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Research Article

Some Nonlinear Weakly Singular Integral Inequalities with Two Variables and Applications

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Some nonlinear weakly singular integral inequalities in two variables which generalize some known results are discussed. The results can be used as powerful tools in the analysis of certain classes of differential equations, integral equations, and evolution equations. An example is presented to show boundedness of solution of a differential equation here.

1. Introduction

Various singular integral inequalities play an important role in the development of the theory of differential equations, functional differential equations, and integral equations. For example, Henry [1] proposed a linear integral inequality with singular kernel to investigate some qualitative properties for a parabolic differential equation, and Sano and Kunimatsu [2] gave a modified version of Henry type inequality. However, such results are expressed by a complicated power series which are sometimes inconvenient for their applications. To avoid the shortcoming of these results, Medveǒ [3] presented a new method to discuss nonlinear singular integral inequalities of Henry type and their Bihari version as follows:

$$\begin{aligned} u(t) &\le a(t) + \int_0^t (t-s)^{\beta-1} s^{\gamma-1} F(s) u(s)\,ds, \\ u(t) &\le a(t) + \int_0^t (t-s)^{\beta-1} F(s) w(u(s))\,ds, \end{aligned} \tag{1.1}$$

and the estimates of solutions are given, respectively. In [4], Medveǒ also generalized his results to an analogue of the Wendroff inequalities for functions in two variables. From then on, more attention has been paid to such inequalities with singular kernel (see [5–9]). In particular, Ma and Yang [8] used a modification of Medveď method to obtain pointwise explicit bounds on solutions of more general weakly singular integral inequalities of the Volterra type, and later Ma and Pečarić [9] used this method to study nonlinear inequalities of Henry type. Recently, Cheung et al. [10] applied the modified Medveď method to investigate some new weakly singular integral inequalities of Wendroff type and applications to fractional differential and integral equations.

In this paper, motivated mainly by the work of Ma et al. [8, 9] and Cheung et al. [10], we discuss more general form of nonlinear weakly singular integral inequality of Wendroff type for functions in two variables

$$u(x,y) \leq a(x,y) + \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(x,y,s,t) w(u(s,t)) ds\, dt. \quad (1.2)$$

Our results can generalize some known results and be used more effectively to study the qualitative properties of the solutions of certain partial differential and integral equations. Moreover, an example is presented to show the usefulness of our results.

2. Main Result

In what follows, R denotes the set of real numbers, and $R_+ = (0,\infty)$. $C(X,Y)$ denotes the collection of continuous functions from the set X to the set Y. $D_1 z(x,y)$ and $D_2 z(x,y)$ denote the first-order partial derivatives of $z(x,y)$ with respect to x and y, respectively.

Before giving our result, we cite the following definition and lemmas.

Definition 2.1 (see [8]). Let $[x,y,z]$ be an ordered parameter group of nonnegative real numbers. The group is said to belong to the first-class distribution and is denoted by $[x,y,z] \in I$ if conditions $x \in (0,1]$, $y \in (1/2,1)$, and $z \geq 3/2 - y$ are satisfied; it is said to belong to the second-class distribution and is denoted by $[x,y,z] \in II$ if conditions $x \in (0,1]$, $y \in (0,1/2]$ and $z > (1-2y^2)/(1-y^2)$ are satisfied.

Lemma 2.2 (see [8]). *Let α, β, γ, and p be positive constants. Then,*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+, \quad (2.1)$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$ $(\mathrm{Re}\,\xi > 0, \mathrm{Re}\,\eta > 0)$ is well-known B-function and $\theta = p[\alpha(\beta-1)+\gamma-1]+1$.

Lemma 2.3 (see [8]). *Suppose that the positive constants α, β, γ, p_1, and p_2 satisfy the following conditions:*

(1) *if $[\alpha,\beta,\gamma] \in I$, $p_1 = 1/\beta$;*

(2) *if $[\alpha,\beta,\gamma] \in II$, $p_2 = (1+4\beta)/(1+3\beta)$.*

Then, for $i = 1, 2$,

$$B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \in (0, +\infty),$$
$$\theta_i = p_i\left[\alpha(\beta-1)+\gamma-1\right]+1 \geq 0 \tag{2.2}$$

are valid.

Assume that

(A_1) $a(x,y) \in C(R_+^2, R_+)$ and $f(x,y,s,t) \in C(R_+^4, R_+)$;

(A_2) $w(u) \in C(R_+, R_+)$ is nondecreasing and $w(0) = 0$.

Let $\tilde{a}(x,y) = \max_{0\le\tau\le x, 0\le\eta\le y} a(\tau,\eta)$ and $\tilde{f}(x,y,s,t) = \max_{0\le\tau\le x, 0\le\eta\le y} f(\tau,\eta,s,t)$.

Theorem 2.4. *Under assumptions (A_1) and (A_2), if* $u(m,n) \in C(R_+^2, R+)$ *satisfies* (1.2), *then*

(1) for $[\alpha,\beta,\gamma] \in I$,

$$u(x,y) \leq \left[W_1^{-1}\left(W_1(A_1(x,y)) + B_1(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{1/(1-\beta)}ds\,dt\right)\right]^{1-\beta} \tag{2.3}$$

for $0 \le x \le X$ *and* $0 \le y \le Y$, *where*

$$M_1 = \frac{1}{\alpha}B\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right],$$
$$A_1(x,y) = 2^{\beta/(1-\beta)}\tilde{a}(x,y)^{1/(1-\beta)}, \tag{2.4}$$
$$B_1(x,y) = 2^{\beta/(1-\beta)}\left(M_1^2(xy)^{(1/\beta)[\alpha(\beta-1)+\gamma-1]+1}\right)^{\beta/(1-\beta)},$$

W_1^{-1} *is the inverse of* W_1,

$$W_1 = \int_{u_0}^{u}\frac{d\xi}{w^{1/(1-\beta)}\left(\xi^{1-\beta}\right)}, \quad u \geq u_0 > 0, \tag{2.5}$$

and $X, Y \in R_+$ *are chosen such that*

$$W_1(A_1(x,y)) + B_1(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{1/(1-\beta)}ds\,dt \in \mathrm{Dom}\left(W_1^{-1}\right), \tag{2.6}$$

(2) for $[\alpha,\beta,\gamma] \in II$,

$$u(x,y) \leq \left[W_2^{-1}\left(W_2(A_2(x,y)) + B_2(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{(1+4\beta)/\beta}ds\,dt\right)\right]^{\beta/(1+4\beta)} \tag{2.7}$$

for $0 \le x \le X$ *and* $0 \le y \le Y$, *where*

$$M_2 = \frac{1}{\alpha} B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right],$$
$$A_2(x,y) = 2^{(1+3\beta)/\beta}\tilde{a}(x,y)^{(1+4\beta)/\beta}, \tag{2.8}$$
$$B_2(x,y) = 2^{(1+3\beta)/\beta}\left(M_2^2(xy)^{((1+4\beta)/(1+3\beta))[\alpha(\beta-1)+\gamma-1]+1}\right)^{(1+3\beta)/\beta},$$

W_2^{-1} *is the inverse of* W_2,

$$W_2 = \int_{u_0}^{u} \frac{d\xi}{w^{(1+4\beta)/\beta}\left(\xi^{\beta/(1+4\beta)}\right)}, \quad u \ge u_0 > 0, \tag{2.9}$$

and $X, Y \in R_+$ *are chosen such that*

$$W_2(A_2(x,y)) + B_2(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{(1+4\beta)/\beta} ds\, dt \in \mathrm{Dom}\left(W_2^{-1}\right). \tag{2.10}$$

Proof. With the definition of $\tilde{a}(x,y)$ and $\tilde{f}(x,y,s,t)$, clearly, $\tilde{a}(x,y)$ and $\tilde{f}(x,y,s,t)$ are nonnegative and nondecreasing in x and y. Furthermore, $\tilde{a}(x,y) \ge a(x,y)$ and $\tilde{f}(x,y,s,t) \ge f(x,y,s,t)$. From (1.2), we have

$$u(x,y) \le \tilde{a}(x,y) + \int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}\tilde{f}(x,y,s,t)w(u(s,t))ds\, dt. \tag{2.11}$$

Next, for convenience, we introduce indices p_i, q_i. Denote that if $[\alpha,\beta,\gamma] \in I$, then let $p_1 = 1/\beta$ and $q_1 = 1/(1-\beta)$; if $[\alpha,\beta,\gamma] \in II$, then let $p_2 = (1+4\beta)/(1+3\beta)$ and $q_2 = (1+4\beta)/\beta$. Then $1/p_i + 1/q_i = 1$ holds for $i = 1, 2$.

Using the Hölder inequality with indices p_i, q_i to (2.11), we get

$$u(x,y) \le \tilde{a}(x,y) + \left(\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)}s^{p_i(\gamma-1)}(y^\alpha - t^\alpha)^{p_i(\beta-1)}t^{p_i(\gamma-1)}ds\, dt\right)^{1/p_i}$$
$$\times \left(\int_0^x\int_0^y \left(\tilde{f}(x,y,s,t)\right)^{q_i}(w(u(s,t)))^{q_i}ds\, dt\right)^{1/q_i}. \tag{2.12}$$

By

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}\left(A_1^r + A_2^r + \cdots + A_n^r\right), \quad A_i \ge 0,\ r \ge 1, \tag{2.13}$$

from (2.12) and Lemma 2.2, we have

$$\begin{aligned} &u^{q_i}(x,y) \\ &\leq 2^{q_i-1}\left[\widetilde{a}(x,y)^{q_i} + \left(\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)} s^{p_i(\gamma-1)} (y^\alpha - t^\alpha)^{p_i(\beta-1)} t^{p_i(\gamma-1)} ds\,dt\right)^{q_i/p_i}\right. \\ &\qquad\qquad \left.\times\left(\int_0^x\int_0^y \left(\widetilde{f}(x,y,s,t)\right)^{q_i} (w(u(s,t)))^{q_i} ds\,dt\right)\right] \\ &= 2^{q_i-1}\widetilde{a}(x,y)^{q_i} + 2^{q_i-1}\left(M_i^2(xy)^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\widetilde{f}(x,y,s,t)\right)^{q_i} (w(u(s,t)))^{q_i} ds\,dt\right), \end{aligned} \tag{2.14}$$

where

$$M_{\mathrm{i}} = \frac{1}{\alpha} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \tag{2.15}$$

and θ_i is given in Lemma 2.3 for $i = 1,2$.

Since $q_i \geq 0$ and $\theta_i \geq 0$ $(i = 1,2)$, then $\widetilde{a}(x,y)^{q_i}$ and $((xy)^{\theta_i})^{q_i/p_i}$ are also nondecreasing in x and y. Taking any arbitrary $\widetilde{x}$ and $\widetilde{y}$ with $\widetilde{x} \leq X, \widetilde{y} \leq Y$, we obtain

$$u^{q_i}(x,y) \leq 2^{q_i-1}\widetilde{a}(\widetilde{x},\widetilde{y})^{q_i} + 2^{q_i-1}\left(M_i^2(\widetilde{x}\widetilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\widetilde{f}(\widetilde{x},\widetilde{y},s,t)\right)^{q_i} (w(u(s,t)))^{q_i} ds\,dt\right) \tag{2.16}$$

for $0 \leq x \leq \widetilde{x}$, $0 \leq y \leq \widetilde{y}$. Denote

$$A_i(\widetilde{x},\widetilde{y}) = 2^{q_i-1}\widetilde{a}(\widetilde{x},\widetilde{y})^{q_i}, \tag{2.17}$$

and let

$$z_i(x,y) = A_i(\widetilde{x},\widetilde{y}) + 2^{q_i-1}\left(M_i^2(\widetilde{x}\widetilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\widetilde{f}(\widetilde{x},\widetilde{y},s,t)\right)^{q_i} (w(u(s,t)))^{q_i} ds\,dt\right). \tag{2.18}$$

Then, $u^{q_i}(x,y) \leq z_i(x,y)$ or $u(x,y) \leq z_i^{1/q_i}(x,y)$. Meanwhile, $z_i(0,y) = A_i(\widetilde{x},\widetilde{y})$, and $z_i(x,y)$ is nondecreasing in x and y. Considering

$$\begin{aligned} D_1 z_i(x,y) &= 2^{q_i-1}\left(M_i^2(\widetilde{x}\widetilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^y \left(\widetilde{f}(\widetilde{x},\widetilde{y},x,t)\right)^{q_i} (w(u(x,t)))^{q_i} ds\,dt\right) \\ &\leq 2^{q_i-1}\left(M_i^2(\widetilde{x}\widetilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^y \left(\widetilde{f}(\widetilde{x},\widetilde{y},x,t)\right)^{q_i} \left(w(z_i(x,t))^{1/q_i}\right)^{q_i} dt\right), \end{aligned} \tag{2.19}$$

we have

$$\frac{D_1 z_i(x,y)}{w^{q_i}\left(z_i^{1/q_i}(x,y)\right)} \le 2^{q_i-1}\left(M_i^2(\tilde{x}\tilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^y \left(\tilde{f}(\tilde{x},\tilde{y},x,t)\right)^{q_i} dt\right), \tag{2.20}$$

where we apply the fact that $w^{q_i}(z_i^{1/q_i}(x,y))$ is nondecreasing in y. Integrating both sides of the above inequality from 0 to x, we obtain

$$\begin{aligned} W_i(z_i(x,y)) &\le W_i(z_i(0,y)) + 2^{q_i-1}\left(M_i^2(\tilde{x}\tilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\tilde{f}(\tilde{x},\tilde{y},s,t)\right)^{q_i} ds\,dt\right) \\ &= W_i(A_i(\tilde{x},\tilde{y})) + 2^{q_i-1}\left(M_i^2(\tilde{x}\tilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\tilde{f}(\tilde{x},\tilde{y},s,t)\right)^{q_i} ds\,dt\right) \end{aligned} \tag{2.21}$$

for $0 \le x \le \tilde{x}$, $0 \le y \le \tilde{y}$, where

$$W_i(u) = \int_{u_0}^u \frac{d\xi}{w^{q_i}(\xi^{1/q_i})}, \quad u \ge u_0 > 0. \tag{2.22}$$

From assumption (A_2), W_i is strictly increasing so its inverse W_i^{-1} is continuous and increasing in its corresponding domain. Replacing x and y by $\tilde{x}$ and $\tilde{y}$, we have

$$W_i(z_i(\tilde{x},\tilde{y})) \le W_i(A_i(\tilde{x},\tilde{y})) + 2^{q_i-1}\left(M_i^2(\tilde{x}\tilde{y})^{\theta_i}\right)^{q_i/p_i}\left(\int_0^{\tilde{x}}\int_0^{\tilde{y}} \left(\tilde{f}(\tilde{x},\tilde{y},s,t)\right)^{q_i} ds\,dt\right). \tag{2.23}$$

Since $\tilde{x}$ and $\tilde{y}$ are arbitrary, we replace $\tilde{x}$ and $\tilde{y}$ by x and y, respectively, and get

$$W_i(z_i(x,y)) \le W_i(A_i(x,y)) + 2^{q_i-1}\left(M_i^2(xy)^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\tilde{f}(x,y,s,t)\right)^{q_i} ds\,dt\right). \tag{2.24}$$

for $0 \le x \le X$ and $0 \le y \le Y$. The above inequality can be rewritten as

$$z_i(x,y) \le W_i^{-1}\left(W_i(A_i(x,y)) + 2^{q_i-1}\left(M_i^2(xy)^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\tilde{f}(x,y,s,t)\right)^{q_i} ds\,dt\right)\right). \tag{2.25}$$

Therefore, we have

$$\begin{aligned} u(x,y) &\le z_i^{1/q_i}(x,y) \\ &\le \left[W_i^{-1}\left(W_i(A_i(x,y)) + 2^{q_i-1}\left(M_i^2(xy)^{\theta_i}\right)^{q_i/p_i}\left(\int_0^x\int_0^y \left(\tilde{f}(x,y,s,t)\right)^{q_i} ds\,dt\right)\right)\right]^{1/q_i} \end{aligned} \tag{2.26}$$

for $0 \le x \le X$ and $0 \le y \le Y$.

Finally, considering two situations for $i = 1,2$ and using parameters $\alpha,\ \beta,\ \gamma$ to denote p_i, q_i, M_i, and θ_i in the above inequality, we can obtain the estimations, respectively. we omit the details here. □

Remark 2.5. *Medveď* [4, *Theorem* 2.2] *investigated the special case* ($\alpha = \gamma = 1$, $f(x,y,s,t) = F(s,t)$) *of inequality* (1.2) *under the assumption that "* $w(u)$ *satisfies the condition* (q)*." However, in our result, the* (q) *condition is eliminated. If we take* $\alpha = 1$ *and* $w(u) = u$*, then we can obtain the result of linear case* [4, *Theorem* 2.4].

Remark 2.6. *Let* $u^p(x,y) = v(x,y)$*, then* $u(x,y) = v^{1/p}(x,y)$ *or* $u^q(x,y) = v^{q/p}(x,y)$*. Therefore, if we take* $w(v) = v^{q/p}$*, the formula* (2.6) *in* [10] *is the special case of inequality* (1.2)*, and we can obtain more concise results than* (2.7) *and* (2.9) *in* [10]*. Moreover, here the condition* $p \ge q$ *also can be eliminated.*

Remark 2.7. *When* $[\alpha,\beta,\gamma]$ *does not belong to* I *or* II*, there are some technical problems which we do not discuss here.*

3. Some Corollaries

Corollary 3.1. *Let functions* $u(x,y)$*,* $a(x,y)$*,* $f(x,y,s,t)$ *be defined as in Theorem 2.4, and let* k *be a constant with* $0 < k \le 1$*. Suppose that*

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(x,y,s,t)(u(s,t))^k ds\, dt. \tag{3.1}$$

Then,

(1) for $[\alpha,\beta,\gamma] \in I$*,*

if $k = 1$*,*

$$u(x,y) \le 2^\beta \tilde{a}(x,y) \exp\left[(1-\beta) B_1(x,y) \int_0^x \int_0^y \left(\tilde{f}(x,y,s,t)\right)^{1/(1-\beta)} ds\, dt\right], \tag{3.2}$$

if $0 < k < 1$*,*

$$u(x,y) \le \left\{2^\beta \tilde{a}(x,y)^{(1-k)/(1-\beta)} + (1-k) B_1(x,y) \int_0^x \int_0^y \left(\tilde{f}(x,y,s,t)\right)^{1/(1-\beta)} ds\, dt\right\}^{(1-\beta)/(1-k)} \tag{3.3}$$

for $x \ge 0$*,* $y \ge 0$*, where* $\tilde{a}(x,y)$*,* $\tilde{f}(x,y,s,t)$*,* $B_1(x,y)$ *are defined as in Theorem 2.4,*

(2) for $[\alpha,\beta,\gamma] \in II$*,*

if $k = 1$*,*

$$u(x,y) \le 2^{(1+3\beta)/(1+4\beta)} \tilde{a}(x,y) \exp\left[\frac{\beta}{1+4\beta} B_2(x,y) \int_0^x \int_0^y \left(\tilde{f}(x,y,s,t)\right)^{(1+4\beta)/\beta} ds\, dt\right], \tag{3.4}$$

if $0 < k < 1$,

$$u(x,y) \le \left\{ \left(2^{1+3\beta}\tilde{a}(x,y)^{1+4\beta}\right)^{(1-k)/\beta} + (1-k)B_2(x,y)\int_0^x\int_0^y \left(\tilde{f}(x,y,s,t)\right)^{(1+4\beta)/\beta} ds\,dt \right\}^{\beta/(1+4\beta)(1-k)}, \tag{3.5}$$

for $x \ge 0$, $y \ge 0$, *where* $\tilde{a}(x,y)$, $\tilde{f}(x,y,s,t)$, $B_2(x,y)$ *are defined as in Theorem 2.4.*

Proof. Clearly, inequality (3.1) is the special case of (1.2). Taking $w(u) = u^k$, we can get (3.1).

(i) If $k = 1$,

$$W_i(u) = \int_{u_0}^{u} \frac{d\xi}{\xi} = \ln\frac{u}{u_0}, \quad u \ge u_0 > 0,$$
$$W_i^{-1}(u) = u_0 e^u, \quad \mathrm{Dom}\left(W_i^{-1}\right) = [0,\infty), \quad i = 1,2. \tag{3.6}$$

(ii) If $0 < k < 1$,

$$W_i(u) = \int_{u_0}^{u} \frac{d\xi}{\xi^k} = \frac{1}{1-k}\left(u^{1-k} - u_0^{1-k}\right),$$
$$W_i^{-1}(u) = \left(u_0^{1-k} + (1-k)u\right)^{1/(1-k)}, \quad \mathrm{Dom}\left(W_i^{-1}\right) = [0,\infty), \quad i = 1,2. \tag{3.7}$$

□

Therefore, the positive numbers X and Y in (2.6) and (2.10) can be taken as ∞, and the results can be obtained by simple computation. We omit the details.

Corollary 3.2. *Let functions* $u(x,y)$, $a(x,y)$, $f(x,y,s,t)$ *be defined as in Theorem 2.4. Suppose that* $g(x,y,s,t) \in C(R_+^4, R_+)$ *and* $u(x,y)$ *satisfies*

$$u(x,y) \le a(x,y) + \int_0^x\int_0^y (x^\sigma - s^\sigma)^{\mu-1}s^{\tau-1}(y^\sigma - t^\sigma)^{\mu-1}t^{\tau-1}g(x,y,s,t)u(s,t)ds\,dt$$
$$+ \int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}f(x,y,s,t)w(u(s,t))ds\,dt. \tag{3.8}$$

Then,

(i) *if* $[\alpha,\beta,\gamma], [\sigma,\mu,\tau] \in I$,

$$u(x,y) \le \Bigg[W_1^{-1}\Bigg(W_1\left(A_1(x,y)\Omega_1(x,y)^{1/(1-\beta)}\right)$$
$$+ \Omega_1(x,y)^{1/(1-\beta)}B_1(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{1/(1-\beta)}ds\,dt\Bigg)\Bigg]^{1-\beta} \tag{3.9}$$

for $0 \le x \le X_1$ *and* $0 \le y \le Y_1$*, where*

$$\Omega_1(x,y) = 2^{\mu} \exp\left[(1-\mu)B_1(x,y)\int_0^x\int_0^y \tilde{g}(x,y,s,t)^{1/(1-\mu)}ds\,dt\right], \tag{3.10}$$

W_1, W_1^{-1}, $A_1(x,y)$, $B_1(x,y)$ *are defined as in Theorem 2.4, and* $X_1, Y_1 \in R_+$ *are chosen such that*

$$\begin{aligned} &W_1\left(A_1(x,y)\Omega_1(x,y)^{1/(1-\beta)}\right) \\ &\quad + \Omega_1(x,y)^{1/(1-\beta)}B_1(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{1/(1-\beta)}ds\,dt \in \operatorname{Dom}\left(W_1^{-1}\right), \end{aligned} \tag{3.11}$$

(ii) *if* $[\alpha,\beta,\gamma], [\sigma,\mu,\tau] \in II$,

$$\begin{aligned} u(x,y) \le \Bigg[W_2^{-1}\Bigg(&W_2\left(A_2(x,y)\Omega_2(x,y)^{(1+4\beta)/\beta}\right) \\ &+ \Omega_2(x,y)^{(1+4\beta)/\beta}B_2(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{(1+4\beta)/\beta}ds\,dt\Bigg)\Bigg]^{\beta/(1+4\beta)} \end{aligned} \tag{3.12}$$

for $0 \le x \le X_2$ *and* $0 \le y \le Y_2$*, where*

$$\Omega_2(x,y) = 2^{(1+3\mu)/(1+4\mu)} \exp\left[\frac{\mu}{1+4\mu}B_2(x,y)\int_0^x\int_0^y \tilde{g}(x,y,s,t)^{(1+4\mu)/\mu}ds\,dt\right], \tag{3.13}$$

W_2, W_2^{-1}, $A_2(x,y)$, $B_2(x,y)$ *are defined as in Theorem 2.4, and* $X_2, Y_2 \in R_+$ *are chosen such that*

$$\begin{aligned} &\left(W_2\left(A_2(x,y)\Omega_2(x,y)^{(1+4\beta)/\beta}\right)\right. \\ &\quad \left.+ \Omega_2(x,y)^{(1+4\beta)/\beta}B_2(x,y)\int_0^x\int_0^y \tilde{f}(x,y,s,t)^{(1+4\beta)/\beta}ds\,dt\right) \in \operatorname{Dom}\left(W_2^{-1}\right). \end{aligned} \tag{3.14}$$

Proof. By the two mentioned lemmas, it follows from (3.8) that

$$u(x,y) \le P_i(x,y) + \int_0^x\int_0^y (x^{\sigma}-s^{\sigma})^{\mu-1}s^{\tau-1}(y^{\sigma}-t^{\sigma})^{\mu-1}t^{\tau-1}\tilde{g}(x,y,s,t)u(s,t)ds\,dt, \tag{3.15}$$

where $\tilde{g}(x,y,s,t) = \max_{0\le\tau\le x, 0\le\eta\le y} g(\tau,\eta,s,t)$ and

$$P_i(x,y) = \tilde{a}(x,y) + \left(M_i^2(xy)^{\theta_i}\right)^{1/p_i}\left[\int_0^x\int_0^y \tilde{f}^{q_i}(x,y,s,t)w(u(s,t))^{q_i}ds\,dt\right]^{1/q_i}. \tag{3.16}$$

(i) For $[\alpha,\beta,\gamma], [\sigma,\mu,\tau] \in I$, applying Corollary 3.1 to (3.15), we have

$$u(x,y) \le 2^{\mu} P_1(x,y) \exp\left[(1-\mu) B_1(x,y) \int_0^x \int_0^y \tilde{g}(x,y,s,t)^{1/(1-\mu)} ds\, dt\right]. \tag{3.17}$$

Letting

$$\Omega_1(x,y) = 2^{\mu} \exp\left[(1-\mu) B_1(x,y) \int_0^x \int_0^y \tilde{g}(x,y,s,t)^{1/(1-\mu)} ds\, dt\right], \tag{3.18}$$

we get

$$\begin{aligned} u(x,y) &\le P_1(x,y)\Omega_1(x,y) \\ &= \tilde{a}(x,y)\Omega_1(x,y) \\ &\quad + \Omega_1(x,y)\left(M_1^2 (xy)^{\theta_1}\right)^{1/p_1} \left[\int_0^x \int_0^y \tilde{f}^{q_1}(x,y,s,t) w(u(s,t))^{q_1} ds\, dt\right]^{1/q_1}. \end{aligned} \tag{3.19}$$

Since inequality (3.19) is similar to (2.12), we can repeat the procedure of proof in Theorem 2.4 and get (3.9).

(ii) As for the case that $[\alpha,\beta,\gamma], [\sigma,\mu,\tau] \in II$, the proof is similar to the argument in the proof of case (i) with suitable modification. We omit the details. □

Remark 3.3. When $[\alpha,\beta,\gamma] \in I$, $[\sigma,\mu,\tau] \in II$ or $[\alpha,\beta,\gamma] \in II$, $[\sigma,\mu,\tau] \in I$, we can get the results which are similar to that in Corollary 3.2 and omit them here.

4. Application

In this section, we will apply our result to discuss the boundedness of certain partial integral equation with weakly singular kernel.

Suppose that $u(x,y) \in C(R_+^2, R+)$ satisfies the inequality as follow:

$$u(x,y) \le \frac{1}{2} + \int_0^x \int_0^y (x-s)^{-1/3} s^{-1/6} (y-t)^{-1/3} t^{-1/6} e^{-s-2t} \sqrt{u(s,t)}\, ds\, dt \tag{4.1}$$

for $x \ge 0$, $y \ge 0$. Then, (4.1) is the special case of inequality (1.2) that is,

$$\begin{gathered} a(x,y) = \frac{1}{2}, \qquad \alpha = 1, \qquad \beta = \frac{2}{3}, \qquad \gamma = \frac{5}{6}, \\ f(x,y,s,t) = e^{-s-2t}, \qquad w(u) = \sqrt{u(s,t)}. \end{gathered} \tag{4.2}$$

Obviously, $[\alpha, \beta, \gamma] = [1, 2/3, 5/6] \in I$. Letting $p_1 = 3/2$, $q_1 = 3$, we have

$$\begin{gathered}
\widetilde{a}(x,y) = \frac{1}{2}, \qquad \widetilde{f}(x,y,s,t) = e^{-s-2t}, \\
A_1(x,y) = 2^2\left(\frac{1}{2}\right)^3 = \frac{1}{2}, \qquad M_1 = B\left[\frac{3}{4}, \frac{1}{2}\right], \\
B_1(x,y) = 2^2\left\{\left(B\left[\frac{3}{4}, \frac{1}{2}\right]\right)^2 (xy)^{1/4}\right\}^2 = 4\left(B\left[\frac{3}{4}, \frac{1}{2}\right]\right)^4 \sqrt{xy}, \\
W_1(u) = \int_{u_0}^{u} \frac{d\xi}{\sqrt{\xi}} = 2\left(\sqrt{u} - \sqrt{u_0}\right), \\
W_1^{-1}(u) = \left(\sqrt{u_0} + \frac{u}{2}\right)^2, \quad \operatorname{Dom}\left(W_1^{-1}\right) = [0, +\infty).
\end{gathered} \tag{4.3}$$

Applying (2.3) in Theorem 2.4, we get for $x \geq 0$, $y \geq 0$

$$\begin{aligned}
u(x,y) &\leq \left[W_1^{-1}\left(W_1(A_1(x,y)) + B_1(x,y)\int_0^x\int_0^y \left(e^{-s-2t}\right)^3 ds\, dt\right)\right]^{1/3} \\
&= \left[W_1^{-1}\left(W_1\left(\frac{1}{2}\right) + 4\left(B\left[\frac{3}{4}, \frac{1}{2}\right]\right)^4 \sqrt{xy}\int_0^x\int_0^y e^{-3}e^{-6t} ds\, dt\right)\right]^{1/3} \\
&= \left[W_1^{-1}\left(\sqrt{2} - 2\sqrt{u_0} + \frac{2}{9}\left(B\left[\frac{3}{4}, \frac{1}{2}\right]\right)^4 \sqrt{xy}\left(1 - e^{-3x}\right)\left(1 - e^{-6y}\right)\right)\right]^{1/3} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{1}{9}\left(B\left[\frac{3}{4}, \frac{1}{2}\right]\right)^4 \sqrt{xy}\left(1 - e^{-3x}\right)\left(1 - e^{-6y}\right)\right)^{2/3}
\end{aligned} \tag{4.4}$$

which implies that $u(x,y)$ in (4.1) is bounded.

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