\nabla f \neq 0$ for every x in R^n. The following statements are equivalent.*

(i) *$f : R^n \to R$ is a C^1 weak subsolution of* (1.1) *with constant 1-tension field.*

(ii) *f is a C^1 weak solution of* (1.1) *on R^n.*

(iii) *f is a C^1 A-harmonic function on R^n.*

Acknowledgments

The author wishes to express sincere gratitude to Professor S. Walter Wei for many helpful suggestions and encouragements without which this would have not been written.

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